Infinite arctangent sums involving Fibonacci and Lucas numbers[*]

Kunle Adegoke[†]

Department of Physics and Engineering Physics,
Obafemi Awolowo University, Ile-Ife, 220005 Nigeria

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Abstract

Using a straightforward elementary approach, we derive numerous infinite arctangent summation formulas involving Fibonacci and Lucas numbers. While most of the results obtained are new, a couple of celebrated results appear as particular cases of the more general formulas derived here.

Contents

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[†]adegoke00@gmail.com

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1 Introduction

It is our goal, in this work, to derive infinite arctangent summation formulas involving Fibonacci and Lucas numbers. The results obtained will be found to be of a more general nature than one finds in earlier literature.

Previously known results containing arctangent identities and/or infinite summation involving Fibonacci numbers can be found in references [1, 2, 3, 4, 5] and references therein.

In deriving the results in this paper, the main identities employed are the trigonometric addition formula

$$\tan^{-1}\left\{\frac{\lambda(y-x)}{xy+\lambda^2}\right\} = \tan^{-1}\frac{\lambda}{x} - \tan^{-1}\frac{\lambda}{y}\,, \tag{1.1}$$

which holds for $\lambda \in \mathbb{R}$ such that either $xy > 0$ or $xy < 0$ and $\lambda^2 < -xy$, and the following identities which resolve products of Fibonacci and Lucas numbers

$$F_{u-v}F_{u+v} = F_u^2 - (-1)^{(u-v)}F_v^2\,, \tag{1.2a}$$
$$L_{u-v}L_{u+v} = L_{2u} + (-1)^{(u-v)}L_{2v}\,, \tag{1.2b}$$
$$L_uF_v = F_{v+u} + (-1)^u F_{v-u}\,, \tag{1.2c}$$
$$F_uL_v = F_{v+u} - (-1)^u F_{v-u}\,, \tag{1.2d}$$
$$L_uL_v = L_{u+v} + (-1)^u L_{v-u}\,, \tag{1.2e}$$
$$5F_{u-v}F_{u+v} = L_{2u} - (-1)^{(u-v)}L_{2v}\,. \tag{1.2f}$$

Also we shall make repeated use of the following identities connecting Fibonacci and Lucas numbers:

$$F_{2u} = F_uL_u\,, \tag{1.3a}$$
$$L_{2u} - 2(-1)^u = 5F_u^2\,, \tag{1.3b}$$
$$5F_u^2 - L_u^2 = 4(-1)^{(u+1)}\,, \tag{1.3c}$$
$$L_{2u} + 2(-1)^u = L_u^2\,. \tag{1.3d}$$

Identities (1.2) and (1.3) or their variations can be found in [6, 7, 8].

On notation, G_i, i integers, denotes generalized Fibonacci numbers defined through the second order recurrence relation $G_i = G_{i-1} + G_{i-2}$, where two boundary terms, usually G_0 and G_1, need to be specified. When $G_0 = 0$ and $G_1 = 1$, we have the Fibonacci numbers, denoted F_i, while when $G_0 = 2$ and $G_1 = 1$, we have the Lucas numbers, denoted L_i.

Throughout this paper, the principal value of the arctangent function is assumed.

Interesting results obtained in this paper, for integers k, $j \neq 0$ and p include

$$\sum_{r=1}^{\infty} \tan^{-1}\left\{\frac{F_j^2 L_j L_{4jr}}{F_{4jr}^2 - F_{2j}^2 + F_j^2}\right\} = \tan^{-1}\left(\frac{1}{L_j}\right), \sum_{r=1}^{\infty} \tan^{-1}\left\{\frac{L_j^2 F_j L_{4jr}}{F_{4jr}^2 - F_{2j}^2 + L_j^2}\right\} = \tan^{-1}\left(\frac{1}{F_j}\right),$$

$$\sum_{r=1}^{\infty} \tan^{-1}\left\{\frac{F_{2j}^2 L_{4jr+2j}}{F_{4jr+2j}^2}\right\} = \tan^{-1}\left(\frac{1}{L_{2j}}\right), \sum_{r=1}^{\infty} \tan^{-1}\left\{\frac{F_{2(2j-1)}}{F_{4jr-2r+2j-1}}\right\} = \tan^{-1}\left(\frac{1}{L_{2j-1}}\right),$$

$$\sum_{r=1}^{\infty} \tan^{-1}\left\{\frac{F_{2(2j-1)}}{F_{4jr-2r+1}}\right\} = \tan^{-1}\left(\frac{F_{2j-1}}{F_{2j}}\right), \sum_{r=1}^{\infty} \tan^{-1}\left\{\frac{1}{F_{2r+2k-1}}\right\} = \tan^{-1}\left(\frac{1}{F_{2k}}\right),$$

$$\sum_{r=p}^{\infty} \tan^{-1}\left\{\frac{1}{5}\frac{L_{2r}}{F_{2r}^2}\right\} = \tan^{-1}\left(\frac{1}{L_{2p-1}}\right), \sum_{r=1}^{\infty} \tan^{-1}\left\{\frac{F_{4j}}{F_{4jr-1}}\right\} = \tan^{-1}\left(\frac{L_{2j}}{L_{2j-1}}\right).$$

We also obtained the following special values

$$\sum_{r=1}^{\infty} \tan^{-1}\left\{\frac{L_{4r-2}}{F_{4r-2}^2}\right\} = \frac{\pi}{2}, \quad \sum_{r=1}^{\infty} \tan^{-1}\left\{\frac{L_{4r}}{F_{4r}^2}\right\} = \frac{\pi}{4}, \quad \sum_{r=1}^{\infty} \tan^{-1}\left\{\frac{\sqrt{35}\,L_{4r-2}}{L_{2(4r-2)}}\right\} = \frac{\pi}{2},$$

$$\sum_{r=1}^{\infty} \tan^{-1}\left\{\frac{\sqrt{3}L_{2r}}{L_{4r}}\right\} = \frac{\pi}{3}, \quad \sum_{r=1}^{\infty} \tan^{-1}\left\{\frac{1}{5}\frac{L_{2r}}{F_{2r}^2}\right\} = \frac{\pi}{4}, \quad \sum_{r=1}^{\infty} \tan^{-1}\left\{\frac{\sqrt{5}}{L_{2r}}\right\} = \tan^{-1}\sqrt{5},$$

$$\sum_{r=1}^{\infty} \tan^{-1}\left\{\frac{\sqrt{35}\,L_{4r}}{L_{8r}}\right\} = \sqrt{\frac{7}{5}}, \sum_{r=1}^{\infty} \tan^{-1}\left\{\frac{\sqrt{5}F_{2r-1}}{L_{2r-1}^2}\right\} = \frac{\pi}{2},$$

$$\sum_{r=1}^{\infty} \tan^{-1}\left(\frac{5\sqrt{7}F_{4r-1}}{L_{2(4r-1)}}\right) = \tan^{-1}\sqrt{7}, \quad \sum_{r=1}^{\infty} \tan^{-1}\left\{\frac{L_{4r+2}}{F_{4r+2}^2}\right\} = \tan^{-1}\left(\frac{1}{3}\right).$$

2 Preliminary result

Taking $x = G_{mr+n-m}$ and $y = G_{mr+n}$ in the arctangent addition formula, identity (1.1), gives

$$\tan^{-1}\left\{\frac{\lambda(G_{mr+n} - G_{mr+n-m})}{G_{mr+n}G_{mr+n-m} + \lambda^2}\right\} = \tan^{-1}\left(\frac{\lambda}{G_{mr+n-m}}\right) - \tan^{-1}\left(\frac{\lambda}{G_{mr+n}}\right). \tag{2.1}$$

Summing each side of identity (2.1) from $r = p \in \mathbb{Z}$ to $r = N \in \mathbb{Z}^+$ and noting that the summation of the terms on the right hand side telescopes, we obtain

$$\sum_{r=p}^{N} \tan^{-1}\left\{\frac{\lambda(G_{mr+n} - G_{mr+n-m})}{G_{mr+n}G_{mr+n-m} + \lambda^2}\right\} = \tan^{-1}\left(\frac{\lambda}{G_{mp+n-m}}\right) - \tan^{-1}\left(\frac{\lambda}{G_{mN+n}}\right) . \tag{2.2}$$

Now taking limit as $N \to \infty$, we have

Lemma.

For $\lambda \in \mathbb{R}$, $n, m, p \in \mathbb{Z}$, $m \neq 0$ holds

$$\sum_{r=p}^{\infty} \tan^{-1}\left\{\frac{\lambda(G_{mr+n} - G_{mr+n-m})}{G_{mr+n}G_{mr+n-m} + \lambda^2}\right\} = \tan^{-1}\left(\frac{\lambda}{G_{mp+n-m}}\right) . \tag{2.3}$$

3 Main Results

3.1 $G \equiv F$ in identity (2.3), that is, $G_0 = 0$, $G_1 = 1$

Choosing $m = 4j$ and $n = 2k + 2j$ and using identities (1.2a) and (1.2d) we prove

THEOREM 3.1. *For $\lambda \in \mathbb{R}$, $j, k, p \in \mathbb{Z}$ and $j \neq 0$ holds*

$$\sum_{r=p}^{\infty} \tan^{-1}\left\{\frac{\lambda F_{2j} L_{4jr+2k}}{F_{4jr+2k}^2 - F_{2j}^2 + \lambda^2}\right\} = \tan^{-1}\left(\frac{\lambda}{F_{4jp+2k-2j}}\right) , \tag{3.1}$$

while taking $m = 4j - 2$ and $n = 2k + 2j - 2$ and using identities (1.2a) and (1.2c) we prove

THEOREM 3.2. *For $\lambda \in \mathbb{R}$ and $j, k, p \in \mathbb{Z}$ holds*

$$\sum_{r=p}^{\infty} \tan^{-1}\left\{\frac{\lambda L_{2j-1} F_{4jr-2r+2k-1}}{F_{4jr-2r+2k-1}^2 - F_{2j-1}^2 + \lambda^2}\right\} = \tan^{-1}\left(\frac{\lambda}{F_{4jp-2p+2k-2j}}\right) . \tag{3.2}$$

3.2 $G \equiv L$ in identity (2.3), that is, $G_0 = 2$, $G_1 = 1$

Choosing $m = 4j$ and $n = 2k + 2j - 1$ and using identities (1.2b) and (1.2f) we prove

THEOREM 3.3. *For $\lambda \in \mathbb{R}$, $j, k, p \in \mathbb{Z}$ and $j \neq 0$ holds*

$$\sum_{r=p}^{\infty} \tan^{-1}\left(\frac{5\lambda F_{2j} F_{4jr+2k-1}}{L_{8jr+4k-2} - L_{4j} + \lambda^2}\right) = \tan^{-1}\left(\frac{\lambda}{L_{4jp+2k-2j-1}}\right), \tag{3.3}$$

while taking $m = 4j - 2$ and $n = 2k + 2j - 1$ and using identities (1.2b) and (1.2e) we prove

THEOREM 3.4. *For $\lambda \in \mathbb{R}$ and $j, k, p \in \mathbb{Z}$ holds*

$$\sum_{r=p}^{\infty} \tan^{-1}\left(\frac{\lambda L_{2j-1} L_{4jr-2r+2k}}{L_{8jr-4r+4k} - L_{4j-2} + \lambda^2}\right) = \tan^{-1}\left(\frac{\lambda}{L_{4jp-2p+2k-2j+1}}\right). \tag{3.4}$$

4 Corollaries and special values

Different combinations of the parameters λ, j, k and p in the above theorems yield a variety of interesting particular cases. In this section we will consider some of the possible choices.

4.1 Results from Theorem 3.1

4.1.1 $\lambda = F_j$, $p = 1$ and $k = 0$ in identity (3.1)

The choice $\lambda = F_j$, $p = 1$ and $k = 0$ in identity (3.1) gives

$$\sum_{r=1}^{\infty} \tan^{-1}\left\{\frac{F_j^2 L_j L_{4jr}}{F_{4jr}^2 - F_{2j}^2 + F_j^2}\right\} = \tan^{-1}\left(\frac{1}{L_j}\right). \tag{4.1}$$

Thus, at $j = 1$, we obtain the special value

$$\boxed{\sum_{r=1}^{\infty} \tan^{-1}\left\{\frac{L_{4r}}{F_{4r}^2}\right\} = \frac{\pi}{4}}. \tag{4.2}$$

4.1.2 $\lambda = L_j$, $p = 1$ and $k = 0$ in identity (3.1)

The above choice gives

$$\sum_{r=1}^{\infty} \tan^{-1}\left\{\frac{L_j^2 F_j L_{4jr}}{F_{4jr}^2 - F_{2j}^2 + L_j^2}\right\} = \tan^{-1}\left(\frac{1}{F_j}\right). \tag{4.3}$$

At $j = 1$, identity(4.2) is reproduced, while at $j = 2$ we have the special value

$$\boxed{\sum_{r=1}^{\infty} \tan^{-1}\left\{\frac{9L_{8r}}{F_{8r}^2}\right\} = \frac{\pi}{4}}. \tag{4.4}$$

Note that identities (4.2) and (4.4) are special cases of identity (4.8) below, at $j = 1$ and $j = 2$, respectively.

4.1.3 $\lambda = F_{2j}$, $k = j$ and $p = 0$ in identity (3.1)

This choice gives

$$\sum_{r=1}^{\infty} \tan^{-1}\left\{\frac{F_{2j}^2 L_{4jr-2j}}{F_{4jr-2j}^2}\right\} = \frac{\pi}{2}, \tag{4.5}$$

which, at $j = 1$, gives the special value

$$\boxed{\sum_{r=1}^{\infty} \tan^{-1}\left\{\frac{L_{4r-2}}{F_{4r-2}^2}\right\} = \frac{\pi}{2}}. \tag{4.6}$$

4.1.4 $\lambda = F_{2j}$ and $p = 1$ in identity (3.1)

This choice gives

$$\sum_{r=1}^{\infty} \tan^{-1}\left\{\frac{F_{2j}^2 L_{4jr+2k}}{F_{4jr+2k}^2}\right\} = \tan^{-1}\left(\frac{F_{2j}}{F_{2j+2k}}\right). \tag{4.7}$$

At $k = 0$ in identity (4.7) we have

$$\sum_{r=1}^{\infty} \tan^{-1}\left\{\frac{F_{2j}^2 L_{4jr}}{F_{4jr}^2}\right\} = \frac{\pi}{4}. \tag{4.8}$$

Note that identities (4.2) and (4.4) are special cases of identity (4.8) at $j = 1$ and $j = 2$, respectively.

At $k = j \neq 0$ in identity (4.7) we have

$$\sum_{r=1}^{\infty} \tan^{-1} \left\{ \frac{F_{2j}^2 L_{4jr+2j}}{F_{4jr+2j}^2} \right\} = \tan^{-1} \left(\frac{1}{L_{2j}} \right) , \tag{4.9}$$

yielding at $j = 1$, the special value

$$\boxed{\sum_{r=1}^{\infty} \tan^{-1} \left\{ \frac{L_{4r+2}}{F_{4r+2}^2} \right\} = \tan^{-1} \left(\frac{1}{3} \right)} . \tag{4.10}$$

Finally, taking limit of identity (4.7) as $j \to \infty$, we obtain

$$\lim_{j \to \infty} \sum_{r=1}^{\infty} \tan^{-1} \left\{ \frac{F_{2j}^2 L_{4jr+2k}}{F_{4jr+2k}^2} \right\} = \tan^{-1} \left(\frac{1}{\phi^{2k}} \right) . \tag{4.11}$$

4.1.5 $5\lambda^2 = L_{4j}$, $p = 0$ and $k = j$ in identity (3.1)

Another interesting particular case of identity (3.1) is obtained by setting $5\lambda^2 = L_{4j}$, $p = 0$ and $k = j$ to obtain

$$\sum_{r=1}^{\infty} \tan^{-1} \left\{ \frac{F_{2j} \sqrt{5L_{4j}} L_{4jr-2j}}{L_{2(4jr-2j)}} \right\} = \frac{\pi}{2} , \tag{4.12}$$

which at $j = 1$ gives the special value

$$\boxed{\sum_{r=1}^{\infty} \tan^{-1} \left\{ \frac{\sqrt{35}\, L_{4r-2}}{L_{2(4r-2)}} \right\} = \frac{\pi}{2}} . \tag{4.13}$$

4.1.6 $5\lambda^2 = L_{4j}$, $p = 0$ and $k = 2j$ in identity (3.1)

In this case Theorem 3.1 reduces to

$$\sum_{r=1}^{\infty} \tan^{-1} \left\{ \frac{F_{2j} \sqrt{5L_{4j}} L_{4jr}}{L_{8jr}} \right\} = \tan^{-1} \left(\frac{1}{\sqrt{5}} \frac{\sqrt{L_{4j}}}{F_{2j}} \right) . \tag{4.14}$$

At $j=1$, we have the special value

$$\boxed{\sum_{r=1}^{\infty}\tan^{-1}\left\{\frac{\sqrt{35}\,L_{4r}}{L_{8r}}\right\}=\sqrt{\frac{7}{5}}}\,. \tag{4.15}$$

4.1.7 $\lambda=L_{2j}/\sqrt{5}$ and $k=j$ in identity (3.1)

Setting $\lambda=L_{2j}/\sqrt{5}$ and $k=j$ in identity (3.1) we have

$$\sum_{r=p}^{\infty}\tan^{-1}\left\{\frac{\sqrt{5}F_{4j}}{L_{4jr+2j}}\right\}=\tan^{-1}\left(\frac{L_{2j}}{F_{4jp}\sqrt{5}}\right)\,, \tag{4.16}$$

which at $p=1$ gives

$$\sum_{r=1}^{\infty}\tan^{-1}\left\{\frac{\sqrt{5}F_{4j}}{L_{4jr+2j}}\right\}=\tan^{-1}\left(\frac{1}{F_{2j}\sqrt{5}}\right) \tag{4.17}$$

and at $p=0$ yields

$$\sum_{r=1}^{\infty}\tan^{-1}\left\{\frac{\sqrt{5}F_{4j}}{L_{4jr-2j}}\right\}=\frac{\pi}{2}\,. \tag{4.18}$$

4.1.8 $\lambda=L_{2j}/\sqrt{5}$, $p=0$ and $k=2j\neq 0$ in identity (3.1)

The above choice yields

$$\sum_{r=1}^{\infty}\tan^{-1}\left\{\frac{\sqrt{5}F_{4j}}{L_{4jr}}\right\}=\tan^{-1}\left(\frac{L_{2j}}{F_{2j}\sqrt{5}}\right)\,. \tag{4.19}$$

4.2 Results from Theorem 3.2

4.2.1 $\lambda=F_{2j-1}$ and $p=1$ in identity (3.2)

The above choice gives

$$\sum_{r=1}^{\infty}\tan^{-1}\left\{\frac{F_{2(2j-1)}}{F_{4jr-2r+2k-1}}\right\}=\tan^{-1}\left(\frac{F_{2j-1}}{F_{2j+2k-2}}\right)\,. \tag{4.20}$$

At $k=j$ in identity (4.20) we have the interesting formula

$$\boxed{\sum_{r=1}^{\infty} \tan^{-1}\left\{\frac{F_{2(2j-1)}}{F_{4jr-2r+2j-1}}\right\} = \tan^{-1}\left(\frac{1}{L_{2j-1}}\right)}. \tag{4.21}$$

Note that identity (4.21), at $j = 1$, includes Lehmer's result (cited in [3, 5]) as a particular case.

Setting $j = 1$ in identity (4.20) we obtain

$$\boxed{\sum_{r=1}^{\infty} \tan^{-1}\left\{\frac{1}{F_{2r+2k-1}}\right\} = \tan^{-1}\left(\frac{1}{F_{2k}}\right)}. \tag{4.22}$$

Note again that identity (4.22) subsumes Lehmer's formula and the result of Melham ($p = 1$ in identity(3.5) of [5]), at $k = 1$ and at $k = 0$ respectively.

Finally, taking limit $j \to \infty$ in identity (4.20), we obtain

$$\lim_{j\to\infty} \sum_{r=1}^{\infty} \tan^{-1}\left\{\frac{F_{2(2j-1)}}{F_{4jr-2r+2k-1}}\right\} = \tan^{-1}\left(\frac{1}{\phi^{2k-1}}\right) . \tag{4.23}$$

4.2.2 $\lambda = L_{2j-1}/\sqrt{5}$ and $k = j$ in identity (3.2)

The above choice gives

$$\sum_{r=p}^{\infty} \tan^{-1}\left\{\frac{\sqrt{5}L_{2j-1}^2 F_{4jr-2r+2j-1}}{L_{4jr-2r+2j-1}^2}\right\} = \tan^{-1}\left(\frac{1}{\sqrt{5}}\frac{L_{2j-1}}{F_{4jp-2p}}\right) . \tag{4.24}$$

Setting $p = 1$ in identity (4.24), we find

$$\sum_{r=1}^{\infty} \tan^{-1}\left\{\frac{\sqrt{5}L_{2j-1}^2 F_{4jr-2r+2j-1}}{L_{4jr-2r+2j-1}^2}\right\} = \tan^{-1}\left(\frac{1}{\sqrt{5}}\frac{1}{F_{2j-1}}\right) , \tag{4.25}$$

while choosing $j = 1$ leads to

$$\sum_{r=p}^{\infty} \tan^{-1}\left\{\frac{\sqrt{5}F_{2r+1}}{L_{2r+1}^2}\right\} = \tan^{-1}\left(\frac{1}{\sqrt{5}}\frac{1}{F_{2p}}\right) , \tag{4.26}$$

which at $p = 0$ gives the special value

$$\boxed{\sum_{r=1}^{\infty}\tan^{-1}\left\{\frac{\sqrt{5}F_{2r-1}}{L_{2r-1}^2}\right\}=\frac{\pi}{2}}. \tag{4.27}$$

4.2.3 $5\lambda^2 = L_{4j-2}$ and $k = j$ in identity (3.2)

The above substitutions give

$$\sum_{r=p}^{\infty}\tan^{-1}\left\{\frac{\sqrt{5L_{4j-2}}L_{2j-1}F_{4jr-2r+2j-1}}{L_{2(4jr-2r+2j-1)}}\right\}=\tan^{-1}\left(\frac{\sqrt{5L_{4j-2}}}{5F_{4jp-2p}}\right). \tag{4.28}$$

At $p=0$ in identity (4.28) we have, for positive integers j,

$$\sum_{r=1}^{\infty}\tan^{-1}\left\{\frac{\sqrt{5L_{4j-2}}L_{2j-1}F_{4jr-2r-2j+1}}{L_{2(4jr-2r-2j+1)}}\right\}=\frac{\pi}{2}, \tag{4.29}$$

giving, at $j=1$, the special value

$$\boxed{\sum_{r=1}^{\infty}\tan^{-1}\left\{\frac{\sqrt{15}F_{2r-1}}{L_{2(2r-1)}}\right\}=\frac{\pi}{2}}. \tag{4.30}$$

At $p=2$ in identity (4.28) we have, for positive integers j,

$$\sum_{r=1}^{\infty}\tan^{-1}\left\{\frac{\sqrt{5L_{4j-2}}L_{2j-1}F_{4jr-2r+6j-3}}{L_{2(4jr-2r+6j-3)}}\right\}=\tan^{-1}\left(\frac{1}{\sqrt{5F_{4j-2}F_{8j-4}}}\right), \tag{4.31}$$

which gives, at $j=1$, the special value

$$\boxed{\sum_{r=1}^{\infty}\tan^{-1}\left\{\frac{\sqrt{15}F_{2r+3}}{L_{2(2r+3)}}\right\}=\tan^{-1}\left(\frac{1}{\sqrt{15}}\right)}. \tag{4.32}$$

4.3 Results from Theorem 3.3

4.3.1 $\lambda = \sqrt{L_{4j}}$, $k = 0$ and $p = 1$ in identity (3.3)

The above choice gives

$$\sum_{r=1}^{\infty} \tan^{-1}\left(\frac{5\sqrt{L_{4j}}F_{2j}F_{4jr-1}}{L_{8jr-2}}\right) = \tan^{-1}\left(\frac{\sqrt{L_{4j}}}{L_{2j-1}}\right), \tag{4.33}$$

which, at $j = 1$, gives

$$\boxed{\sum_{r=1}^{\infty} \tan^{-1}\left(\frac{5\sqrt{7}F_{4r-1}}{L_{2(4r-1)}}\right) = \tan^{-1}\sqrt{7}}. \tag{4.34}$$

4.3.2 $\lambda = L_{2j}$ and $p = 1$ in identity (3.3)

Setting $\lambda = L_{2j}$ and $p = 1$ in identity (3.3) gives

$$\sum_{r=1}^{\infty} \tan^{-1}\left\{\frac{F_{4j}}{F_{4jr+2k-1}}\right\} = \tan^{-1}\left(\frac{L_{2j}}{L_{2j+2k-1}}\right). \tag{4.35}$$

Taking limit as $j \to \infty$ in identity (4.35) gives

$$\lim_{j\to\infty} \sum_{r=1}^{\infty} \tan^{-1}\left\{\frac{F_{4j}}{F_{4jr+2k-1}}\right\} = \tan^{-1}\left(\frac{1}{\phi^{2k-1}}\right). \tag{4.36}$$

4.3.3 $\lambda = \sqrt{5}F_{2j}$, $p = 1$ and $k = 0$ in identity (3.3)

Setting $\lambda = \sqrt{5}F_{2j}$, $p = 1$ and $k = 0$ in identity (3.3) we obtain

$$\sum_{r=1}^{\infty} \tan^{-1}\left(\frac{5\sqrt{5}F_{2j}^2F_{4jr-1}}{L_{4jr-1}^2}\right) = \tan^{-1}\left(\frac{\sqrt{5}F_{2j}}{L_{2j-1}}\right), \tag{4.37}$$

which gives the special value

$$\sum_{r=1}^{\infty} \tan^{-1}\left(\frac{5\sqrt{5}F_{4r-1}}{L_{4r-1}^2}\right) = \tan^{-1}\sqrt{5}, \tag{4.38}$$

at $j = 1$.

4.4 Results from Theorem 3.4

4.4.1 $\lambda = \sqrt{L_{4j-2}}$ and $j = 0 = k$ in identity (3.4)

With the above choice we obtain

$$\sum_{r=p}^{\infty} \tan^{-1}\left\{\frac{\sqrt{3}L_{2r}}{L_{4r}}\right\} = \tan^{-1}\left(\frac{\sqrt{3}}{L_{2p-1}}\right), \tag{4.39}$$

which gives rise, at $p = 1$, to the special value

$$\boxed{\sum_{r=1}^{\infty} \tan^{-1}\left\{\frac{\sqrt{3}L_{2r}}{L_{4r}}\right\} = \frac{\pi}{3}}. \tag{4.40}$$

4.4.2 $\lambda = L_{2j-1}$ and $p = 1$ in identity (3.4)

With the above choice we have

$$\sum_{r=1}^{\infty} \tan^{-1}\left\{\frac{L_{2j-1}^2}{5}\frac{L_{4jr-2r+2k}}{F_{4jr-2r+2k}^2}\right\} = \tan^{-1}\left(\frac{L_{2j-1}}{L_{2j+2k-1}}\right). \tag{4.41}$$

$k = 0$ in identity (4.41) gives

$$\sum_{r=1}^{\infty} \tan^{-1}\left\{\frac{L_{2j-1}^2}{5}\frac{L_{4jr-2r}}{F_{4jr-2r}^2}\right\} = \frac{\pi}{4}, \tag{4.42}$$

which at $j = 1$ gives the special value

$$\boxed{\sum_{r=1}^{\infty} \tan^{-1}\left\{\frac{1}{5}\frac{L_{2r}}{F_{2r}^2}\right\} = \frac{\pi}{4}}. \tag{4.43}$$

$j = 1$ in identity (4.41) leads to

$$\sum_{r=1}^{\infty} \tan^{-1}\left\{\frac{1}{5}\frac{L_{2r+2k}}{F_{2r+2k}^2}\right\} = \tan^{-1}\left(\frac{1}{L_{2k+1}}\right), \tag{4.44}$$

which gives the special value

$$\boxed{\sum_{r=1}^{\infty} \tan^{-1}\left\{\frac{1}{5}\frac{L_{2r+2}}{F_{2r+2}^2}\right\} = \tan^{-1}\left(\frac{1}{4}\right)}, \tag{4.45}$$

at $k = 1$.

Taking limit $j \to \infty$ in identity (4.41), we obtain

$$\lim_{j\to\infty} \sum_{r=1}^{\infty} \tan^{-1} \left\{ \frac{L_{2j-1}^2}{5} \frac{L_{4jr-2r+2k}}{F_{4jr-2r+2k}^2} \right\} = \tan^{-1} \left(\frac{1}{\phi^{2k}} \right) . \tag{4.46}$$

4.4.3 $\lambda = L_{2j-1}$ and $j = 0 = k$ in identity (3.4)

This choice gives

$$\sum_{r=p}^{\infty} \tan^{-1} \left\{ \frac{1}{5} \frac{L_{2r}}{F_{2r}^2} \right\} = \tan^{-1} \left(\frac{1}{L_{2p-1}} \right) , \tag{4.47}$$

Note that identities (4.43) and (4.45) are special cases of (4.47) at $p = 1$ and at $p = 2$.

4.4.4 $\lambda = \sqrt{5} F_{2j-1}$ and $j = 0 = k$ in identity (3.4)

The above choice gives

$$\sum_{r=p}^{\infty} \tan^{-1} \left\{ \frac{\sqrt{5}}{L_{2r}} \right\} = \tan^{-1} \left(\frac{\sqrt{5}}{L_{2p-1}} \right) , \tag{4.48}$$

which at $p = 1$ gives the special value

$$\boxed{\sum_{r=1}^{\infty} \tan^{-1} \left\{ \frac{\sqrt{5}}{L_{2r}} \right\} = \tan^{-1} \sqrt{5}} . \tag{4.49}$$

5 Conclusion

Using a fairly straightforward technique, we have derived numerous infinite arctangent summation formulas involving Fibonacci and Lucas numbers. While most of the results obtained are new, a couple of 'celebrated' results appear as particular cases of more general formulas derived in this paper.

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