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Neuberg's Orthogonal Circles

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In this article, we highlight some metric properties in connection with **Neuberg's circles** and **triangle**.

We recall some results that are necessary.

1st Definition.

It's called Brocard's point of the triangle ABC the point Ω with the property: $\sphericalangle \Omega AB = \sphericalangle \Omega BC = \sphericalangle \Omega CA$. The measure of the angle ΩAB is denoted by ω and it is called Brocard's angle. It occurs the relationship:

$\text{ctg}\omega = \text{ctg}A + \text{ctg}B + \text{ctg}C$ (see [1]).

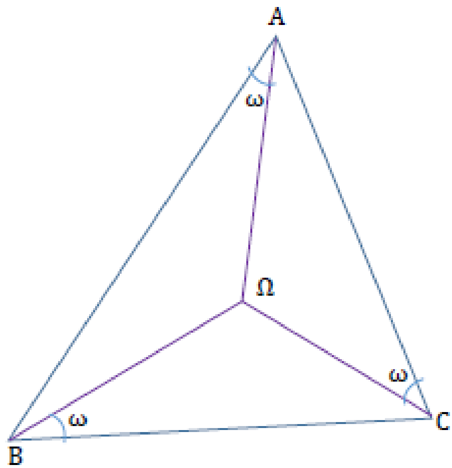

Figure 1.

2nd Definition.

Two triangles are called equibrocardian if they have the same Brocard's angle.

3rd Definition.

The locus of points M from the plane of the triangle located on the same side of the line BC as A and forming with BC an equibrocardian triangle with ABC, containing the vertex A of the triangle, it's called A-Neuberg' circle of the triangle ABC.

We denote by N_a the center of A-Neuberg' circle by radius n_a (analogously, we define B-Neuberg' and C-Neuberg' circles).

We get that $m(BN_aC) = 2\omega$ and $n_a = \frac{a}{2}\sqrt{\text{ctg}^2\omega - 3}$ (see [1]).

The triangle $N_aN_bN_c$ formed by the centers of Neuberg's circles is called Neuberg's triangle.

1st Proposition.

The distances from the center circumscribed to the triangle ABC to the vertex of Neuberg's triangle are proportional to the cubes of ABC triangle's sides lengths.

Proof.

Let O be the center of the circle circumscribed to the triangle ABC (see *Figure 2*).

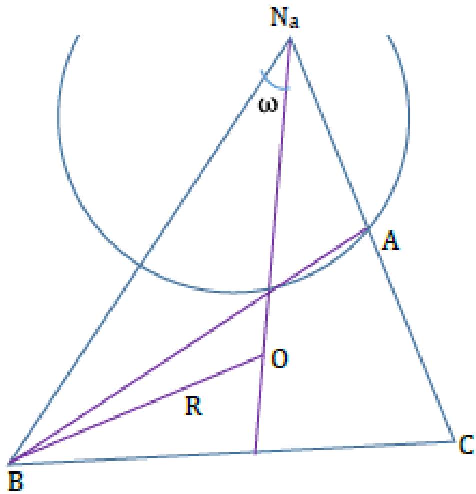

Figure 2.

The law of cosines applied in the triangle ON_aB provides:

$$\frac{ON_a}{\sin(N_aBO)} = \frac{R}{\sin\omega}.$$

But $m(\sphericalangle N_aBO) = m(\sphericalangle N_aBC) - m(\sphericalangle OBC) = A - \omega$.

We have that $\frac{ON_a}{\sin(A-\omega)} = \frac{R}{\sin\omega}$.

But

$$\frac{\sin(A-\omega)}{\sin\omega} = \frac{C\Omega}{A\Omega} = \frac{\frac{a}{c}\cdot 2R\sin\omega}{\frac{b}{a}\cdot 2R\sin\omega} = \frac{a^3}{abc} = \frac{a^3}{4RS},$$

S being the area of the triangle ABC.

It follows that $ON_a = \frac{a^3}{4S}$, and we get that $\frac{ON_a}{a^3} = \frac{ON_b}{b^3} = \frac{ON_c}{c^3}$.

Consequence.

In a triangle ABC, we have:

1) $ON_a \cdot ON_b \cdot ON_c = R^3$;

2) $\text{ctg}\omega = \frac{ON_a}{a} + \frac{ON_b}{b} + \frac{ON_c}{c}$.

2nd Proposition.

If $N_aN_bN_c$ is the Neuberg's triangle of the triangle ABC, then:

$$N_aN_b{}^2 = \frac{(a^2+b^2)(a^4+b^4)-a^2b^2c^2}{2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4}.$$

(The formulas for N_bN_c and N_cN_a are obtained from the previous one, by circular permutations.)

Proof.

We apply the law of cosines in the triangle N_aON_c:

$$ON_a = \frac{a^3}{4S}, ON_b = \frac{b^3}{4S}, m(\sphericalangle N_aON_b) = 180^0 - \hat{c}.$$

$$N_aN_b{}^2 = \frac{a^6+b^6-2a^3b^3\cos(180^0-c)}{16S^2}$$

$$= \frac{a^6+b^6+2a^3b^3\cos C}{16S^2}.$$

But the law of cosines in the triangle ABC provides

$$2\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

and, from din Heron's formula, we find that

$$16S^2 = 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4.$$

Substituting the results above, we obtain, after a few calculations, the stated formula.

4th Definition.

Two circles are called orthogonal if they are secant and their tangents in the common points are perpendicular.

3rd Proposition.

(Gaultier – 18B)

Two circles $\mathcal{C}(O_1, r_1)$, $\mathcal{C}(O_2, r_2)$ are orthogonal if and only if

$$r_1^2 + r_2^2 = O_1O_2^2.$$

Proof.

Let $\mathcal{C}(O_1, r_1)$, $\mathcal{C}(O_2, r_2)$ be orthogonal (see *Figure 3*); then, if A is one of the common points, the triangle O_1AO_2 is a right triangle and the Pythagorean Theorem applied to it, leads to ${r_1}^2 + {r_2}^2 = {O_1O_2}^2$.

Reciprocally.

If the metric relationship from the statement occurs, it means that the triangle O_1AO_2 is a right triangle, therefore A is their common point (the relationship $r_1{}^2 + r_2{}^2 = O_1O_2{}^2$ implies $r_1{}^2 + r_2{}^2 > O_1O_2{}^2$), then $O_1A \perp O_2A$, so O_1A is tangent to the circle $\mathcal{C}(O_2, r_2)$ because it is perpendicular in A on radius O_2A, and as well O_2A is tangent to the circle $\mathcal{C}(O_1, r_1)$, therefore the circles are orthogonal.

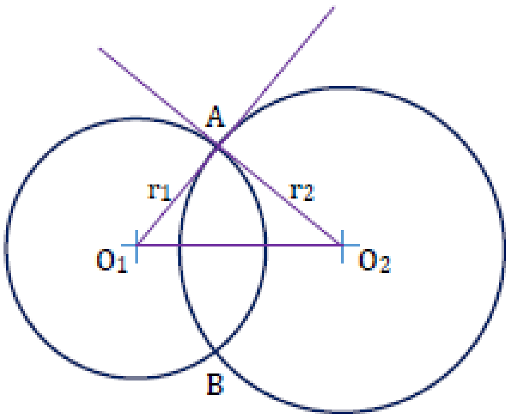

Figure 3.

4th Proposition.

B-Neuberg's and C-Neuberg's circles associated to the right triangle ABC (in A) are orthogonal.

Proof.

If $m(\hat{A}) = 90^0$, then $N_bN_c^2 = \frac{b^6+c^6}{16S}$.

$n_b = \frac{b}{2}\sqrt{\text{ctg}^2\omega - 3}\,; n_c = \frac{c}{2}\sqrt{\text{ctg}^2\omega - 3}$.

But $\text{ctg}\omega - \frac{a^2+b^2+c^2}{4S} = \frac{b^2+c^2}{2S} = \frac{a^2}{bc}$.

It was taken into account that $a^2 = b^2 + c^2$ and $2S = bc$.

$$\text{ctg}^2\omega - 3 = \frac{a^4}{b^2c^2} - 3 = \frac{(b^2+c^2)^2 - 3b^2c^2}{b^2c^2}$$

$$\text{ctg}^2\omega - 3 = \frac{b^4 + c^4 - b^2c^2}{b^2c^2}$$

$$n_b^2 + n_c^2 = \frac{b^4 + c^4 - b^2c^2}{b^2c^2}\left(\frac{b^2+c^2}{4}\right)$$

$$= \frac{(b^2+c^2)(b^4+c^4-b^2c^2)}{4b^2c^2} = \frac{b^6+c^6}{16S^2}.$$

By $N_b^2 + N_c^2 = N_bN_c^2$, it follows that B-Neuberg's and C-Neuberg's circles are orthogonal.

References.

[1] F. Smarandache, I. Patrascu: *The Geometry of Homological Triangles*. Columbus: The Educational Publisher, Ohio, USA, 2012.
[2] T. Lalescu: *Geometria triunghiului* [The Geometry of the Triangle]. Craiova: Editura Apollo, 1993.
[3] R. A. Johnson: *Advanced Euclidian Geometry*. New York: Dover Publications, 2007.