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One Application of Wallis Theorem

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One Application Of Wallis Theorem[1]

Theorem 1. (Wallis, 1616-1703)

$$\int_0^{\pi/2} \sin^{2n+1} x dx = \int_0^{\pi/2} \cos^{2n+1} x dx = \frac{2.4\ldots(2n)}{1.3\ldots(2n+1)}.$$

Proof. Using integration by parts, we obtain

$$I_n = \int_0^{\pi/2} \sin^{2n+1} x dx = \int_0^{\pi/2} \sin^{2n} x \sin x dx = -\cos x \cdot \sin 2nx \Big|_0^{\pi/2} +$$

$$+ 2n \int_0^{\pi/2} \sin^{2n-1} x\left(1-\sin^2 x\right)dx = 2nI_{n-1} - 2nI_n$$ from where $I_n = \frac{2n}{2n+1} I_{n-1}$.

By multiplication we obtain the statement. We prove in the same way for cos x.

Theorem 2. $\int_0^{\pi/2} \sin^{2n} x dx = \int_0^{\pi/2} \cos^{2n} x dx = \frac{1.3\ldots(2n-1)}{2.4\ldots(2n)} \cdot \frac{\pi}{2}.$

Proof. Same as the first theorem.

Theorem 3. If $f(x) = \sum_{k=0}^{\infty} a_{2k} x^{2k}$, then

$$\int_0^{\pi/2} f(\sin x)dx = \int_0^{\pi/2} f(\cos x)dx = \frac{\pi}{2} a_0 + \frac{\pi}{2} \sum_{k=1}^{\infty} a_{2k} \frac{1.3\ldots(2k-1)}{2.4\ldots(2k)}.$$

Proof. In the $f(x) = \sum_{k=0}^{\infty} a_{2k} x^{2k}$ function we substitute x by $\sin x$ and then integrate from 0 to $\pi/2$, and we use the second theorem.

Theorem 4. If $g(x) = \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1}$, then

$$\int_0^{\pi/2} g(\sin x)dx = \int_0^{\pi/2} g(\cos x)dx = a_1 + \sum_{k=1}^{\infty} a_{2k+1} \frac{2.4\ldots(2k)}{1.3\ldots(2k+1)}.$$

Theorem 5. If $h(x) = \sum_{k=0}^{\infty} a_k x^k$, then

$$\int_0^{\pi/2} h(\sin x)dx = \int_0^{\pi/2} h(\cos x)dx = \frac{\pi}{2} a_0 + a_1 + \sum_{k=1}^{\infty}\left(\frac{\pi}{2} a_{2k} \frac{1.3\ldots(2k-1)}{2.4\ldots(2k)} + a_{2k+1} \frac{2.4\ldots(2k)}{1.3\ldots(2k+1)}\right).$$

Application 1.

$$\int_0^{\pi/2} \sin(\sin x)dx = \int_0^{\pi/2} \sin(\cos x)dx = \sum_{k=0}^{\infty} (-1)^k \frac{1}{1^2 3^2 \ldots (2k+1)^2}.$$

Proof. We use that $\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$.

Application 2.

$$\int_0^{\pi/2} \cos(\sin x)dx = \int_0^{\pi/2} \cos(\cos x) = \frac{\pi}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k (k!)^2}.$$

Proof. We use that $\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$.

Application 3.

$$\int_0^{\pi/2} sh(\sin x)dx = \int_0^{\pi/2} sh(\cos x)dx = \sum_{k=0}^{\infty} \frac{1}{1^2 3^2 \ldots (2k+1)^2}$$

Proof. We use that $sh\ x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$

[1] Together with Mihály Bencze.

Application 4. $\int_0^{\pi/2} ch(\sin x)dx = \int_0^{\pi/2} ch(\cos x)dx = \frac{\pi}{2}\sum_{k=0}^{\infty}\frac{1}{4^k (k!)^2}$.

Proof. We use that $ch\ x = \sum_{k=0}^{\infty}\frac{x^{2k}}{(2k)\ !}$.

Application 5.

$\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}$.

Proof. In the $\arcsin x = x + \sum_{k=1}^{\infty}\frac{1.3.....(2k-1)x^{2k+1}}{2.4.....(2k)(2k+1)}$ expression we substitute x by sin x, and use theorem 4. It results that $\pi^2/8 = \sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}$. Because $\sum_{k=1}^{\infty} 1/k^2 = \sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} + \frac{1}{4}\sum_{k=1}^{\infty} 1/k^2$

we get $\sum_{k=1}^{\infty} 1/k^2 = \pi/6$.

Application 6.

$$\int_0^{\pi/2}\sin x\ ctg(\sin x)dx = \int_0^{\pi/2}\cos x\ ctg(\cos x)dx = \frac{\pi}{2} - \frac{\pi}{2}\sum_{k=1}^{\infty}\frac{B_k}{(k!)^2}$$

where B_k is the k-th Bernoulli type number (see[1]).

Proof. We use that $xctg\ x = 1 - \sum_{k=1}^{\infty}\frac{4^k B_k}{(2k)\ !}x^{2k}$.

Application 7.

$$\int_0^{\pi/2} arctg(\sin x)dx = \int_0^{\pi/2} arctg(\cos x)dx =$$
$$= 1 + \sum_{k=1}^{\infty}(-1)^k\frac{2.4.....(2k)}{1.3.....(2k-1)(2k+1)^2} .$$

Proof. We use that $arctg\ x = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{2k+1}$.

Application 8.

$$\int_0^{\pi/2}\arg th(\sin x)dx = \int_0^{\pi/2}\arg th(\cos x)dx = 1 + \sum_{k=1}^{\infty}\frac{2.4....(2k)}{1.3....(2k-1)}\frac{1}{(2k+1)^2} .$$

Proof. We use that $\arg th\ x = \sum_{k=0}^{\infty}\frac{x^{2k+1}}{2k+1}$.

Application 9.

$$\int_0^{\pi/2}\arg sh(\sin x)dx = \int_0^{\pi/2}\arg sh(\cos x)dx = \sum_{k=0}^{\infty}(-1)^k\frac{1}{(2k+1)^2} .$$

Proof. We use that $\arg sh\ x = \sum_{k=0}^{\infty}(-1)^k\frac{1.3....(2k-1)x^{2k+1}}{2.4....(2k)(2k+1)}$.

Application 10.

$$\int_0^{\pi/2} tg(\sin x)dx = \int_0^{\pi/2} tg(\cos x)dx = \sum_{k=1}^{\infty}\frac{2^{2k-1}(4^k-1)B_k}{1^2 3^2 ..(2k-1)^2 k} .$$

Proof. We use that

$$tg\ x = \sum_{k=1}^{\infty}\frac{2^{2k}(4^k-1)B_k}{(2k)\ !}x^{2k-1} .$$

Application 11.

$$\int_0^{\pi/2} \frac{\sin x}{\sin(\sin x)}dx = \int_0^{\pi/2} \frac{\cos x}{\sin(\cos x)}dx = \frac{\pi}{2} + \pi\sum_{k=1}^{\infty} \frac{\left(2^{2k-1}-1\right)B_k}{2^{2k}(k!)^2} .$$

Proof. We use that $\frac{x}{\sin x} = 1 + 2\sum_{k=1}^{\infty} \frac{\left(2^{2k-1}-1\right)B_k}{(2k)!}x^{2k}$.

Application 12.

$$\int_0^{\pi/2} \frac{\sin x}{sh(\sin x)}dx = \int_0^{\pi/2} \frac{\cos x}{sh(\cos x)}dx = \frac{\pi}{2} + \pi\sum_{k=1}^{\infty} \frac{\left(2^{2k-1}-1\right)B_k}{2^{2k}(k!)^2} .$$

Proof. We use that $\frac{x}{sh\ x} = 1 + 2\sum_{k=1}^{\infty}(-1)^k \frac{\left(2^{2k-1}-1\right)B_k x^{2k}}{(2k)!}$.

Application 13.

$$\int_0^{\pi/2} \sec(\sin x)dx = \int_0^{\pi/2} \sec(\cos x)dx = \frac{\pi}{2} + \pi\sum_{k=1}^{\infty} \frac{E_k}{2^{2k+1}(k!)^2} ,$$

where E_k is the k-th Euler type number (see([1]).

Proof. We use that $\sec x = 1 + \sum_{k=1}^{\infty} \frac{E_k}{(2k)!}x^{2k}$.

Application 14.

$$\int_0^{\pi/2} \sec h(\sin x)dx = \int_0^{\pi/2} \sec h(\cos x)dx = \frac{\pi}{2} + \pi\sum_{k=1}^{\infty}(-1)^k \frac{E_k}{2^{2k+1}(k!)^2} .$$

Proof. We use that

$$\sec h\ x = 1 + \sum_{k=1}^{\infty}(-1)^k \frac{E_k}{(2k)!}x^{2k} .$$

References:

[1] Octav Mayer: Teoria funcţiilor de o variabilă complexă, Ed. Academică, Bucureşti, 1981.

[2] Mihály Bencze, About Taylor-formula (manuscript).

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