Some results on pi formulas

by

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Abstract

In this note we show some formulas for the number pi

Resumen

Se muestran algunas fórmulas para la constante pi:

$$\pi = 2\sqrt{2} + 2\int_{1}^{(1+\sqrt{2})/2} \frac{\sqrt{1+\sqrt{1+4x-4x^2}} - \sqrt{1-\sqrt{1+4x-4x^2}}}{\sqrt{x}}\,dx$$

$$\pi = 3.1415926535\ldots$$

Introducción

Este documento está dividido en ocho partes, cada parte se ha desarrollado de manera independiente y tiene numeración propia.

Part 1: pi formulas

Notación: $i = \sqrt{-1}$, $Re(x+iy) = x, Im(x+iy) = y$, $\mathbb{N} = \{1,2,3,\ldots\}$

Fórmulas

(1) $$\pi = 6\sum_{n=1}^{\infty}\frac{(-1)^n}{n}Im\left(\left(\frac{2}{-1+\sqrt{4\sqrt{3}-3}+2i}\right)^n\right)$$

(2) $$\pi = 8\sum_{n=1}^{\infty}\frac{(-1)^n}{n}Im\left(\left(\frac{2}{-1+\sqrt{4\sqrt{2}+1}+2i}\right)^n\right)$$

(3) $$\pi = 8\sum_{n=1}^{\infty}\frac{(-1)^n}{n}Im\left(\left(\frac{2}{-1+\sqrt{8\sqrt{2}-7}+4i}\right)^n\right)$$

(4) $$\pi = 12\sum_{n=1}^{\infty}\frac{(-1)^n}{n}Im\left(\left(\frac{2}{-1+\sqrt{4\sqrt{3}+5}+2i}\right)^n\right)$$

(5) $$\pi = 12\sum_{n=1}^{\infty}\frac{(-1)^n}{n}Im\left(\left(\frac{2}{-1+\sqrt{8\sqrt{3}+1}+4i}\right)^n\right)$$

(6) $$\pi = 12\sum_{n=1}^{\infty}\frac{(-1)^n}{n}Im\left(\left(\frac{2}{-1+\sqrt{12\sqrt{3}-11}+6i}\right)^n\right)$$

(7) $$\pi = 16\sum_{n=1}^{\infty}\frac{(-1)^n}{n}Im\left(\left(\frac{2}{-1+\sqrt{4\sqrt{4+2\sqrt{2}}+4\sqrt{2}+1}+2i}\right)^n\right)$$

(8) $$\pi = 16\sum_{n=1}^{\infty}\frac{(-1)^n}{n}Im\left(\left(\frac{2}{-1+\sqrt{8\sqrt{4+2\sqrt{2}}+8\sqrt{2}-7}+4i}\right)^n\right)$$

(9) $$\pi = 16\sum_{n=1}^{\infty}\frac{(-1)^n}{n}Im\left(\left(\frac{2}{-1+\sqrt{12\sqrt{4+2\sqrt{2}}+12\sqrt{2}-23}+6i}\right)^n\right)$$

(10) $$\pi = 16\sum_{n=1}^{\infty}\frac{(-1)^n}{n}Im\left(\left(\frac{2}{-1+\sqrt{16\sqrt{4+2\sqrt{2}}+16\sqrt{2}-47+8i}}\right)^n\right)$$

(11) $$\pi = 16\sum_{n=1}^{\infty}\frac{(-1)^n}{n}Im\left(\left(\frac{2}{-1+\sqrt{20\sqrt{4+2\sqrt{2}}+20\sqrt{2}-79+10i}}\right)^n\right)$$

(12) $$\pi = -8\sum_{n=0}^{\infty}\frac{1}{2n+1}Im\left(\left(\frac{1}{\sqrt{1+2x-x^2}+x\,i}\right)^{2n+1}\right), 0 < x < 1+\sqrt{2}$$

(13) $$\pi = -8\sum_{n=0}^{\infty}\frac{1}{2n+1}Im\left(\left(\frac{1}{x+i\left(1+\sqrt{2-x^2}\right)}\right)^{2n+1}\right), 0 < x < \sqrt{2}$$

(14) $$\pi = -8\sum_{n=0}^{\infty}\frac{1}{2n+1}Im\left(\left(\frac{1}{x+i\left(1-\sqrt{2-x^2}\right)}\right)^{2n+1}\right), 1 < x < \sqrt{2}$$

(15) $$\pi = -12\sum_{n=0}^{\infty}\frac{1}{2n+1}Im\left(\left(\frac{1}{\sqrt{1+2x\sqrt{3}-x^2}+i\,x}\right)^{2n+1}\right), 0 < x < 2+\sqrt{3}$$

(16) $$\pi = -12\sum_{n=0}^{\infty}\frac{1}{2n+1}Im\left(\left(\frac{1}{x+i\left(\sqrt{3}+\sqrt{4-x^2}\right)}\right)^{2n+1}\right), 0 < x < 2$$

(17) $$\pi = -12\sum_{n=0}^{\infty}\frac{1}{2n+1}Im\left(\left(\frac{1}{x+i\left(\sqrt{3}-\sqrt{4-x^2}\right)}\right)^{2n+1}\right), 1 < x < 2$$

(18) $$\pi = 4\sum_{n=1}^{\infty}\frac{1}{n}Im\left(\left(1-\frac{1}{x(1+i)}\right)^n\right), x > \frac{1}{2\sqrt{2}}$$

(19) $$\pi = 6\sum_{n=1}^{\infty}\frac{1}{n}Im\left(\left(1-\frac{1}{x\left(\sqrt{3}+i\right)}\right)^n\right), x > \frac{1}{4}$$

(20) $$\pi = 3\sum_{n=1}^{\infty}\frac{1}{n}Im\left(\left(1-\frac{1}{x\left(1+i\sqrt{3}\right)}\right)^n\right), x > \frac{1}{4}$$

$$(21)\qquad \pi = -4k\sum_{n=1}^{\infty}\frac{1}{n}Im\left(\left(1 - x\,2^{1/2k}\cos\left(\frac{\pi}{4k}\right) - i\,x\,2^{1/2k}\sin\left(\frac{\pi}{4k}\right)\right)^n\right)$$

$$k\in\mathbb{N}, 0 < x < 2^{1-\frac{1}{2k}}\cos\left(\frac{\pi}{4k}\right)$$

$$(22)\qquad \pi = -8\sum_{n=1}^{\infty}\frac{1}{n}Im\left(\left(1 - \frac{x}{2}\sqrt[4]{2}\sqrt{2+\sqrt{2}} - i\,\frac{x}{2}\sqrt[4]{2}\sqrt{2-\sqrt{2}}\right)^n\right)$$

$$0 < x < \frac{\sqrt{2+\sqrt{2}}}{\sqrt[4]{2}}$$

$$(23)\qquad \pi = 4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}Im\left(\left(\frac{1+2i}{2-i}\right)^n\right)$$

$$(24)\qquad \pi = 4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}Im\left(\left(\frac{-1+2i}{3-i}\right)^n\right)$$

$$(25)\qquad \pi = 4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}Im\left(\left(\frac{-1+4i}{5-3i}\right)^n\right)$$

$$(26)\qquad \pi = \frac{3\sqrt{3}}{2}\sum_{n=0}^{\infty}\sum_{\substack{k+p+q=n\\ k,p,q\in\mathbb{N}\cup\{0\}}}\frac{n!\,(-1)^p}{k!\,p!\,q!\,(k+2p+1)}\left(\frac{1}{1+a}\right)^{k+p+1}\left(\frac{a}{1+a}\right)^q\ , a > -\frac{5}{8}$$

La formula (26) se puede escribir como:

$$\pi = \frac{3\sqrt{3}}{2}\sum_{n=0}^{\infty}n!\sum_{k=0}^{n}\sum_{p=0}^{n-k}f(k,p,n-k-p,a)$$

donde

$$f(k,p,q,a) = \frac{(-1)^p}{k!\,p!\,q!\,(k+2p+1)}\left(\frac{1}{1+a}\right)^{k+p+1}\left(\frac{a}{1+a}\right)^q$$

$$(27)\qquad \pi = 3\sqrt{3}\sum_{n=0}^{\infty}\sum_{\substack{k+p+q=n\\ k,p,q\in\mathbb{N}\cup\{0\}}}\frac{n!\,(-1)^{n+q}}{k!\,p!\,q!\,(k+2p+1)}\left(\frac{1}{1+a}\right)^{k+p+1}\left(\frac{a}{1+a}\right)^q\ , a > \frac{1}{2}$$

La fórmula (27) se puede escribir como:

$$\pi = 3\sqrt{3}\sum_{n=0}^{\infty}(-1)^n\, n!\sum_{k=0}^{n}\sum_{p=0}^{n-k} f(k,p,n-k-p,a)$$

donde

$$f(k,p,q,a) = \frac{(-1)^q}{k!\,p!\,q!\,(k+2p+1)}\left(\frac{1}{1+a}\right)^{k+p+1}\left(\frac{a}{1+a}\right)^q$$

(28) $\pi = \frac{3\sqrt{3}}{4}\int_0^1 \left(\sqrt[3]{1+\sqrt{1-x}} - \sqrt[3]{1-\sqrt{1-x}}\right) x^{-2/3}\, dx$

(29) $\pi = \sqrt{2}\int_0^1 \left(\sqrt{1+\sqrt{1-x^2}} - \sqrt{1-\sqrt{1-x^2}}\right) x^{-1/2}\, dx$

(30) $\pi = 2\sqrt{3}\left(\sum_{n=0}^{\infty}\binom{2n}{n}\frac{(-1)^n 12^{-n}}{2n+1} - \sum_{n=0}^{\infty}\frac{(-1)^n 3^{-n-1}}{2n+3}\sum_{k=0}^{n}\binom{2k}{k}\frac{2^{-2k-1}}{k+1}\right)$

(31) $\pi = 2\sqrt{3}\sum_{n=0}^{\infty}\sum_{\substack{k+p+q=n \\ k,p,q\in\mathbb{N}\cup\{0\}}}\frac{(-1)^{n+q} n!}{k!\,p!\,q!}\left(\frac{1}{1+a}\right)^{k+p+1}\left(\frac{a}{1+a}\right)^q f(k,p)\ ,a > \frac{1}{2}$

donde

$$f(k,p) = \frac{1}{2k+4p+1} + \frac{1}{2k+4p+3}$$

La fórmula (31) se puede escribir como:

$$\pi = 2\sqrt{3}\sum_{n=0}^{\infty}(-1)^n n!\sum_{k=0}^{n}\sum_{p=0}^{n-k} g(k,p,n-k-p,a)$$

donde

$$g(k,p,q,a) = \frac{(-1)^q}{k!\,p!\,q!}\left(\frac{1}{1+a}\right)^{k+p+1}\left(\frac{a}{1+a}\right)^q f(k,p)$$

(32) $$\pi = 2\sum_{n=0}^{\infty} \sum_{\substack{k+p+q+r=n \\ k,p,q,r \in \mathbb{N}\cup\{0\}}} \frac{(-1)^{n+r} n!}{k!\,p!\,q!\,r!} \left(\frac{1}{1+a}\right)^{k+p+q+1} \left(\frac{a}{1+a}\right)^{r} f(k,p,q) \ , a > 1$$

donde

$$f(k,p,q) = \frac{3}{k+2p+3q+1} - \frac{1}{k+2p+3q+3}$$

(33) $$\pi = 2\sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{p=0}^{k} \sum_{q=0}^{p} \frac{(-1)^{k} n!}{(n-k)!\,(k-p)!\,(p-q)!\,q!} \left(\frac{a}{1+a}\right)^{n-k} \left(\frac{1}{1+a}\right)^{k+1} f(k,p,q)$$

donde

$$f(k,p,q) = \frac{3}{k+p+q+1} - \frac{1}{k+p+q+3} \ , a > 1$$

(34) $$\pi = 4\sum_{k=0}^{n} \tan^{-1}\left(\frac{Im(z_k)}{Re(z_k)}\right) + 4\tan^{-1}\left(\frac{Im((1+i)^{n+2})}{2^{n+2} - Re((1+i)^{n+2})}\right) \ , n \in \mathbb{N}$$

donde

$$z_{n+1} = \frac{3+i}{2} - \frac{1+i}{2z_n} \ , z_0 = \frac{3+i}{2} \ , n \in \mathbb{N}\cup\{0\}$$

Algunos ejemplos de (34):

$$\pi = 4\tan^{-1}\left(\frac{1}{2}\right) + 4\tan^{-1}\left(\frac{1}{3}\right)$$

$$\pi = 4\tan^{-1}\left(\frac{1}{3}\right) + 4\tan^{-1}\left(\frac{3}{11}\right) + 4\tan^{-1}\left(\frac{1}{5}\right)$$

$$\pi = 4\tan^{-1}\left(\frac{1}{3}\right) + 4\tan^{-1}\left(\frac{3}{11}\right) + 4\tan^{-1}\left(\frac{1}{5}\right) + 4\tan^{-1}\left(\frac{1}{9}\right) + 4\tan^{-1}\left(\frac{1}{73}\right) - 4\tan^{-1}\left(\frac{1}{8}\right)$$

(35) $$\pi = 4\tan^{-1}\left(\frac{Im(z_n)}{Re(z_n)}\right) - 4\tan^{-1}\left(\frac{Im(1+z_n-z_{n+1})}{Re(1+z_n-z_{n+1})}\right) \ , n \in \mathbb{N}$$

donde

$$z_{n+2} = \left(\frac{3+i}{2}\right) z_{n+1} - \left(\frac{1+i}{2}\right) z_n \ , z_0 = 1, z_1 = \frac{3+i}{2}, n \in \mathbb{N}\cup\{0\}$$

Algunos ejemplos de (35):

$$\pi = 4\tan^{-1}\left(\frac{2}{3}\right) + 4\tan^{-1}\left(\frac{1}{5}\right)$$

$$\pi = 4\tan^{-1}\left(\frac{5}{4}\right) - 4\tan^{-1}\left(\frac{1}{9}\right)$$

$$\pi = 4\tan^{-1}\left(\frac{9}{7}\right) - 4\tan^{-1}\left(\frac{1}{8}\right)$$

$$\pi = 4\tan^{-1}\left(\frac{8}{7}\right) - 4\tan^{-1}\left(\frac{1}{15}\right)$$

(36) $$\pi = 4\sum_{n=0}^{\infty}\tan^{-1}\left(\frac{Im(z_n)}{Re(z_n)}\right)$$

$$z_{n+1} = 1 + \left(\frac{1+i}{2}\right)\left(1 - \frac{1}{z_n}\right), z_0 = \frac{3+i}{2}, n \in \mathbb{N} \cup \{0\}$$

(37) $$\pi = 6\sum_{n=0}^{\infty}\tan^{-1}\left(\frac{Im(z_n)}{Re(z_n)}\right)$$

$$z_{n+1} = 1 + \left(\frac{1+i\sqrt{3}}{4}\right)\left(1 - \frac{1}{z_n}\right), z_0 = \frac{5+i\sqrt{3}}{4}, n \in \mathbb{N} \cup \{0\}$$

(38) $$\pi = 3\sum_{n=0}^{\infty}\tan^{-1}\left(\frac{Im(z_n)}{Re(z_n)}\right)$$

$$z_{n+1} = 1 + \left(\frac{3+i\sqrt{3}}{4}\right)\left(1 - \frac{1}{z_n}\right), z_0 = \frac{7+i\sqrt{3}}{4}, n \in \mathbb{N} \cup \{0\}$$

(39) $$\frac{\pi}{8} + \frac{1}{4}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{k}\sum_{r=0}^{n-k}\binom{k}{m}\binom{n-k}{r}\left(\frac{a}{1+a}\right)^{k-m}\left(\frac{b}{1+b}\right)^{n-k-r}\left(\frac{1}{1+a}\right)^{m+1}\left(\frac{1}{1+b}\right)^{r+1} f(m,r)$$

$$f(m,r) = \frac{(-1)^{m+r}}{2m+2r+1}, a > 0, b > 0$$

(40) $$\pi = 2\sum_{n=0}^{\infty}\left(\frac{3}{4n+1} - \frac{3}{4n+2} - \frac{1}{4n+3} + \frac{1}{4n+4}\right)$$

(41) $$\pi = 8\int_0^1 \frac{(1+x^2)\sqrt{2} - (1-x^2)}{1+6x^2+x^4}\,dx$$

(42) $$\pi = 8\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}\left(\frac{a}{1+a}\right)^{n-k}\left(\frac{1}{1+a}\right)^{k+1}(-1)^k 6^{k-m} f(k,m)$$

$$f(k,m) = \frac{\sqrt{2}-1}{2k+2m+1} + \frac{\sqrt{2}+1}{2k+2m+3}\ , a > 3$$

(43) $$\pi = 24\tan^{-1}\left(\frac{1}{10}\right) + 8\tan^{-1}\left(\frac{1030301\sqrt{2} - 1431559}{271439}\right)$$

(44) $$\pi = 32\tan^{-1}\left(\frac{1}{10}\right) - 8\tan^{-1}\left(\frac{104060401\sqrt{2} - 147153121}{1758719}\right)$$

(45) $$\pi = 30\tan^{-1}\left(\frac{1}{10}\right) + 6\tan^{-1}\left(\frac{90050\sqrt{3} - 147003}{49001\sqrt{3} + 270150}\right)$$

(46) $$\pi = 36\tan^{-1}\left(\frac{1}{10}\right) - 6\tan^{-1}\left(\frac{3\left(580060 - 283833\sqrt{3}\right)}{2554497 + 580060\sqrt{3}}\right)$$

(47) $$\pi = 24\tan^{-1}\left(\frac{1}{10}\right) + 12\tan^{-1}\left(\frac{18802 - 10201\sqrt{3}}{18121}\right)$$

(48) $$\pi = 36\tan^{-1}\left(\frac{1}{10}\right) - 12\tan^{-1}\left(\frac{1030301\sqrt{3} - 1702998}{2190421}\right)$$

(49) $$\pi = 28\tan^{-1}\left(\frac{1}{10}\right) + 4\tan^{-1}\left(\frac{11\cdot 139\cdot 839}{3^2\cdot 29\cdot 55889}\right)$$

(50) $$\pi = 32\tan^{-1}\left(\frac{1}{10}\right) - 4\tan^{-1}\left(\frac{1758719}{147153121}\right)$$

(51) $$\pi = 8\tan^{-1}\left(\frac{1}{3}\right) + 8\tan^{-1}\left(5\sqrt{2} - 7\right)$$

(52) $\pi = 8\tan^{-1}\left(\frac{1}{2}\right) - 8\tan^{-1}\left(5\sqrt{2} - 7\right)$

(53) $\pi = 8\sum_{n=0}^{\infty}\frac{(-1)^n c_n}{(2n+1)3^{2n+1}}$

$$c_{n+1} = \left(891 - 630\sqrt{2}\right)c_n + 630\sqrt{2} - 890\,,\, c_0 = 15\sqrt{2} - 20, n \in \mathbb{N} \cup \{0\}$$

(54) $\pi = 8\sum_{n=0}^{\infty}\frac{(-1)^n c_n}{(2n+1)2^{2n+1}}$

$$c_{n+1} = \left(396 - 280\sqrt{2}\right)c_n + 280\sqrt{2} - 395\,,\, c_0 = 15 - 10\sqrt{2}\,, n \in \mathbb{N} \cup \{0\}$$

(55) $\pi = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{2}{n}\sqrt{2n(2k-1) - 2k(k-1) - 2\sqrt{k(k-1)(2n-k)(2n-k+1)}}$

(56) $\pi = 8\sum_{n=1}^{\infty} n\binom{2n}{n}(-1)^{n-1}4^{-n}\, I_n$

donde

$$I_n = \int_0^1 \left((1+x^2)^{2/3} - 1\right)^{n-1} dx\,, n \in \mathbb{N}$$

$$I_n = \sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^{n-k-1}F\left(-\frac{2k}{3}, \frac{1}{2}; \frac{3}{2}; -1\right)\,, n \in \mathbb{N}$$

$$0 < I_n < \left(2^{2/3} - 1\right)^{n-1}\,, n \in \mathbb{N}$$

$F(a, b; c; x)$, es la función hypergeométrica de Gauss.

(57) $\pi = 4\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\int_0^1 \left(\frac{k}{n}\right)^{x^2} dx$

Part 2: formula for pi

Se muestra una fórmula para la constante pi

Fórmula

(1) $$\pi = 4\sum_{n=0}^{\infty}\int_{1}^{2} x^{-n-1}(-x^2+3x-2)^n\,dx$$

Algunas fórmulas relacionadas con (1)

(2) $$I_n = \int_{1}^{2} x^{-n-1}(-x^2+3x-2)^n\,dx = \int_{1}^{2} x^{-n-1}(x-1)^n(2-x)^n\,dx \;, n \in \mathbb{N}\cup\{0\}$$

(3) $$I_0 = \ln 2$$

En fórmulas (4)-(11) , para $n \in \mathbb{N}$, se tiene:

(4) $$I_n = \left(\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{n}\right)\ln 2 + \sum_{k=0}^{n}\sum_{m=0, m\neq n}^{n+k}\binom{n}{k}\binom{n+k}{m}\frac{(-1)^{n+m}(2^{m-n}-1)}{m-n}$$

(5) $$I_n = \left(\sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k}2^k\right)\ln 2 + \sum_{\substack{0\le k,m\le n\\ k+m\neq n}}\binom{n}{k}\binom{n}{m}\frac{(-1)^{n+k+m}(2^k-2^{n-m})}{k+m-n}$$

(6) $$0 < I_n \le \left(3-2\sqrt{2}\right)^n \ln 2$$

(7) $$P_n(x) = (-x^2+3x-2)^n = \sum_{k=0}^{2n} c_{n,k}x^k \Rightarrow c_{n,k} = \frac{1}{k!}\left(\left.\frac{d^k P_n}{dx^k}\right|_{x=0}\right) , k = 0 \ldots 2n$$

(8) $$I_n = c_{n,n}\ln 2 + \sum_{k=0}^{n-1} c_{n,k}\frac{2^{k-n}-1}{k-n} + \sum_{k=n+1}^{2n} c_{n,k}\frac{2^{k-n}-1}{k-n}$$

(9) $$I_n = c_{n,n}\ln 2 + \sum_{k=0}^{n-1}\left(c_{n,k} + c_{n,k+n+1}\right)\frac{2^{k-n}-1}{k-n}$$

(10) $$\begin{cases} c_{1,0} = -2, c_{1,1} = 3, c_{1,2} = -1 \\ c_{n+1,0} = -2c_{n,0} \\ c_{n+1,1} = -2c_{n,1} + 3c_{n,0} \\ c_{n+1,k} = -2c_{n,k} + 3c_{n,k-1} - c_{n,k-2}\ , 2 \le k \le 2n \\ c_{n+1,2n+1} = 3c_{n,2n} - c_{n,2n-1} \\ c_{n+1,2n+2} = -c_{n,2n} \end{cases}$$

(11) $$\begin{cases} c_{1,0} = -2, c_{1,1} = 3, c_{1,2} = -1 \\ c_{n,0} = (-2)^n\ , n \ge 1 \\ c_{n,1} = -\frac{3}{2}nc_{n,0}\ , n \ge 1 \\ c_{n,2} = \frac{1}{4}\left((3-3n)c_{n,1} + 2nc_{n,0}\right)\ , n \ge 2 \\ c_{n,k} = \frac{1}{2k}\left((2n+2-k)c_{n,k-2} + (3k-3-3n)c_{n,k-1}\right)\ , 3 \le k \le 2n-1 \\ c_{n,2n} = -\frac{1}{3n}c_{n,2n-1}\ , n \ge 2 \end{cases}$$

(12) $$P_n(x) = \sum_{\substack{m+q+r=n \\ m,q,r\in N\cup\{0\}}} \frac{n!\,(-1)^{m+r}\,2^m\,3^q}{m!\,q!\,r!}x^{q+2r}\ , n \in \mathbb{N}$$

(13) $$P_n(x) = \sum_{\substack{m+k-r=n \\ 2r\le k}} \frac{n!\,(-1)^{m+r}\,2^m\,3^{k-2r}}{m!\,(k-2r)!\,r!}x^{k}\ , n \in \mathbb{N}$$

(14) $$c_{n,k} = \sum_{r=0}^{[k/2]} \frac{n!\,(-1)^{n-k}\,2^{n-k+r}\,3^{k-2r}}{(n-k+r)!\,r!\,(k-2r)!}\ , n \in \mathbb{N}, 0 \le k \le 2n$$

En la fórmula (14) , $\frac{1}{(-n)!} = 0\ , n \in \mathbb{N}$.

(15) $$\pi = 4\sum_{n=0}^{\infty} I_n = 4\left(\ln 2 + (3\ln 2 - 2) + (13\ln 2 - 9) + \left(63\ln 2 - \frac{131}{3}\right) + \cdots\right)$$

(16) $$e^{\pi} = 2^4 \times (2^{12}e^{-8}) \times (2^{52}e^{-36}) \times \left(2^{252}e^{-524/3}\right) \times \ldots$$

Part 3: sequence for pi

Se muestra una sucesión que converge al número pi

Sea $z_1 = \frac{3}{2}\sqrt{3}$, sea z_n , la sucesión definida por la ecuación cúbica:

(1) $$4z_{n+1}^3 - 3^{2n+3}z_{n+1} + 3^{2n+3}z_n = 0\ , n \in \mathbb{N}$$

Para $n = 1$, fijo , la ecuación (1) posee tres raíces reales: $x0 < x1 < x2$, eligiendo $z_2 = x1$ (raíz intermedia) , y siguiendo este proceso: $z_{n+1} = x1$ (raíz intermedia de la correspondiente ecuación para z_n), se prueba usando trigonometría que:

(2) $$\pi = \lim_{n\to\infty} z_n = 3.141592\ldots$$

Analogamente, poniendo: $w_1 = \frac{2}{3\sqrt{3}}$, $w_n = \frac{1}{z_n}$, se tiene la ecuación:

(3) $$3^{2n+3}w_{n+1}^3 - 3^{2n+3}w_n w_{n+1}^2 + 4w_n = 0\ , n \in \mathbb{N}$$

Para n fijo , la ecuación (3) posee tres raíces reales: $y0 < y1 < y2$, eligiendo $w_{n+1} = y2$, , la mayor raíz de (3) , y siguiendo este proceso, se obtiene que:

(4) $$\frac{1}{\pi} = \lim_{n\to\infty} w_n = 0.318309\ldots$$

Algunos valores para las sucesiones z_n , w_n , son:

$z_n : n = 1..6$

$x0$	$x1$	$x2$
0	2.598076 ...	0
−8.86 ...	3.078181 ...	5.78 ...
−24.79 ...	3.134508 ...	21.65 ...
−71.66 ...	3.140805 ...	68.52 ...
−211.99 ...	3.141505 ...	208.85 ...
−632.89 ...	3.141582 ...	629.75 ...

$w_n : n = 1..6$

$y0$	$y1$	$y2$
0	0	0.384900 ...
−0.11 ...	0.17 ...	0.324867 ...
−0.04 ...	0.04 ...	0.319029 ...
−0.01 ...	0.01 ...	0.318389 ...
−0.00 ...	0.00 ...	0.318318 ...
−0.00 ...	0.00 ...	0.318310 ...

El proceso anterior se puede explicitar como sigue:

Sea $z_{n,k}$; $n, k \in \mathbb{N}$, la sucesión definida como sigue:

(1) $$z_{n+1,k+1} = \frac{z_{n,\infty} - 8z_{n+1,k}^3 3^{-2n-3}}{1 - 4z_{n+1,k}^2 3^{-2n-2}} \ , z_{n+1,1} = z_{n,\infty}, z_{1,\infty} = \frac{3\sqrt{3}}{2}$$

La sucesión definida por (1) , cumple la siguiente condición:

(2) $$\pi = \lim_{n\to\infty} z_{n,\infty}$$

Analogamente se tiene: Sea $w_{n,k}$; $n, k \in \mathbb{N}$, la sucesión definida por:

(3) $$w_{n+1,k+1} = \frac{2w_{n+1,k}^3 - w_{n,\infty} w_{n+1,k}^2 - 4w_{n,\infty} 3^{-2n-3}}{3w_{n+1,k}^2 - 2w_{n,\infty} w_{n+1,k}} \ , w_{n+1,1} = w_{n,\infty}, w_{1,\infty} = \frac{2}{3\sqrt{3}}$$

La sucesión definida por (3) , cumple la siguiente condición:

(4) $$\frac{1}{\pi} = \lim_{n\to\infty} w_{n,\infty}$$

La prueba de (1),(2),(3),(4) , utiliza argumentos trigonometrícos y el método de Newton-Raphson para resolver ecuaciones.

Codigo Para (1) y (2)

$$\pi(\varepsilon z,\varepsilon x,nmax,kmax)=\left\{\begin{array}{c} z_0 \leftarrow 0 \\ z_1 \leftarrow \dfrac{3\sqrt{3}}{2} \\ for\ n\ \in 1..nmax \\ \left\{\begin{array}{c} break\ if\ |z_n - z_{n-1}| < \varepsilon z \\ x_0 \leftarrow 0 \\ x_1 \leftarrow z_n \\ for\ k\ \in 1..kmax \\ \left\{\begin{array}{c} break\ if\ |x_k - x_{k-1}| < \varepsilon x \\ x_{k+1} \leftarrow \dfrac{x_1 - 8\,x_k^3\,3^{-2n-3}}{1-4\,x_k^2\,3^{-2n-2}} \end{array}\right. \\ z_{n+1} \leftarrow x_k \end{array}\right. \\ z \end{array}\right.$$

$$\pi(10^{-6},10^{-4},9,9)=\begin{pmatrix} 0 \\ 2.59807621\ldots \\ 3.07818129\ldots \\ 3.13450868\ldots \\ 3.14080507\ldots \\ 3.14150514\ldots \\ 3.14158293\ldots \\ 3.14159157\ldots \\ 3.14159253\ldots \end{pmatrix}$$

Codigo Para (3) y (4)

$$\pi 1(\varepsilon w,\varepsilon x,nmax,kmax)=\left\{\begin{array}{c} w_0 \leftarrow 0 \\ w_1 \leftarrow \dfrac{2}{3\sqrt{3}} \\ for\ n\ \in 1..nmax \\ \left\{\begin{array}{c} break\ if\ |w_n - w_{n-1}| < \varepsilon w \\ x_0 \leftarrow 0 \\ x_1 \leftarrow w_n \\ for\ k\ \in 1..kmax \\ \left\{\begin{array}{c} break\ if\ |x_k - x_{k-1}| < \varepsilon x \\ x_{k+1} \leftarrow \dfrac{2\,x_k^3 - x_1 x_k^2 - 4\,x_1 3^{-2n-3}}{3\,x_k^2 - 2x_1 x_k} \end{array}\right. \\ w_{n+1} \leftarrow x_k \end{array}\right. \\ w \end{array}\right.$$

$$\pi 1(10^{-6},10^{-4},9,9)=\begin{pmatrix}0\\0.38490018\ldots\\0.32486717\ldots\\0.31902928\ldots\\0.31838972\ldots\\0.31831880\ldots\\0.31831092\ldots\\0.31831004\ldots\end{pmatrix}$$

Part 4: symmetric formula for pi

Fórmula simétrica para la constante pi

Esta nota está relacionada con la fórmula (Jonathan Sondow , 1997):

(1) $$\pi=\frac{\prod_{n=1}^{\infty}\left(1+\frac{1}{4n^2-1}\right)}{\sum_{n=1}^{\infty}\frac{1}{4n^2-1}}$$

donde

(2) $$\prod_{n=1}^{\infty}\left(1+\frac{1}{4n^2-1}\right)=\frac{\pi}{2}\ ,\sum_{n=1}^{\infty}\frac{1}{4n^2-1}=\frac{1}{2}$$

En esta nota se muestran algunas fórmulas alternativas , similares a (1).

Fórmulas

(3) $$\frac{\sqrt{2}}{\pi}-\frac{1}{2\sqrt{2}}=\frac{\sum_{n=1}^{\infty}\frac{1}{16n^2-1}}{\prod_{n=1}^{\infty}\left(1+\frac{1}{16n^2-1}\right)}$$

(4) $$\frac{\pi}{8\sqrt{2}}=\frac{\sum_{n=1}^{\infty}\frac{1}{4(2n-1)^2-1}}{\prod_{n=1}^{\infty}\left(1+\frac{1}{4(2n-1)^2-1}\right)}$$

(5) $$\frac{4}{\pi} = 1 + \frac{\sum_{n=1}^{\infty} \frac{1}{4(2n)^2 - 1}}{\sum_{n=1}^{\infty} \frac{1}{4(2n-1)^2 - 1}}$$

(6) $$\frac{k\sin(\pi/k)}{2\pi} - \frac{\cos(\pi/k)}{2} = \frac{\sum_{n=1}^{\infty} \frac{1}{k^2n^2 - 1}}{\prod_{n=1}^{\infty}\left(1 + \frac{1}{k^2n^2 - 1}\right)} \ , k \in \mathbb{N} - \{1\}$$

Para $k = 2$, se tiene la fórmula (1).

(7) $$\frac{2^{k-1}}{\pi}\underbrace{\sqrt{2 - \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}}_{k-radicales} - \frac{1}{4}\underbrace{\sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}}_{k-radicales} = \frac{\sum_{n=1}^{\infty} \frac{1}{2^{2k+2}n^2 - 1}}{\prod_{n=1}^{\infty}\left(1 + \frac{1}{2^{2k+2}n^2 - 1}\right)} \ , k \in \mathbb{N}$$

En (7) con $k = 2$, se tiene:

(8) $$\frac{2}{\pi}\sqrt{2 - \sqrt{2}} - \frac{1}{4}\sqrt{2 + \sqrt{2}} = \frac{\sum_{n=1}^{\infty} \frac{1}{64n^2 - 1}}{\prod_{n=1}^{\infty}\left(1 + \frac{1}{64n^2 - 1}\right)}$$

(9) $$\pi = 4 - \sum_{n=1}^{\infty} 2^{2n}(n!)^2\left(\frac{1}{x_n} - \frac{4(n+1)^2}{x_{n+1}}\right)$$

donde

$$x_{n+1} = (4(n+1)^2 - 1)x_n + \prod_{k=1}^{n}(4k^2 - 1) \ , x_1 = 1$$

(10) $$\frac{1}{\pi} = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{1}{2^{2n}(n!)^2}\left(\frac{x_{n+1}}{4(n+1)^2} - x_n\right)$$

x_n , se define como en (9).

(11) $$\pi = 4\prod_{n=1}^{\infty}\left(\frac{4(n+1)^2}{4(n+1)^2 - 1 + \frac{1}{\sum_{k=1}^{n} \frac{1}{4k^2 - 1}}}\right)$$

(12) $$\frac{1}{\pi} = \frac{1}{4}\prod_{n=1}^{\infty}\left(\frac{1}{1 - \frac{1}{4(n+1)^2} + \frac{1}{4(n+1)^2 \sum_{k=1}^{n} \frac{1}{4k^2 - 1}}}\right)$$

(13) $$\pi = 4\left(\frac{2^3}{3^2}\right)\left(\frac{2^3 3}{5^2}\right)\left(\frac{2^4 3}{7^2}\right)\left(\frac{2^4 5}{3^4}\right)\left(\frac{2^3 3\ 5}{11^2}\right)\left(\frac{2^3 3\ 7}{13^2}\right)\dots$$

(14) $$\pi = 4\left(\frac{2^6}{3\ 5^2}\right)\left(\frac{2^8 5}{3^3 7^2}\right)\left(\frac{2^6 3^2\ 5\ 7}{11^2 13^2}\right)\left(\frac{2^{10} 7}{5^2 17^2}\right)\left(\frac{2^6 5^2 11}{7^2 19^2}\right)\left(\frac{2^8 3^2 11\ 13}{5^4 23^2}\right)\dots$$

Part 5: formula for pi

Una fórmula para la constante pi

Fórmula

(1) $$\sum_{n=1}^{\infty}\binom{2n}{n}H_n(-1)^{n-1}x^n = \frac{2}{a^2+b^2}\left(ar + b\frac{\pi}{64}\right) + \frac{2i}{a^2+b^2}\left(a\frac{\pi}{64} - br\right)$$

donde

(2) $$x = \frac{1}{4}\left(a^2 - \frac{101}{100} + \frac{a}{5}i\right), a = \frac{1}{2}\left(-1 + \sqrt{\frac{24}{25} + \frac{2}{5t}}\right), r = \ln\left(\frac{2\sqrt{(a(1+a)+b^2)^2 + b^2}}{(1+a)^2 + b^2}\right)$$

(3) $$H_n = \sum_{k=1}^{n}\frac{1}{k}, i = \sqrt{-1}, b = \frac{1}{10}$$

(4) $$t = \sqrt{\frac{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}}{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}}}$$

Part 6: pi,radicals,gamma function

Una fórmula que involucra a la constante pi , a la función gamma y radicales del tipo:

$$r_k = \underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}_{(k-1)-radicales} \quad , k \in \mathbb{N}$$

Fórmula

Para $k \in \mathbb{N}$, se tiene:

(1) $$\frac{1}{\pi}\left(\frac{\left(\Gamma(1+2^{-k-1})\right)^6}{\left(\Gamma(1+2^{-k})\right)^3}\right)\frac{2^{4+2^{-(k-2)}}}{2+r_k}$$
$$= \sum_{n=0}^{\infty}\frac{(2^{-1}+2^{-k-1})_n^3\ (3\cdot 2^{k+1}n+2^k+3)}{2^{2n+k}(1+2^{-k-1})_n^3} + \frac{4}{2^k(2^k-1)}\sum_{n=0}^{\infty}\frac{(1/2)_n(2^{-1}+2^{-k-1})_n}{(1+2^{-k-1})_n(3\cdot 2^{-1}-2^{-k-1})_n}$$

donde

$$(a)_0 = 1, (a)_n = a(a+1)\ldots(a+n-1) \quad , n \in \mathbb{N}$$

$$\frac{\left(\Gamma(1+2^{-k-1})\right)^6}{\left(\Gamma(1+2^{-k})\right)^3} = \frac{(1+2^{-k})^3}{(1+2^{-k-1})^6}\prod_{n=1}^{\infty}\left(1+\frac{2^{-k}}{n+1}\right)^3\left(1+\frac{2^{-k-1}}{n+1}\right)^{-6}$$

$$\Gamma(x) = \int_0^{\infty} e^{-t}\, t^{x-1}\, dt \, , x > 0$$

Ejemplos:

(2) $$\frac{32}{\pi}\left(\frac{\left(\Gamma(5/4)\right)^6}{\left(\Gamma(3/2)\right)^3}\right) = \sum_{n=0}^{\infty}\frac{(3/4)_n^3\ (12n+5)}{2^{2n+1}(5/4)_n^3} + 2\sum_{n=0}^{\infty}\frac{(1/2)_n(3/4)_n}{(5/4)_n^2}$$

(3) $$\frac{1}{\pi}\left(\frac{\left(\Gamma(9/8)\right)^6}{\left(\Gamma(5/4)\right)^3}\right)\frac{32}{2+\sqrt{2}} = \sum_{n=0}^{\infty}\frac{(5/8)_n^3\ (24n+7)}{2^{2n+2}(9/8)_n^3} + \frac{1}{3}\sum_{n=0}^{\infty}\frac{(1/2)_n(5/8)_n}{(9/8)_n(11/8)_n}$$

(4) $$\frac{1}{\pi}\left(\frac{\left(\Gamma(17/16)\right)^6}{\left(\Gamma(9/8)\right)^3}\right)\frac{16\sqrt{2}}{2+\sqrt{2+\sqrt{2}}} = \sum_{n=0}^{\infty}\frac{(9/16)_n^3\ (48n+11)}{2^{2n+3}(17/16)_n^3} + \frac{1}{14}\sum_{n=0}^{\infty}\frac{(1/2)_n(9/16)_n}{(17/16)_n(23/16)_n}$$

Part 7: infinite products

Productos infinitos para $e^{\pi^2/8}$ y π^π

Sean a_n , b_n ,$n \in \mathbb{N}$ sucesiones definidas como:

$$a_n = \underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}_{n-radicales}$$

$$b_n = \underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}_{n-radicales}$$

Se tiene:

(1) $$e^{\pi^2/8} = 4\prod_{n=1}^{\infty}\left(\frac{2\sqrt{2}\sqrt{a_n}}{2+a_n}\right)^{2^{2n+1}} = 4\left(\frac{2\sqrt{2}\sqrt[4]{2}}{2+\sqrt{2}}\right)^{8}\left(\frac{2\sqrt{2}\sqrt[4]{2+\sqrt{2}}}{2+\sqrt{2+\sqrt{2}}}\right)^{32}\ldots$$

(2) $$\pi^\pi = \left(2\sqrt{2}\right)^{2\sqrt{2}}\prod_{n=1}^{\infty}\left(\frac{2^{n+1}b_{n+1}}{(2^n b_n)^{a_{n+1}/2}}\right)^{2^{n+1}b_{n+1}} = \left(2\sqrt{2}\right)^{2\sqrt{2}}\left(\frac{4\sqrt{2-\sqrt{2}}}{\left(2\sqrt{2}\right)^{\sqrt{2+\sqrt{2}}/2}}\right)^{4\sqrt{2-\sqrt{2}}}\ldots$$

Part 8: pi,radicals,arctangents

En esta nota mostramos fórmulas que involucran el número pi y los números:

$$z_n = \sqrt[n]{1+\sqrt[n]{1+\sqrt[n]{1+\cdots}}} \quad ,n = 2,3,4,5,\ldots$$

Sea $n \in \mathbb{N} - \{1\}$, se tiene:

$$(1)\qquad \pi = 4\ \tan^{-1}\left(\sqrt[n]{1+\sqrt[n]{1+\sqrt[n]{1+\cdots}}}\right) - 4\ \tan^{-1}\left(\frac{1}{3n-2}\right) - 4\sum_{n=1}^{\infty}\tan^{-1}(q_n)$$

donde

$$q_n = \frac{(a_{n+1})^2 - a_n a_{n+2}}{a_{n+1}(a_n + a_{n+2})}\ \ , n \in \mathbb{N}$$

$$a_{k+n} = c_{n-1}a_{k+n-1} + c_{n-2}a_{k+n-2} + \cdots + c_1 a_{k+1} + a_k$$

$$c_k = \binom{n}{k} - 2(-1)^{n-k}\binom{n-1}{k-1}\ , k = 1,2,3,\ldots,n-1$$

$$a_1 = 1, a_0 = a_{-1} = a_{-2} = \cdots = a_{-n+2} = 0$$

Ejemplos 1: $n = 2$

$$(2)\qquad \pi = 4\ \tan^{-1}\left(\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}\right) - 4\ \tan^{-1}\left(\frac{1}{4}\right) - 4\sum_{n=1}^{\infty}\tan^{-1}(q_n)$$

$$a_{k+2} = 4a_{k+1} + a_k\ \ , a_1 = 1, a_2 = 4$$

Ejemplo 2: $n = 3$

$$(3)\qquad \pi = 4\ \tan^{-1}\left(\sqrt[3]{1+\sqrt[3]{1+\sqrt[3]{1+\cdots}}}\right) - 4\ \tan^{-1}\left(\frac{1}{7}\right) - 4\sum_{n=1}^{\infty}\tan^{-1}(q_n)$$

$$a_{k+3} = 7a_{k+2} + a_{k+1} + a_k\ \ , a_1 = 1, a_2 = 7, a_3 = 50$$

Ejemplo 3: $n = 5$

$$(4)\qquad \pi = 4\ \tan^{-1}\left(\sqrt[5]{1+\sqrt[5]{1+\sqrt[5]{1+\cdots}}}\right) - 4\ \tan^{-1}\left(\frac{1}{13}\right) - 4\sum_{n=1}^{\infty}\tan^{-1}(q_n)$$

$$a_{k+5} = 13a_{k+4} - 2a_{k+3} + 18a_{k+2} + 3a_{k+1} + a_k$$

$$a_1 = 1, a_2 = 13, a_3 = 167, a_4 = 2163, a_5 = 28022$$

2. Sea $z_m = \sqrt[m]{1+\sqrt[m]{1+\sqrt[m]{1+\cdots}}}\ \ , m \in \mathbb{N} - \{1\}$, se tiene:

(5) $$\pi = 3\sum_{n=0}^{\infty}(z_m)^{-n}\sum_{k=[n/m]}^{[n/(m-1)]}\binom{2k}{k}\binom{k}{n-(m-1)k}\frac{2^{-4k}}{2k+1}$$

3. Para $z_5 = \sqrt[5]{1+\sqrt[5]{1+\sqrt[5]{1+\cdots}}}$, se tiene:

(6) $$\pi = 2\tan^{-1} z_5 + 2\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+k}{n-k}\frac{(-1)^k}{2k+1}(z_5)^{-5n-5k-5}$$

(7) $$\pi = 2\tan^{-1} z_5 + 2\sum_{n=0}^{\infty}(z_5)^{-5n-5}\sum_{k=0}^{n}\binom{n}{n-2k}\frac{(-1)^k}{2k+1}$$

(8) $$\pi = 4\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}2^{n-k}(-1)^k\left(\frac{4(z_5)^{-5n-3k-2m-4}}{5n+3k+2m+4}+\frac{(z_5)^{-5n-3k-2m-9}}{5n+3k+2m+9}\right)$$

4. Para $n \in \mathbb{N} - \{1\}$, se tiene:

(9) $$\pi\sqrt[n]{1+\sqrt[n]{1+\sqrt[n]{1+\cdots}}} = \sqrt[n]{\pi^n + \pi^{n-1}\sqrt[n]{\pi^n + \pi^{n-1}\sqrt[n]{\pi^n + \cdots}}}$$

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