An elementary proof that $\sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}$ *

Hervé G. (also known as FDP)

August 1, 2016

Abstract

It is presented a simple proof that,

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}$$

1 Introduction

It is known that Léonard Euler evaluated the series,

$$\sum_{n=1}^{+\infty} \frac{1}{n^2}$$

answering the Basel problem [1].

It is little known that he evaluated empirically also the value of what is known today as

$$\beta(3) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)^3}$$

the Dirichlet Beta function evaluated at 3.

*Thanks go to `http://les-mathematiques.net` and `http://math.stackexchange.com` for inspiration.

Actually, he empirically stated that [2],

$$\frac{1}{m^3}+\sum_{k=1}^{+\infty}\left(\frac{1}{(2kn+m)^3}-\frac{1}{(2kn-m)^3}\right)=\frac{1+\left(\tan\left(\frac{m\pi}{2n}\right)\right)^2}{8n^3\left(\tan\left(\frac{m\pi}{2n}\right)\right)^3}\pi^3$$

And from this formula, evaluated at $m=1, n=2$, he stated explicitly a value for $\beta(3)$ [2].

Assuming only that,

$$\zeta(2)=\frac{\pi^2}{6}$$

It's proven below, using elementary methods of analysis, that

$$\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^3}=\frac{\pi^3}{32}$$

Detailed computations are provided to show there is no catch.

2 Some preliminary computations

Lemma 2.1. *For $n\geq 0$, integer, and $\beta>\alpha>0$, real, the following formula holds.*

$$\boxed{\int_\alpha^\beta x^n\ln x dx=\frac{1}{n+1}(\beta^{n+1}\ln\beta-\alpha^{n+1}\ln\alpha)-\frac{1}{(n+1)^2}(\beta^{n+1}-\alpha^{n+1})} \tag{2.1}$$

Proof.

For $n \geq 0$, integer,

$$\int_\alpha^\beta x^n \ln x dx = \left[\frac{x^{n+1}}{n+1} \ln x\right]_\alpha^\beta - \int_\alpha^\beta \frac{x^{n+1}}{n+1} \times \frac{1}{x} dx$$

$$= \frac{1}{n+1}(\beta^{n+1} \ln \beta - \alpha^{n+1} \ln \alpha) - \frac{1}{n+1}\left[\frac{x^{n+1}}{n+1}\right]_\alpha^\beta$$

$$= \frac{1}{n+1}(\beta^{n+1} \ln \beta - \alpha^{n+1} \ln \alpha) - \frac{1}{(n+1)^2}(\beta^{n+1} - \alpha^{n+1})$$

□

Lemma 2.2.

$$\boxed{\int_0^1 \frac{\ln x}{x-1} dx = \frac{\pi^2}{6}} \tag{2.2}$$

Proof.

$0 < \alpha < \beta < 1$, real,

$$\int_\alpha^\beta \frac{\ln x}{x-1} dx = \int_\alpha^\beta \frac{x^{N+1} \ln x}{x-1} dx - \int_\alpha^\beta \left(\frac{x^{N+1}-1}{x-1}\right) \ln x dx$$

From 2.1 , it follows that,

$$\int_\alpha^\beta \left(\frac{x^{N+1}-1}{x-1}\right) \ln x dx = \int_\alpha^\beta \left(\sum_{k=0}^N x^k\right) \ln x dx$$

$$= \sum_{k=0}^N \left(\int_\alpha^\beta x^k \ln x dx\right)$$

$$= \ln \beta \sum_{k=0}^N \frac{\beta^{k+1}}{k+1} - \sum_{k=0}^N \frac{\alpha^{k+1} \ln \alpha}{k+1} - \sum_{k=0}^N \frac{\beta^{k+1}}{(k+1)^2} + \sum_{k=0}^N \frac{\alpha^{k+1}}{(k+1)^2}$$

and for $k \geq 0$,

$$\boxed{\lim_{\alpha \to 0} \alpha^{k+1} \ln(\alpha) = 0} \tag{2.3}$$

Therefore,

$$\lim_{\alpha \to 0, \beta \to 1} \left(\ln \beta \sum_{k=0}^N \frac{\beta^{k+1}}{k+1} - \sum_{k=0}^N \frac{\alpha^{k+1} \ln \alpha}{k+1} - \sum_{k=0}^N \frac{\beta^{k+1}}{(k+1)^2} + \sum_{k=0}^N \frac{\alpha^{k+1}}{(k+1)^2}\right) = -\sum_{k=0}^N \frac{1}{(k+1)^2}$$

and then, for $n \geq 0$, integer,

$$\int_0^1 \left(\frac{x^{N+1}-1}{x-1}\right) \ln x dx = -\sum_{k=1}^{N+1} \frac{1}{k^2}$$

Define the function f for $x \geq 0$, real,

$$\begin{aligned} &x = 0, f(0) = 0 \\ &x = 1, f(1) = 1 \\ &x \neq 0, 1, f(x) = \frac{x \ln x}{x-1} \end{aligned}$$

Observe that,

$$\begin{aligned} \lim_{x \to 1} \frac{\ln x}{x-1} &= \lim_{x \to 1} \frac{\ln x - \ln 1}{x-1} \\ &= (\ln)'(1) \\ &= 1 \end{aligned}$$

and, from 2.3 , it follows that f is continuous.

and then, there exist m, M, real, such that, for $x \in [0,1]$,

$$m \leq f(x) \leq M$$

Therefore, for $x \in [0;1]$, $N \geq 0$, integer,

$$mx^N \leq x^N f(x) \leq Mx^N$$

and then, for $N \geq 0$, integer,

$$\frac{m}{N+1} \leq \int_0^1 \frac{x^{N+1} \ln x}{x-1} dx \leq \frac{M}{N+1}$$

Therefore,

$$\lim_{N \to +\infty} \int_0^1 \frac{x^{N+1} \ln x}{x-1} dx = 0$$

Since,

$$\lim_{N\to+\infty}\sum_{k=0}^{N}\frac{1}{(k+1)^2}=\zeta(2)$$

then,

$$\int_0^1\frac{\ln x}{x-1}dx=\frac{\pi^2}{6}$$

□

Lemma 2.3.

$$\boxed{\sum_{k=1}^{+\infty}\frac{(-1)^k}{k^2}=-\frac{\pi^2}{12}} \tag{2.4}$$

Proof.
Since, for $k\geq 1$,

$$\left|\frac{(-1)^k}{k^2}\right|\leq\frac{1}{k^2}$$

the series is convergent.

For $N\geq 1$, integer,

$$\begin{aligned}\sum_{k=1}^{2N+1}\frac{(-1)^k}{k^2}&=\sum_{k=1}^{N}\frac{1}{(2k)^2}-\sum_{k=0}^{N}\frac{1}{(2k+1)^2}\\&=\sum_{k=1}^{N}\frac{1}{(2k)^2}-\left(\sum_{k=1}^{2N+1}\frac{1}{k^2}-\sum_{k=1}^{N}\frac{1}{(2k)^2}\right)\\&=\frac{1}{2}\sum_{k=1}^{N}\frac{1}{k^2}-\sum_{k=1}^{2N+1}\frac{1}{k^2}\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{k=1}^{+\infty}\frac{(-1)^n}{n^2}&=\frac{1}{2}\zeta(2)-\zeta(2)\\&=-\frac{1}{2}\zeta(2)\\&=-\frac{1}{2}\times\frac{\pi^2}{6}\\&=-\frac{\pi^2}{12}\end{aligned}$$

Lemma 2.4.

$$\boxed{\int_0^1 \frac{\ln x}{1+x}dx = -\frac{\pi^2}{12}} \tag{2.5}$$

Let $\epsilon > 0$, real, and $N \geq 0$, integer,

$$\int_\epsilon^1 \frac{\ln x}{1+x}dx = \int_\epsilon^1 \left(\frac{1-(-1)^{N+1}x^{N+1}}{1+x}\right)\ln x dx + \int_\epsilon^1 \frac{(-1)^{N+1}x^{N+1}\ln x}{1+x}dx$$

From 2.1 , it follows that,

$$\begin{aligned}
\int_\epsilon^1 \left(\frac{1-(-1)^{N+1}x^{N+1}}{1+x}\right)\ln x dx &= \int_\epsilon^1 \left(\sum_{k=0}^{N}(-1)^k x^k\right)\ln x dx \\
&= \sum_{k=0}^{N}\left(\int_\epsilon^1 (-1)^k x^k \ln x dx\right) \\
&= \sum_{k=0}^{N}(-1)^k \left(\frac{\epsilon^{k+1}}{(k+1)^2} - \frac{1}{(k+1)^2} - \frac{\epsilon^{k+1}\ln\epsilon}{k+1}\right)
\end{aligned}$$

From 2.3 , it follows,

$$\begin{aligned}
\lim_{\epsilon\to 0}\int_\epsilon^1 \left(\frac{1-(-1)^{N+1}x^{N+1}}{1+x}\right)\ln x dx &= \lim_{\epsilon\to 0}\sum_{k=0}^{N}(-1)^k \left(\frac{\epsilon^{k+1}}{(k+1)^2} - \frac{1}{(k+1)^2} - \frac{\epsilon^{k+1}\ln\epsilon}{k+1}\right) \\
&= -\sum_{k=0}^{N}\frac{(-1)^k}{(k+1)^2} \\
&= \sum_{k=1}^{N+1}\frac{(-1)^k}{k^2}
\end{aligned}$$

Therefore,

$$\int_0^1 \left(\frac{1-(-1)^{N+1}x^{N+1}}{1+x}\right)\ln x dx = \sum_{k=1}^{N+1}\frac{(-1)^k}{k^2}$$

Define the function g for $x \geq 0$, real,

$$x = 0,\ g(0) = 0$$

$$x \neq 0,\ g(x) = \frac{x\ln x}{1+x}$$

From 2.3 , it follows that g is continuous and then, there exist m, M, real, such that for $x \in [0,1]$,

$$m \leq g(x) \leq M$$

Therefore, for $x \in [0,1]$, $N \geq 0$, integer,

$$mx^N \leq x^N g(x) \leq Mx^N$$

and then, for $N \geq 0$, integer,

$$\frac{m}{N+1} \leq \int_0^1 \frac{x^{N+1}\ln x}{1+x}dx \leq \frac{M}{N+1}$$

Therefore,

$$\lim_{N\to+\infty} \int_0^1 \frac{(-1)^{N+1}x^{N+1}\ln x}{1+x}dx = 0$$

and then, from 2.4 , it follows that,

$$\int_0^1 \frac{\ln x}{1+x}dx = \lim_{N\to+\infty} \left(\sum_{k=1}^{N+1} \frac{(-1)^k}{k^2}\right)$$
$$= -\frac{\pi^2}{12}$$

□

Lemma 2.5.

(2.6) $$\boxed{\int_0^1 \frac{x\ln x}{1+x^2}dx = -\frac{\pi^2}{48}}$$

Proof.

From 2.3 , it follows that the function $x \to \dfrac{x\ln x}{1+x^2}$, defined and continuous on $]0,+\infty[$, can be extended to a defined and continuous function on $[0,+\infty[$.

$N \geq 0$ integer. The same applies to the functions $x \to \dfrac{x\left(1-\left(-x^2\right)^{N+1}\right)\ln x}{1+x^2}$ and $x \to \dfrac{(-1)^{N+1}x^{2N+3}\ln x}{1+x^2}$, defined and continuous on $]0,+\infty[$.

From 2.1 , it follows that for $N \geq 0$, integer,

$$\begin{aligned}
\int_0^1 \frac{x \ln x}{1+x^2} dx &= \int_0^1 \left(\frac{x\left(1-(-x^2)^{N+1}\right)}{1+x^2} \right) \ln x dx + \int_0^1 \frac{(-1)^{N+1} x^{2N+3} \ln x}{1+x^2} dx \\
&= \int_0^1 \left(\sum_{k=0}^{N} (-1)^k x^{2k+1} \ln x \right) dx + \int_0^1 \frac{(-1)^{N+1} x^{2N+3} \ln x}{1+x^2} dx \\
&= \sum_{k=0}^{N} \left(\int_0^1 (-1)^k x^{2k+1} \ln x dx \right) + \int_0^1 \frac{(-1)^{N+1} x^{2N+3} \ln x}{1+x^2} dx \\
&= \sum_{k=0}^{N} \frac{(-1)^{k+1}}{(2k+2)^2} + \int_0^1 \frac{(-1)^{N+1} x^{2N+3} \ln x}{1+x^2} dx \\
&= \frac{1}{4} \sum_{k=1}^{N+1} \frac{(-1)^k}{k^2} + \int_0^1 \frac{(-1)^{N+1} x^{2N+3} \ln x}{1+x^2} dx
\end{aligned}$$

$N \geq 0$ integer. Define the function h for $x \geq 0$, real,

$$x = 0,\ h(0) = 0$$

$$x \neq 0,\ h(x) = \frac{x^3 \ln x}{1+x^2}$$

From 2.3 , it follows that h is continuous and then, there exist m, M, real, such that for $x \in [0,1]$,

$$m \leq h(x) \leq M$$

Therefore, for $x \in [0;1]$, $N \geq 0$, integer,

$$m x^{2N} \leq x^{2N} h(x) \leq M x^{2N}$$

and then, for $N \geq 0$, integer,

$$\frac{m}{2N+1} \leq \int_0^1 \frac{x^{2N+3} \ln x}{1+x^2} dx \leq \frac{M}{2N+1}$$

Therefore,

$$\lim_{N \to +\infty} \int_0^1 \frac{(-1)^{N+1} x^{2N+3} \ln x}{1+x^2} dx = 0$$

and then, from 2.4 , it follows that,

$$\begin{aligned}\int_0^1 \frac{x\ln x}{1+x^2}dx &= \lim_{N\to+\infty}\left(\frac{1}{4}\sum_{k=1}^{N+1}\frac{(-1)^k}{k^2}\right)\\ &= \frac{1}{4}\times -\frac{\pi^2}{12}\\ &= -\frac{\pi^2}{48}\end{aligned}$$

□

Lemma 2.6.

For $x \geq 0$, real, and $N \geq 0$, integer,

$$\boxed{\arctan x = \sum_{k=0}^{N}\frac{(-1)^k x^{2k+1}}{2k+1} + R_N(x) \mathit{and}\ |R_N(x)| \leq \frac{x^{2N+3}}{2N+3}} \tag{2.7}$$

R_N is a continuous function on $[0;+\infty[$.

Proof.

Define for $x \geq 0$ and $N \geq 0$, integer,

$$R_N(x) = \int_0^x \frac{(-t^2)^{N+1}}{1+t^2}dt$$

The function R_N is an antiderivative of $t \to \dfrac{(-t^2)^{N+1}}{1+t^2}$, defined and continuous on $[0,+\infty[$. Therefore, R_N is continuous on $[0;+\infty[$.

$$
\begin{aligned}
\arctan x &= \int_0^x \frac{1}{1+t^2}dt \\
&= \int_0^x \frac{1-(-t^2)^{N+1}}{1+t^2}dt + \int_0^x \frac{(-t^2)^{N+1}}{1+t^2}dt \\
&= \int_0^x \left(\sum_{k=0}^{N}(-1)^k t^{2k}\right) dt + \int_0^x \frac{(-t^2)^{N+1}}{1+t^2}dt \\
&= \sum_{k=0}^{N}(-1)^k \left(\int_0^x t^{2k}dt\right) + \int_0^x \frac{(-t^2)^{N+1}}{1+t^2}dt \\
&= \sum_{k=0}^{N}(-1)^k \frac{x^{2k+1}}{2k+1} + \int_0^x \frac{(-t^2)^{N+1}}{1+t^2}dt
\end{aligned}
$$

Since for t, real, $0 < \dfrac{1}{1+t^2} < 1$ then for $N \geq 0$, integer,

$$
\begin{aligned}
\left|\int_0^x \frac{(-t^2)^{N+1}}{1+t^2}dt\right| &\leq \int_0^x t^{2N+2}dt \\
&\leq \frac{x^{2N+3}}{2N+3}
\end{aligned}
$$

□

Lemma 2.7.

$$
\boxed{\int_0^1 \frac{\arctan x \ln x}{x}dx = -\sum_{k=1}^{+\infty} \frac{(-1)^k}{(2k+1)^3}} \tag{2.8}
$$

Proof.

Since for $k \geq 1$, integer,

$$
\left|\frac{(-1)^k}{(2k+1)^3}\right| \leq \frac{1}{(2k+1)^3}
$$

then the series $\displaystyle\sum_{k=1}^{+\infty} \frac{(-1)^k}{(2k+1)^3}$ is convergent.

Since,

$$
\begin{aligned}
\lim_{x\to 0} \frac{\arctan x}{x} &= \lim_{x\to 0} \frac{\arctan x - \arctan(0)}{x-0} \\
&= (\arctan)'(0)
\end{aligned}
$$

That is,

$$(2.9) \quad \boxed{\lim_{x\to 0} \frac{\arctan x}{x} = 1}$$

Thus, it follows that the function $x \to \dfrac{\arctan x}{x}$, defined and continuous on $]0, +\infty[$, can be extended to a defined and continuous function on $[0, +\infty[$.

and then, there exist, m, M, real, such for $x \in]0; 1]$,

$$m \ln x \leq \frac{\arctan x \ln x}{x} \leq M \ln x$$

Thus, the integral $\displaystyle\int_0^1 \frac{\arctan x \ln x}{x} dx$ is convergent.

From 2.7 , it follows that for $N \geq 0$,integer, $x > 0$,

$$\left| \frac{R_N(x) \ln x}{x} \right| \leq x^{2N+2} |\ln x|$$

From 2.3 , it follows that $x \to \dfrac{R_N(x) \ln x}{x}$, a defined and continuous function on $]0; \infty[$, can be extended to a defined and continuous function on $[0; +\infty[$.

The same applies to the function $x \to x^{2k} \ln x$, defined on $]0; +\infty[$ for $k > 0$, integer.

Moreover, for α, β, real, such that $[\alpha; \beta] \subset [0; 1]$, $\displaystyle\int_\alpha^\beta \ln x dx$ is convergent.

Therefore, for $N \geq 0$, integer,

$$\begin{aligned}
\int_0^1 \frac{\arctan x \ln x}{x} &= \int_0^1 \left(\sum_{k=0}^{N} \frac{(-1)^k x^{2k}}{2k+1} \right) \ln x dx + \int_0^1 \frac{R_N(x) \ln x}{x} dx \\
&= \sum_{k=0}^{N} \left(\frac{(-1)^k}{2k+1} \left(\int_0^1 x^{2k} \ln x dx \right) \right) + \int_0^1 \frac{R_N(x) \ln x}{x} dx \\
&= -\sum_{k=0}^{N} \frac{(-1)^k}{(2k+1)^3} + \int_0^1 \frac{R_N(x) \ln x}{x} dx
\end{aligned}$$

From 2.7 , it follows that,

$$\begin{aligned}\left|\int_0^1 \frac{R_N(x)\ln x}{x}dx\right| &\leq \int_0^1 \left|\frac{R_N(x)\ln x}{x}\right| dx \\ &\leq \int_0^1 x^{2N+2}\left|\ln x\right| dx\end{aligned}$$

and, from 2.1 , it follows that,

$$\begin{aligned}\int_0^1 x^{2N+2}\left|\ln x\right| dx &= -\int_0^1 x^{2N+2}\ln x dx \\ &= \frac{1}{(2N+3)^2}\end{aligned}$$

Therefore,

$$\lim_{N\to+\infty}\left(\int_0^1 \frac{R_N(x)\ln x}{x}dx\right) = 0 \tag{2.10}$$

and,

$$\int_0^1 \frac{\arctan x \ln x}{x}dx = -\sum_{k=1}^{+\infty}\frac{(-1)^k}{(2k+1)^3}$$

□

3 Main result

Let,

$$\begin{aligned}G &= -\int_0^1 \frac{\log y}{1+y^2}dy \\ J &= \int_0^1 \frac{y\arctan y \log y}{1+y^2}dy \\ K &= \int_0^1 \frac{\ln y \ln(1+y^2)}{1+y^2}dy \\ L &= \int_0^1 \frac{\arctan y \log y}{1+y}dy \\ M &= \int_0^1 \frac{\ln y \arctan y}{y-1}dy\end{aligned}$$

Since for $x \in]0,1]$, $\left|\frac{\ln x}{1+x^2}\right| \leq -\ln x$ and for α, β such that $[\alpha,\beta] \subset [0,1]$, $\int_\alpha^\beta \ln x dx$ is convergent then $\int_0^1 \frac{\ln x}{1+x^2}dx$ is convergent.

From 2.3 , it follows that the integral J is convergent since
$x \to \dfrac{y \arctan y \ln y}{1+y^2}$ can be extended to a defined and continuous function on $[0;1]$.

Since,

$$\begin{aligned}\lim_{y\to 0}\frac{\ln(1+y^2)}{y} &= \lim_{y\to 0}\frac{\ln(1+y^2)-\ln(1+0^2)}{y-0}\\ &= \lim_{y\to 0}\frac{2y}{1+y^2}\\ &= 0\end{aligned}$$

then, from 2.3 , it follows that the function
$y \to \dfrac{\ln y \ln(1+y^2)}{1+y^2}$ can be extended to a defined and continuous function on $[0;1]$.

Indeed, for $y \in]0;1]$,
$\dfrac{\ln y \ln(1+y^)}{1+y^2} = y \ln y \times \dfrac{\ln(1+y^2)}{y} \times \dfrac{1}{1+y^2}$
Therefore, K is convergent.

For $y \in]0;1]$,
$\dfrac{\arctan y \ln y}{1+y} = y \ln y \times \dfrac{\arctan y}{y} \times \dfrac{1}{1+y}$
From 2.3 and 2.9 , it follows that this function can be extended to a defined and continuous function on $[0;1]$. Therefore, L is convergent.

Lemma 3.1.

$$\boxed{L = \frac{G\ln 2}{2} - \frac{\pi^3}{64}} \tag{3.1}$$

Proof.
Define for $x \in [0,1]$ the function R:

$$R(x) = \int_0^x \frac{\log t}{1+t}dt = \int_0^1 \frac{x\log(xy)}{1+xy}dy$$

Observe that R is continuous and, from 2.5 , it follows that, $R(1) = -\dfrac{\pi^2}{12}$.

$$L = \left[R(x)\arctan x\right]_0^1 - \int_0^1 \frac{R(x)}{1+x^2}dx$$
$$L = -\frac{\pi^3}{48} - \int_0^1 \frac{R(x)}{1+x^2}dx$$
$$L = -\frac{\pi^3}{48} - \int_0^1 \frac{x\log(xy)}{(1+xy)(1+x^2)}dxdy$$
$$L = -\frac{\pi^3}{48} - \int_0^1 \frac{x\log(x)}{(1+xy)(1+x^2)}dxdy - \int_0^1 \frac{x\log(y)}{(1+xy)(1+x^2)}dxdy$$
$$L = -\frac{\pi^3}{48} - \int_0^1 \left[\frac{\log x\log(1+xy)}{1+x^2}\right]_{y=0}^{y=1} dx-$$
$$\int_0^1 \left[-\frac{\log y\log(1+xy)}{1+y^2} + \frac{\log y\log(1+x^2)}{2(1+y^2)} + \frac{y\log y\arctan x}{1+y^2}\right]_{x=0}^{x=1} dy$$
$$L = -\frac{\pi^3}{48} - \int_0^1 \frac{\log x\log(1+x)}{1+x^2}dx + \int_0^1 \frac{\log x\log(1+x)}{1+x^2}dx-$$
$$\frac{\log 2}{2}\int_0^1 \frac{\log y}{1+y^2}dy - \frac{\pi}{4}\int_0^1 \frac{y\log y}{1+y^2}dy$$

From 2.6, it follows that,

$$\int_0^1 \frac{y\log y}{1+y^2}dy = -\frac{\pi^2}{48}$$

Therefore,

$$L = \frac{G \ln 2}{2} - \frac{\pi^3}{64}$$

□

Lemma 3.2.

$$\boxed{J = -\frac{1}{4} G \ln 2 - \frac{\pi^3}{128} - \frac{1}{2} K + \frac{1}{2} M} \tag{3.2}$$

Proof.

Define for all $y \in [0, 1]$,

$$\begin{aligned} S(y) &= \int_0^y \frac{t \ln(t)}{1+t^2} dt \\ &= \int_0^1 \frac{t y^2 \ln(ty)}{1+t^2 y^2} dt \end{aligned}$$

Observe that S is continuous and, from 2.6, it follows that, $S(1) = -\dfrac{\pi^2}{48}$.

$$\begin{aligned} J &= \Big[S(y) \arctan y \Big]_0^1 - \int_0^1 \frac{S(y)}{1+y^2} dy \\ &= -\frac{\pi^3}{192} - \int_0^1 \int_0^1 \frac{t y^2 \ln(ty)}{(1+t^2y^2)(1+y^2)} dt dy \\ &= -\frac{\pi^3}{192} - \int_0^1 \int_0^1 \frac{t y^2 \ln y}{(1+t^2y^2)(1+y^2)} dt dy - \int_0^1 \int_0^1 \frac{t y^2 \ln t}{(1+t^2y^2)(1+y^2)} dt dy \\ &= -\frac{\pi^3}{192} - \frac{1}{2} \int_0^1 \left[\frac{\ln y \ln(1+t^2y^2)}{1+y^2} \right]_{t=0}^{t=1} dt - \\ &\frac{1}{2} \int_0^1 \left[\frac{\ln t \arctan y + \ln t \arctan(ty)}{1+t} - \frac{\ln t \arctan(ty) - \ln t \arctan y}{t-1} \right]_{y=0}^{y=1} dy \\ &= -\frac{\pi^3}{192} - \frac{1}{2} K - \frac{\pi}{8} \int_0^1 \frac{\ln y}{1+y} dy - \frac{1}{2} L + \frac{1}{2} M - \frac{\pi}{8} \int_0^1 \frac{\ln t}{t-1} dt \end{aligned}$$

Since, from 2.2 and 2.5 , it follows that,

$$\int_0^1 \frac{\ln t}{t-1}dt = \frac{\pi^2}{6}$$
$$\int_0^1 \frac{\ln t}{1+t}dt = -\frac{\pi^2}{12}$$

then,

$$J = -\frac{1}{4}G\ln 2 - \frac{\pi^3}{128} - \frac{1}{2}K + \frac{1}{2}M$$

□

Lemma 3.3.

$$\boxed{K = M - 2J - \frac{5\pi^3}{64} - \frac{G\ln 2}{2} + 2\beta(3)} \tag{3.3}$$

Proof.

Define for all $y \in [0,1]$,

$$\begin{aligned} T(y) &= \int_0^y \frac{\ln(t)}{1+t^2} dt \\ &= \int_0^1 \frac{y\ln(ty)}{1+t^2y^2} dt \end{aligned}$$

Observe that T is continuous and, $T(1) = -G$.

$$\begin{aligned} K &= \int_0^1 \frac{\ln y \ln(1+y^2)}{1+y^2} dy = \Big[T(y)\ln(1+y^2) \Big]_0^1 - \int_0^1 \frac{2yT(y)}{1+y^2} dy \\ &= -G\ln 2 - \int_0^1 \int_0^1 \frac{2y^2\ln(ty)}{(1+t^2y^2)(1+y^2)} dtdy \\ &= -G\ln 2 - \int_0^1 \int_0^1 \frac{2y^2\ln y}{(1+t^2y^2)(1+y^2)} dtdy - \int_0^1 \int_0^1 \frac{2y^2\ln t}{(1+t^2y^2)(1+y^2)} dtdy \\ &= -G\ln 2 - \int_0^1 \left[\frac{2y\ln y \arctan(ty)}{1+y^2} \right]_{t=0}^{t=1} dy + \\ & \int_0^1 \left[\frac{\ln t\arctan(ty) + \ln t\arctan y}{1+t} + \frac{\ln t\arctan(ty) - \ln t\arctan y}{t-1} - \frac{2\ln t\arctan(ty)}{t} \right]_{y=0}^{y=1} dt \\ &= -G\ln 2 - 2J + L + \frac{\pi}{4}\int_0^1 \frac{\ln t}{1+t} dt + M - \frac{\pi}{4}\int_0^1 \frac{\ln t}{t-1} dt - 2\int_0^1 \frac{\ln t\arctan t}{t} dt \end{aligned}$$

Since, from 2.8, $\displaystyle\int_0^1 \frac{\ln t\arctan t}{t} dt = -\beta(3)$, then,

$$K = M - 2J - \frac{5\pi^3}{64} - \frac{G\ln 2}{2} + 2\beta(3)$$

□

From 3.2 and 3.3 , it follows that,

$$\boxed{\beta(3) = \frac{\pi^3}{32}}$$

4 References

References

[1] Letter from Daniel Bernoulli to Léonard Euler, 1736,
Correspondance mathématique et physique de quelques célèbres géomètres du XVIIIème siècle, tome 2, page 433. http://docnum.u-strasbg.fr/cdm/ref/collection/coll7/id/69547.

[2] Léonard Euler, 1748, E101,
Introductio in analysin infinitorum, tomus primus, Caput X, De usu factorum inventorum in definiendis summis serierum infinitarum.. Pages 136-137. http://dx.doi.org/10.3931/e-rara-8740.