Some identities with hyperbolic functions

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Abstract. In this note we show some identities with hyperbolic functions

Resumen. Se muestra una colección de Identidades que involucran funciones hiperbólicas.

1. Introducción

Las funciones hiperbólicas se definen por las siguientes fórmulas:

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \cosh(x) = \frac{e^x + e^{-x}}{2}, \tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)}, \operatorname{sech}(x) = \frac{1}{\cosh(x)}, \coth(x) = \frac{1}{\tanh(x)}$$

Algunas propiedades básicas son:

$$(\cosh x)^2 - (\sinh x)^2 = 1 \quad , \quad e^{\pm x} = \cosh x \pm \sinh x$$

$$\cosh x \geq 1 \quad \forall\, x \in \mathbb{R}, \cosh(-x) = \cosh x, \sinh(-x) = -\sinh x$$

Las fórmulas (2.4) y (2.26) son válidas para $x > 0, k > 0$; (2.5) es válida para $0 < a < b$; (2.8) y (2.9) son válidas para $x > \ln\left(1+\sqrt{2}\right)$, y los coeficientes a_n estan definidos por: $a_{n+2} = -2a_{n+1} + a_n, a_0 = 1, a_1 = -2$; (2.13) es válida para $x > 0, m > 2$; (2.29) y (2.30) son válidas para $x > 0, -1 < y < 1$; el resto de las fórmulas son válidas para $x > 0$.

2. Identidades

2.1.

$$\sum_{n=1}^{\infty} (\coth(nx) - \tanh(nx)) = 2 \sum_{n=1}^{\infty} \left(\coth\left((2n-1)x\right) - 1\right)$$

2.2.

$$\sum_{n=1}^{\infty}(1-\tanh(nx)) = \sum_{n=1}^{\infty}(-1)^{n-1}\,(\coth(nx)-1)$$

2.3.

$$\sum_{n=1}^{\infty} e^{-nx}\tanh(nx) = \frac{1}{2}\left(\coth\left(\frac{x}{2}\right)-1\right)+\sum_{n=1}^{\infty}(-1)^n\left(\coth\left(\left(\frac{2n+1}{2}\right)x\right)-1\right)$$

2.4.

$$\sum_{n=1}^{\infty}(\operatorname{sech}(nx))^k = 2^{k-1}\sum_{n=0}^{\infty}\frac{(-1)^n(k)_n}{n!}\left(\coth\left(\left(\frac{2n+k}{2}\right)x\right)-1\right)$$

2.5.

$$\sum_{n=1}^{\infty}\frac{(\tanh(nb)-\tanh(na))}{n} = 2\ln\prod_{n=1}^{\infty}\frac{\sinh(nb)}{\sinh(na)}e^{-n(b-a)}$$

2.6.

$$\sum_{n=1}^{\infty}\frac{\cosh(nx)}{\cosh(2nx)} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^{\left[\frac{n}{2}\right]}\left(\coth\left(\left(\frac{2n+1}{2}\right)x\right)-1\right)$$

2.7.

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cosh(nx)}{\cosh(2nx)} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^{\left[\frac{n}{2}\right]}\left(1-\tanh\left(\left(\frac{2n+1}{2}\right)x\right)\right)$$

2.8.

$$\sum_{n=1}^{\infty}\frac{1}{1+\sinh(nx)} = \sum_{n=0}^{\infty}a_n\left(\coth\left(\left(\frac{n+1}{2}\right)x\right)-1\right)$$

2.9.

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{1+\sinh(nx)} = \sum_{n=0}^{\infty}a_n\left(1-\tanh\left(\left(\frac{n+1}{2}\right)x\right)\right)$$

2.10.

$$\sum_{n=1}^{\infty}\frac{1}{1+\cosh(nx)} = \sum_{n=0}^{\infty}(-1)^n\,(n+1)\left(\coth\left(\left(\frac{n+1}{2}\right)x\right)-1\right)$$

2.11.

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{1+\cosh(nx)} = \sum_{n=0}^{\infty}(-1)^n\,(n+1)\left(1-\tanh\left(\left(\frac{n+1}{2}\right)x\right)\right)$$

2.12.

$$\sum_{n=1}^{\infty} \ln(\tanh(nx)) = -\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(\coth\left((2n+1)x\right) - 1\right)$$

2.13.

$$\sum_{n=0}^{\infty} e^{-m\,n\,x}\,(\cosh(nx))^2 = \frac{1}{2}\left(1 + \coth\left(\frac{mx}{2}\right)\right) + \sum_{n=0}^{\infty} e^{-m\,n\,x}\,(\sinh(nx))^2$$

2.14.

$$\sum_{n=1}^{\infty} e^{-n\,x}\coth(nx) = \frac{1}{2}\left(\coth\left(\frac{x}{2}\right) - 1\right) + \sum_{n=1}^{\infty}\left(\coth\left(\left(\frac{2n+1}{2}\right)x\right) - 1\right)$$

2.15.

$$\sum_{n=1}^{\infty} \operatorname{sech}(nx) = \sum_{n=0}^{\infty}(-1)^n\left(\coth\left(\left(\frac{2n+1}{2}\right)x\right) - 1\right)$$

2.16.

$$\sum_{n=1}^{\infty}(-1)^{n-1}\operatorname{sech}(nx) = \sum_{n=0}^{\infty}(-1)^n\left(1 - \tanh\left(\left(\frac{2n+1}{2}\right)x\right)\right)$$

2.17.

$$\sum_{n=1}^{\infty}(\operatorname{csch}(nx) - \operatorname{sech}(nx)) = 2\sum_{n=0}^{\infty}\left(\coth\left(\left(\frac{4n+3}{2}\right)x\right) - 1\right)$$

2.18.

$$\sum_{n=1}^{\infty} \ln(1 + \operatorname{sech}(nx)) = \sum_{n=1}^{\infty}\frac{1}{2n+1}\left(\coth\left(\left(\frac{2n-1}{2}\right)x\right) - \coth\left((2n-1)x\right)\right)$$

2.19.

$$\sum_{n=1}^{\infty} \ln(1 - \operatorname{sech}(nx)) = -\sum_{n=1}^{\infty}\frac{1}{2n-1}\left(\coth\left(\left(\frac{2n-1}{2}\right)x\right) + \coth\left((2n-1)x\right) - 2\right)$$

2.20.

$$\sum_{n=0}^{\infty} \sinh(e^{-n\,x}) = \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(\coth\left(\left(\frac{2n+1}{2}\right)x\right) + 1\right)$$

2.21.

$$\sum_{n=0}^{\infty}(-1)^n \sinh(e^{-n\,x}) = \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(1+\tanh\left(\left(\frac{2n+1}{2}\right)x\right)\right)$$

2.22.

$$\sum_{n=0}^{\infty}(\cosh(e^{-n\,x})-1) = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{(2n)!}(1+\coth(nx))$$

2.23.

$$\sum_{n=0}^{\infty}(-1)^n\,(\cosh(e^{-n\,x})-1) = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{(2n)!}(1+\tanh(nx))$$

2.24.

$$\sum_{n=1}^{\infty}\operatorname{csch}(nx) = \sum_{n=0}^{\infty}\left(\coth\left(\left(\frac{2n+1}{2}\right)x\right)-1\right)$$

2.25.

$$\sum_{n=1}^{\infty}(-1)^{n-1}\operatorname{csch}(nx) = \sum_{n=0}^{\infty}\left(1-\tanh\left(\left(\frac{2n+1}{2}\right)x\right)\right)$$

2.26.

$$\sum_{n=1}^{\infty}(\operatorname{csch}(nx))^k = 2^{k-1}\sum_{n=0}^{\infty}\frac{(k)_n}{n!}\left(\coth\left(\left(\frac{2n+k}{2}\right)x\right)-1\right)$$

2.27.

$$\sum_{n=1}^{\infty}\ln(2e^{-n\,x}\sinh(nx)) = -\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}(\coth(nx)-1)$$

2.28.

$$\sum_{n=1}^{\infty}\ln(2e^{-n\,x}\cosh(nx)) = \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}(\coth(nx)-1)$$

2.29.

$$\tan^{-1}(y) + 2\sum_{n=1}^{\infty}(-1)^n\tan^{-1}(ye^{-n\,x}) = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}y^{2n+1}\tanh\left(\frac{2n+1}{2}x\right)$$

$$\pi + 12\sum_{n=1}^{\infty}(-1)^n\tan^{-1}\left(\frac{e^{-n\,x}}{\sqrt{3}}\right) = 2\sqrt{3}\sum_{n=0}^{\infty}\frac{(-1/3)^n}{2n+1}\tanh\left(\frac{2n+1}{2}x\right)$$

2.30.

$$\tan^{-1}(y)+2\sum_{n=1}^{\infty}\tan^{-1}(ye^{-n\,x})=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}y^{2n+1}\coth\left(\frac{2n+1}{2}x\right)$$

$$\pi+12\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{e^{-n\,x}}{\sqrt{3}}\right)=2\sqrt{3}\sum_{n=0}^{\infty}\frac{(-1/3)^n}{2n+1}\coth\left(\frac{2n+1}{2}x\right)$$

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