C. DUMIRESCU
N. VÎRLAN
ȘT. ZAMFIR
E. RĂDESCU
N. RĂDESCU
FLORENTIN SMARANDACHE

Smarandache Type Function Obtained by Duality

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SMARANDACHE TYPE FUNCTION OBTAINED BY DUALITY[1]

Abstract. In this paper we extended the Smarandache function from the set N^* of positive integers to the set Q of rationals.

Using the inversion formula this function is also regarded as a generating function. We make in evidence a procedure to construct (numerical) function starting from a given function in two particular cases. Also conections between the Smarandache function and Euler's totient function as with Riemann's zeta function are etablished.

1. Introduction

The Smarandache function [13] is a numerical function $S : N^* \to N^*$ defined by $S(n) = \min\{m | m! \text{ is divisible by } n\}$.

From the definition it results that if

$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_t^{\alpha_t} \tag{1}$$

is the decomposition of n into primes then

$$S(n) = \max S(p_i^{\alpha_i}) \tag{2}$$

and moreover, if $[n_1, n_2]$ is the smallest common multiple of n_1 and n_2 then

$$S([n_1, n_2]) = \max\{S(n_1), S(n_2)\} \tag{3}$$

The Smarandache function characterizes the prime in the sense that a positive integer $p \geq 4$ is prime if and only if it is a fixed point of S.

From Legendre's formula:

$$m! = \prod_p p^{\sum_{i \geq 1} [\frac{m}{p^i}]} \tag{4}$$

it results [2] that if $a_n(p) = \dfrac{(p^n - 1)}{(p-1)}$ and $b_n(p) = p^n$ then considering the standard numerical scale

$$[p] : b_0(p), b_1(p), \ldots, b_n(p), \ldots$$

[1]Together with C.Dumitrescu, N.Vîrlan, Şt. Zamfir, E.Rădescu and N.Rădescu

and the generalized scale

$$[p] : a_0(p), a_1(p), \ldots, a_n(p), \ldots$$

we have

$$S(p^k) = p(\alpha_{[p]})_{(p)} \tag{5}$$

that is $S(p^k)$ is calculated multiplying by p the number obtained writing the exponent α in the generalised scale $[p]$ and "reading" it in the standard scale (p).

Let us observe that the calculus in the generalised scale $[p]$ is essentilly different from the calculus in the usual scale (p), becuase the usual relationship $b_{n+1}(p) = pb_n(p)$ is modified in $a_{n+1}(p) = pa_n(p) + 1$ (for more detals see [2]).

In the following let us note $S_p(\alpha) = S(p^\alpha)$. In [3] it is proved that

$$S_p(\alpha) = (p-1)\alpha + \sigma_{[p]}(\alpha) \tag{6}$$

where $\sigma_{[p]}(\alpha)$ is the sum of the digits of α written in the scale $[p]$, and also that

$$S_p(\alpha) = \frac{(p-1)^2}{p}(E_p(\alpha) + \alpha) + \frac{p-1}{p}\sigma_{(p)}(\alpha) + \sigma_{[p](\alpha)} \tag{7}$$

where $\sigma_{(p)}(\alpha)$ is the sum of the digits of α written in the standard scale (p) and $E_p(\alpha)$ is the exponent of p in the decomposition into primes of α! From (4) it results that $E_p(\alpha) = \sum_{i\geq 1} \left[\frac{\alpha}{p^i}\right]$, where $[h]$ is the integral part of h. It is also said [11] that

$$E_p(\alpha) = \frac{\alpha - \sigma_{(p)}(\alpha)}{p-1} \tag{8}$$

We can observe that this equality may be writen as

$$E_p(\alpha) = \left(\left[\frac{\alpha}{p}\right]_{(p)}\right)_{[p]}$$

that is the exponent of p in the decomposition into primes of α! is obtained writing the integral part of α/p in the base (p) and reading in the scale $[p]$.

Finally we note that in [1] it is proved that

$$S_p(\alpha) = p\left(\alpha - \left[\frac{\alpha}{p}\right] + \left[\frac{\sigma_{[p]}(\alpha)}{p}\right]\right) \tag{9}$$

From the definition of S it results that $S_p(E_p(\alpha)) = p\left[\frac{\alpha}{p}\right] = \alpha - \alpha_p$ (α_p is the remainder of α with respect to the modulus m) and also that

$$E_p(S_p(\alpha)) \geq \alpha; \quad E_p(S_p(\alpha) - 1) < \alpha \tag{10}$$

so

$$\frac{S_P(\alpha)-\sigma_{(p)}(S_P(\alpha))}{p-1}\geq\alpha;\ \frac{S_p(\alpha)-1-\sigma_{(p)}(S_p(\alpha)-1)}{p-1}<\alpha.$$

Using (6) we obtain that $S_p(\alpha)$ is the unique solution of the system

$$\sigma_{(p)}(x)\leq\sigma_{[p]}(\alpha)\leq\sigma_{(p)}(x-1)+1 \tag{11}$$

2. Connections with classical numerical functions

It is said that Riemann's zeta function is

$$\zeta(s)=\sum_{n\geq 1}\frac{1}{n^s}.$$

We may establish a connection between the function S_p and Riemann's function as follows:

Proposition 2.1. *If $n=\prod_{i=1}^{t_n}p_i^{\alpha_{i_n}}$ is the dcomposition into primes of the pozitive integer n then*

$$\frac{\zeta(s-1)}{\zeta(s)}=\sum_{n\geq 1}\prod_{i=1}^{t_n}\frac{S_{p_i}(p_i^{\alpha_{i_n}-1})-p_i}{p_i^{s\alpha_{i_n}}}$$

Proof. We firs establish a connection with Euler's totient function φ. Let us observe that, for $\alpha\geq 2, p^{\alpha-1}=(p-1)a_{\alpha-1}(p)+1$, so $\sigma_{[p]}(p^{\alpha-1})=p$. Then by means of (6) it results (for $\alpha\geq 2$) that

$$S_p(p^{\alpha-1})=(p-1)p^{\alpha-1}+\sigma_{[p]}(p^{\alpha-1})=\varphi(p^\alpha)+p$$

Using the well known relation between φ and ζ given by

$$\frac{\zeta(s-1)}{\zeta(s)}=\sum_{n\geq 1}\frac{\varphi(n)}{n^n}$$

and (12) it results the required relation.

Let us remark also that, if n is given by (1), then

$$\varphi(n)=\prod_{i=1}^{t}\varphi(p_i^{\alpha_i})=\prod_{i=1}^{t}(S_{p_i}(p_i^{\alpha_i-1})-p_i)$$

and

$$S(n)=\max(\varphi(p^{\alpha_i+1})+p_i)$$

Now it is said that $1+\varphi(p_i)+\ldots+\varphi(p_i^{\alpha_i})=p_i^{\alpha_i}$ and then

$$\sum_{k=1}^{\alpha_i-1} Sp_i(p_i^k)-(\alpha_i-1)p_i=p_i^{\alpha_i}.$$

Consequently we may write

$$S(n)=\max(S\sum_{k=0}^{\alpha_i-1} Sp_i(p_i^k)-(\alpha_i-1)p_i)$$

To establish a connection with Mangolt's function let us note $\bigwedge=\min, \bigvee=\max$, $\underset{d}{\bigwedge}$ =the greatest common divisor and $\overset{d}{\bigvee}$=the smallest common multiple.

We shall write also $n_1\underset{d}{\bigwedge}n_2=(n_1,n_2)$ and $n_1\overset{d}{\bigvee}n_2=[n_1,n_2]$.

The Smarandache function S may be regarded as function from the lattice $\mathcal{L}_d=(N^*,\underset{d}{\bigwedge},\overset{d}{\bigvee})$, into lattice $\mathcal{L}=(N^*,\bigwedge,\bigvee)$ so that

$$S(\bigvee_{i=\overline{1,k}} n_i)=\bigvee_{i=\overline{1,k}} S(n_i). \tag{12}$$

Of course S is also order preserving in the sense that $n_1\leq_d n_2\rightarrow S(n_1)<S(n_2)$.

It is said [10] that if $(V,\bigwedge,\bigvee)$ is a finite lattice, $V=\{x_1,x_2,\ldots,x_n\}$ with the induced order $\leq$, then for every function $f:V\rightarrow N$ the asociated generating function is defined by

$$F(x)=\sum_{y\leq x} f(y) \tag{13}$$

Magolt's function Λ is

$$\Lambda(n)=\begin{cases}\ln p \text{ if } n=p^i\\ 0 \text{ otherwise}\end{cases}$$

The generating function of Λ in the lattice $\mathcal{L}_d$ is

$$F^d(n)=\sum_{k\leq_d n}\Lambda(k)=ln\ n \tag{14}$$

The last equality follows from the fact that

$$k\leq_d n\Leftrightarrow k\underset{d}{\bigwedge}n=k\Leftrightarrow k\backslash n(k \text{ divides } n)$$

The generating function of Λ in the lattice $\mathcal{L}$ is the function Ψ

$$F(n)=\sum_{k\leq n}\Lambda(k)=\Psi_{(n)}=\ln[1,2,\ldots,n] \tag{15}$$

Then we have the diagram from below.

We observe that the definition of S is in a closed connection with the equalities (1.1) and (2.2) in this diagram. If we note the Mangolt's function by f then the relations

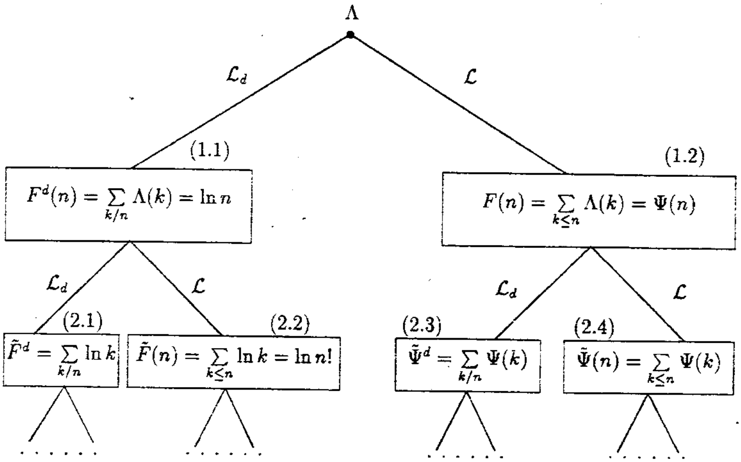
$$[1, 2, \ldots, n] = e^{F(n)} = e^{f(1)} e^{f(2)} \cdots e^{f(n)} = e^{\Psi(n)}$$

$$n! = e^{\tilde{F}} = e^{F^d(1)} e^{F^d(2)} \cdots e^{F^d(n)}$$

together with the definition of S suggest us to consider numerical functions of the from:

$$\nu(n) = \min\{m / n \leq_d [1, 2, \ldots, m]\} \qquad (16)$$

where will be detailed in section 5.



3. The Smarandache function as generating function

Let V be a partitial order set. A function $f: V \to N$ may be obtained from its generating function F, defined as in (15), by the inversion formula

$$f(x) = \sum_{z \leq x} F(z)\mu(z, x) \tag{17}$$

where μ is Moebius function on V, that is $\mu : VXV \to N$ satisfies:

$$(\mu_1)\ \mu(x, y) = 0 \qquad \text{if } x \not\leq y$$

$$(\mu_2)\ \mu(x, x) = 1$$

$$(\mu_3)\ \sum_{x \leq y \leq z} \mu(x, y) = 0 \quad \text{if } x < z$$

It is said [10] that if $V = \{1, 2, \ldots, n\}$ then for $(V, \leq_d)$ we have $\mu(x, y) = \mu\left(\frac{y}{x}\right)$, where $\mu(k)$ is the numerical Meobius function $\mu(1) = 1, \mu(k) = (-1)^i$ if $k = p_1 p_2 \ldots p_k$ and $\mu(k) = 0$ if k is divisible by the square of an integer $d > 1$.

If f is the Smarandache function it results

$$F_S(n) = \sum_{d/n} S(n).$$

Until now it is not known a closed formula for F_S, but in[8] it is proved that

$(i) F_S(n) = n$ if and only if n is prime, $n = 9, n = 16$ or $n = 24$.

$(ii) F_S(n) > n$ if and only if $n \in \{8, 12, 18, 20\}$ or $n = 2p$ with p a prime (hence it results $F_S(n) \leq n + 4$ for every pozitive integer n) and in [2] it is showed that

$(iii) F(p_1 p_2 \ldots p_t) = \sum_{i=1}^{t} 2^{i-1} p_i$.

In this section we shall regard the Smarandache function as a generating function that is using the inversion formula we shall construct the function s so that

$$s(n) = \sum_{d/n} \mu(d) S\left(\frac{n}{d}\right). \tag{18}$$

If n is given by (1) it results that

$$s(n) = \sum_{p_{i_1} p_{i_2} \cdots p_{i_r}} (-1)^r S\left(\frac{n}{p_{i_1} p_{i_2} \ldots p_{i_r}}\right).$$

Let us consider $S(n) = \max S(p_i^{\alpha_i}) = S(p_{i_0}^{\alpha_{i_0}})$. We distinguish the following cases:

(a_1) if $S(p_{i_0}^{\alpha_{i_0}}) \geq S(p_i^{\alpha_i})$ foe all $i \neq i_0$ then we observe that the divisors d for which $\mu(d) \neq 0$ are of the form $d = 1$ or $d = p_{i_1} p_{i_2} \dots p_{i_r}$. A divisor of the last form may contain p_{i_0} or not, so using (2) it results

$$s(n) = S(p_{i_0}^{\alpha_{i_0}})(1-C_{t-1}^1+C_{t-1}^2+\ldots+(-1)^{t-1}C_{t-1}^{t-1})+S(p_{i_0}^{\alpha_{i_0}-1})(-1+C_{t-1}^1-C_{t-1}^2+\ldots+(-1)^t C_{t-1}^{t-1})$$

that is $s(n) = 0$ if $t \geq 2$ or $S(p_{i_0}^{\alpha_{i_0}-1})$ and $s(n) = p_{i_0}$ otherwise.

(a_2) if there exists, j_0 so that $S(p_{i_0}^{\alpha_{i_0}-1}) < S(p_{j_0}^{\alpha_{j_0}})$ and

$$S(p_{j_0}^{\alpha_{j_0}-1}) \geq S(p_i^{\alpha_i}) \text{ for } i \neq i_0, j_0$$

we also suppose that $S(p_{j_0}^{\alpha_{j_0}}) = \max\{S(p_j^{\alpha_j}) / S(p_{i_0}^{\alpha_{i_0}-1}) < S(p_j^{\alpha_j})\}$.

Then

$$\begin{aligned} s(n) &= S(p_{i_0}^{\alpha_{i_0}})(1 - C_{t-1}^1 + C_{t-1}^2 - \ldots + (-1)^{t-1}C_{t-1}^{t-1}) + \\ &+ S(p_{j_0}^{\alpha_{j_0}})(-1 + C_{t-2}^1 - C_{t-2}^2 - \ldots + (-1)^{t-1}C_{t-2}^{t-2}) + \\ &+ S(p_{j_0}^{\alpha_{j_0}-1})(1 - C_{t-2}^1 + C_{t-2}^2 - \ldots + (-1)^{t-2}C_{t-2}^{t-2}) \end{aligned}$$

so $s(n) = 0$ if $t \geq 3$ or $S(p_{j_0}^{\alpha_{j_0}-1}) = S(p_{j_0}^{\alpha_{j_0}})$ and $s(n) = -p_{j_0}$ otherwise.

Consequently, to obtain $s(n)$ we construct as above a maximal sequence $i_1, i_2, \ldots, i_k$, so that $S(n) = S(p_{i_1}^{\alpha_{i_1}}), S(p_{i_1}^{\alpha_{i_1}-1}) < S(p_{i_2}^{\alpha_{i_2}}), \ldots, S(p_{i_{k-1}}^{\alpha_{i_{k-1}}-1}) < S(p_{i_k}^{\alpha_{i_k}})$ and it results that $s(n) = 0$ if $t \geq k+1$ or $S(p_{i_k}^{\alpha_{i_k}}) = S(p_{i_k}^{\alpha_{i_k}-1})$ and $s(n) = (-1)^{k+1}$ otherwise.

Let us observe that

$$S(p^\alpha) = S(p^{\alpha-1}) \Leftrightarrow (p-1)\alpha + \sigma_{[p]}(\alpha) = (p-1)(\alpha-1) + \sigma_{[p]}(\alpha-1) \Leftrightarrow \sigma_{[p]}(\alpha-1) - \sigma_{[p]}(\alpha) = p-1.$$

Otherwise we have $\sigma_{[p]}(\alpha - 1) - \sigma_{[p]}(\alpha) = -1$. So we may write

$$s(n) = \begin{cases} o \text{ if } t \geq k+1 \text{ or } \sigma_{[p]}(\alpha_k - 1) - \sigma_{[p]}(\alpha_k) = p - 1 \\ (-1)^{k+1} p_k \text{ otherwise} \end{cases}$$

Application. It is said [10] that $(V, \wedge, \vee)$ is a finit lattice, with the indused order $\leq$ and for the function $f : V \to N$ we consider the generating function F defined as in (15) then if $g_{ij} = F(x_i \wedge x_j)$ it results $\det g_{ij} = f(x_1) \cdot f(x_2) \cdot \ldots \cdot f(x_n)$. In [10] it is shown also that this assertion may be generalized for partial ordered set by defining

$$g_{ij} = \sum_{\substack{x \leq x_i \\ x \leq x_j}} f(x).$$

Using these results, if we denote by (i,j) the greatest common divisor of i and j, and $\Delta(r) = \det(S((i,j)))$ for $i,j = \overline{1,r}$ then $\Delta(r) = s(1)\cdot s(2)\cdot\ldots\cdot s(r)$. That is for a suffisient large r we have $\Delta(r) = 0$ (in fact for $r \geq 8$). Moreover, for every n there exists a sufficient large r so that $\Delta(n,r) = \det(S(n+i,n+j)) = 0$, for $i,j = \overline{1,r}$ because $\Delta(n,r) = \prod_{i=1}^{n} S(n+1)$.

4. The extension of S to the rational numbers

To obtain this extension we shalll define first a dual function of the Smarandache function.

In [4] and [6] a duality principale is used to obtain, starting from a given lattice on the unit interval, other lattices on the same set. The results are used to propose a definition of bitopological spaces and to introduce a new point of view for studying the fuzzy sets. In [5] the method to obtain news lattices on the unit interval is generalised for an arbitrary lattice.

In the following we adopt a method from [5] to construct all the functions tied in a certain sense by duality to the Smarandache function.

Let us observe that if we note $\Re_d(n) = \{m/n \leq_d m!\}$, $\mathcal{L}_d(n) = \{m/m! \leq_d n\}, \Re(n) = \{m/n \leq m!\}, \mathcal{L}(n) = \{m/m! \leq n\}$ then we may say that the function S is defined by the triplet $(\bigwedge, \in, \Re_d)$, because $S(n) = \bigwedge\{m/m \in \Re_d(n)\}$. Now we may investigate all the functions defined by means of a triplet (a,b,c), where a is one of the symbols $\bigvee, \bigwedge, \overset{d}{\bigwedge}, \underset{d}{\bigvee}$, b is one of the symbols $\in$ and $\notin$, and c is one of the sets $\Re_d(n), \mathcal{L}_d(n), \Re(n), \mathcal{L}(n)$ defined above.

Not all of these functions are non-trivial. As we have already seen the triplet $(\bigwedge, \in, \Re_d)$ defined the function $S_1(n) = S(n)$, but the thriplet $(\bigwedge, \in, \mathcal{L}_d)$ defines the function $S_2(n) = \bigwedge\{m/m! \leq_d n\}$, wich is identically one.

Many of the functions obtained by this method are step functions. For instance let S_3 be the function defined by $(\bigwedge, \in, \Re)$. We have $S_3(n) = \bigwedge\{m/n \leq m!\}$ so $S_3(n) = m$ if only if $n \in [(m-1)!+1, m!]$. Let us focus the attention on the function defined by $(\bigwedge, \in, \mathcal{L}_d)$

$$S_4(4) = \bigvee\{m/m! \leq_d n\} \tag{19}$$

where there is, in a certain sense, the dual of Smarandache function.

Proposition 4.1. *The function S_4 satisfies*

$$S_4(n_1 \underset{d}{\bigvee} n_2) = S_4(n_1) \bigvee S_4(n_2) \tag{20}$$

so is a morphism from $(\mathbf{N}^*, \underset{d}{\bigvee})$ *to* $(\mathbf{N}^*, \bigvee)$

Proof. Let us denote by $p_1, p_2, \ldots, p_i, \ldots$ the sequence of the prime numbers and let

$$n_1 = \prod p_i^{\alpha_i},\ n_2 = \prod p_i^{\beta_i}.$$

The $n_1 \bigwedge_d n_2 = \prod p_i^{\min(\alpha_i,\beta_i)}$. If $S_4(n_1 \bigvee_d n_2) = m, S_4(n_i) = m_i$, for $i = 1,2$ and we suppose $m_1 \leq m_2$ then the right hand in (22) is $m_1 \bigwedge m_2 = m$. By the definition S_4 we have $E_{p_i}(m) \leq \min(\alpha_i, \beta_i)$ for $i \geq 1$ and there exists j so that $E_{p_i}(m+1) > \min(\alpha_i, \beta_i)$. Then $\alpha_i > E_{p_i}(m)$ and $\beta_i \geq E_{p_i}(m)$ for all $i \geq 1$. We also wave $E_{p_i}(m_r) \leq \alpha_i$ for $r = 1,2$. In addition there exist h and k so that $E_{p_h}(m+1) > \alpha_h, e_{p_j}(m+1) > \alpha_k$.

Then $\min(\alpha_i, \beta_i) \geq \min(\varepsilon_{p_1}(m_1), \varepsilon_{p_i}(m_2)) = E_{p_i}(m_1)$, because $m_1 \leq m_2$, so $m - 1 \leq m$. If we assume $m_1 < m$ it results that $m! \leq n_1$, so it exists h that $E_{p_h}(m) > \alpha_h$ and we have the contradiction $E_{p_h}(m) > \min\{\alpha_h, \beta_h\}$. Of course $S_4(2n+1) = 1$ and

$$S_4(n) > 1 \text{ if and only if } n \text{ is even.} \tag{21}$$

Proposition 4.2. *Let $p_1, p_2, \ldots, p_i, \ldots$ be the sequence of all consecutive primes and*

$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_k^{\alpha_k} \cdot q_1^{\beta_1} \cdot q_2 \beta_2 \cdot \ldots \cdot q_r^{\beta_r}$$

the decomposition of $n \in N^*$ inte primes such that the first part of the decomposition contains the (eventualy) consecutive primes, and let

$$t_i = \begin{cases} S(p_i^{\alpha_i}) - 1 \text{ if } E_{p_i}(S(p_i^{\alpha_i})) > \alpha_i \\ S(p_i^{\alpha_i}) + p_i - 1 \text{ if } E_{p_i}(S(p_i^{\alpha_i})) = \alpha_i \end{cases} \tag{22}$$

then $S_n(n) = \min\{t_1, t_2, \ldots, t_k, p_{k+1} - 1\}$.

Proof. If $E_{p_i}(S(p_i^{\alpha_i})) > \alpha_i$, then from the definition of the function S results that $S(p_i^{\alpha_i}) - 1$ is the greatest positive integer m much than $E_{p_i}(m) \leq \alpha_i$. Also if $E_{p_i}(S(p_i^{\alpha_i})) = \alpha_i$ then $S(p_i^{\alpha_i}) + p_i - 1$ is the greatest integer m with the property that $E_{p_i}(m) = \alpha_i$.

It results that $\min\{t_1, t_2, \ldots, t_k, p_{k+1} - 1\}$ is the greatest integer m much that $E_{p-i}(m!) \leq \alpha_i$, for $i = 1, 2, \ldots, k$.

Proposition 4.3. *The function S_4 satisfies*

$$S_4((n_1 + n_2)) \bigwedge S_4([n_1, n_2]) = S_4(n_1) \bigwedge S_4(n_2)$$

for all positive integers n_1 and n_2.

Proof. The equality results using (22) from the fact that $(n_1+n_2,[n_1,n_2])=(n_1,n_2))$.

We point out now some morphism properties of the functions defined bu a triplet (a,b,c) as above.

Proposition 4.4. *(i) The functions* $S_5:\mathbf{N}^*\to\mathbf{N}^*, S_5(n)=\overset{d}{\bigvee}\{m/m!\le_d n\}$ *satisfies*

$$S_5(n_1\bigwedge_d n_2)=S_5(n_1)\bigwedge_d S_5(n_2)=S_5(n_1)\bigwedge S_5(n_2) \tag{23}$$

(ii) The function $\mathbf{S}_6:\mathbf{N}^*\to\mathbf{N}^*$, $S_6(n)=\overset{d}{\bigvee}\{m/n\le_d m!\}$ *satisfies*

$$S_6(n_1\overset{d}{\bigvee}n_2)=S_6(n_1)\overset{d}{\bigvee}S_6(n_2) \tag{24}$$

(iii) The function $S_7:\mathbf{N}^*\to\mathbf{N}^*, S_7(n)=\overset{d}{\bigvee}\{m/m!\le n\}$ *satisfies*

$$S_7(n_1\bigwedge n_2)=S_7(n_1)\bigwedge S_7(n_2); S_7(n_1\bigvee n_2)=S_7(n_1)\bigvee S_7(n_2). \tag{25}$$

Proof. (i) Let $A=\{a_i/a_i!\le_d n_1\}, B=\{b_j/b_j!\le_d n_2\}$ and $C=\{c_k/c_k!\le_d n_1\bigvee_d n_2\}$. Then we have $A\subset B$ or $B\subset A$. Indeed, let $A=\{a_1,a_2,\dots,a_h\}, B=\{b_1,b_2,\dots,b_r\}$ so that $a_i<a_{i+1}$and $b_j<b_{j+1}$. Then if $a_h\le b_r$ it results that $a_i\le b_r$ for $i=\overline{1,h}$ so $a_i!\le_d b_r!\le_d n_2$. That minds $A\subset B$. Analogously, if $b_r\le a_h$ it results $B\subset A$. Of course we have $C=A\dot{\cup}B$ so if $A\subset B$ it results

$$S_5(n_1\bigwedge_d n_2)=\overset{d}{\bigvee}c_k=\overset{d}{\bigvee}a_i=S_5(n_1)=\min\{S_5(n_1),S_5(n_2)\}=S_5(n_1)\bigwedge_d S_5(n_2)$$

From (25) it results that S_5 is order preserving in $\mathcal{L}_d$ (but not in $\mathcal{L}$, becuase $m!<m!+1$ but $S_5(m!)=[1,2,\dots,m]$ and $S_5(m!+1)=1$, because $m!+1$ is odd).

(ii) Let us observe that $S_6(n)=\overset{d}{\bigvee}\{m/\exists i\in\overline{1,t}$ so that $E_{p_i}(m)<\alpha_i\}$. If $a=\bigvee\{m/n\le_d m!\}$ then $n\le_d(a+1)!$ and $a+1=\bigwedge\{m/n\le_d m!\}=S(n)$, so $S_6(n)=[1,2,\dots,S(n)-1]$.

Then we have $S_6(n_1\overset{d}{\bigvee}n_2)=[1,2,\dots,S(n_1\overset{d}{\bigvee}n_2)-1]=[1,,2\dots,S(n_1)\bigvee S(n_2)-1]$ and $S_6(n_1)\overset{d}{\bigvee}S_6(n_2)=[[1,2,\dots,S_6(n_1)-1],[1,2,\dots,S_6(n_2)-1]]=[1,2,\dots,S_6(n_1)\bigvee S_6(n_2)-1]$.

(iii) The relations (27) result from the fact that $S_7(n)=[1,2,\dots,m]$ if and only if $n\in[m!,(m+1)!-1]$.

Now we may extend the Smarandache function to the rational numbers. Every positive rational number a possesses a unique prime decomposition of the form

$$a=\prod_p p^{\alpha_p} \tag{26}$$

with integer exponents α_p, of which only finitely many are nonzero. Multiplication of rational numbers is reduced to addition of their integer exponent systems. As a consequence of this reduction questions concerning divisibility of rational numbers are reduced to questions concerning ordering of the corresponding exponent systems. That is if $b = \prod_p P^{\beta_p}$ then b divides a if and only if $\beta_p \leq \alpha_p$ for all p. The greatest common divisors d and the least common multiple e are given by

$$d = (a, b, \ldots) = \prod_p p^{\min(\alpha_p, \beta_p, \ldots)}, \quad e = [a, b, \ldots] = \prod_p p^{\max(\alpha_p, \beta_p, \ldots)} \tag{27}$$

Futhermore, the least coomon multiple of nonzero numbers (multiplicatively bounded above) is reduced by the rule

$$[a, b, \ldots] = \frac{1}{\left(\frac{1}{a}, \frac{1}{b}, \ldots\right)} \tag{28}$$

to the greatest common divisor of their reciprocals (multiplicatively bounded below).

Of course we may write every positive rational a under the form $a = n/n_1$, with n and n_1 positive integers.

Definition 4.5. *The extencion $S : Q_+^* \to Q_+^*$ of the Smarandache function is defined by*

$$S\left(\frac{n}{n_1}\right) = \frac{S_1(n)}{S_4(n_1)}. \tag{29}$$

A consequence of this definition is that if n_1 and n_2 are positive integers then

$$S\left(\frac{1}{n_1} \overset{d}{\bigvee} \frac{1}{n_2}\right) = S\left(\frac{1}{n_1}\right) \bigvee S\left(\frac{1}{n_2}\right). \tag{30}$$

Indeed

$$S\left(\frac{1}{n_1} \overset{d}{\bigvee} \frac{1}{n_2}\right) = S\left(\frac{1}{n_1 \underset{d}{\bigwedge} n_2}\right) = \frac{1}{S_4(n_1 \underset{d}{\bigwedge} n_2)} = \frac{1}{S_4(n_1) \bigwedge S_4(n_2)} = \frac{1}{S_4(n_1)} \bigvee \frac{1}{S_4(n_2)} =$$

$$= S\left(\frac{1}{n}\right) \bigvee S\left(\frac{1}{n_2}\right)$$

and we can imediately deduce that

$$S\left(\frac{n}{n_1} \overset{d}{\bigvee} \frac{m}{m_1}\right) = (S(n) \bigvee S(m)) \cdot \left(S\left(\frac{1}{n_1}\right) \bigvee S\left(\frac{1}{m_1}\right)\right) \tag{31}$$

It results that function $\tilde{S}$ defined by $\tilde{S}(a) = \dfrac{1}{S(\frac{1}{a})}$ satisfies

$$\tilde{S}(n_1 \bigwedge_d n_2) = \tilde{S}(n_1) \bigwedge \tilde{S}(n_2) \text{ and}$$

$$\tilde{S}\left(\frac{1}{n_1} \bigwedge_d \frac{1}{n_2}\right) = \tilde{S}\left(\frac{1}{n_1}\right) \bigwedge \tilde{S}\left(\frac{1}{n_2}\right) \tag{32}$$

for every positive integers n_1 and n_2. Moreover, it results that

$$\tilde{S}\left(\frac{n_1}{m_1} \bigwedge_d \frac{n_2}{m_2}\right) = (\tilde{S}(n_1) \bigwedge \tilde{S}(n_2)) \cdot \left(\tilde{S}\left(\frac{1}{m_1}\right) \bigwedge \tilde{S}\left(\frac{1}{m_2}\right)\right)$$

and of course the restriction of $\tilde{S}$ to the positive integers is S_4. The extention of S to all the rationals is given by $S(-a) = S(a)$.

5. Numerical functions inspired from the definition of the Smarandache function

We shall use now the equality (21) and the relation (18) to consider numerical functions as the Smarandache function.

We may say that $m!$ is the product of all positive "smaller" than m in the lattice $\mathcal{L}$. Analogously the product p_m of all the divisors of m is the product of all the elements "smaller" than m in the lattice $\mathcal{L}$. So we may consider functions of the form

$$\Theta(n) = \bigwedge\{m | n \geq_d p(m)\}. \tag{33}$$

It is said that if $m = p_1^{x_1} \cdot p_2^{x_2} \cdot \ldots \cdot p_t^{x_t}$ then the product of all the divisors of m is $p(m) = \sqrt{m^{\tau(m)}}$ where $\tau(m) = (x_1+1)(x_2+1)\ldots(x_t+1)$ is the number of all the divisors of m.

If n is given as in (1) then $n \geq_d p(m)$ id and only if

$$\begin{aligned} g_1 &= x_1(x_1+1)(x_2+1)\ldots(x_t+1) - 2\alpha_1 \geq 0 \\ g_2 &= x_2(x_1+1)(x_2+1)\ldots(x_t+1) - 2\alpha_2 \geq 0 \\ g_t &= x_t(x_1+1)(x_2+1)\ldots(x_t+1) - 2\alpha_t \geq 0 \end{aligned} \tag{34}$$

so $\Theta(n)$ may be obtained solving the problem of non linear programming

$$(\min) f = p_1^{x_1} \cdot p_2^{x_2} \cdot \ldots \cdot p_t^{x_t} \tag{35}$$

under the restrictions (37).

The solutions of this problem may be obtained applying the algorithm SUMT (Sequencial Unconstrained Minimization Techniques) due to Fiacco and Mc Cormick [7].

Examples

1. For $n = 3^4 \cdot 5^{12}$, (37) and (38) become $(\min) f(x) = 3^{x_1} 5^{x_2}$ with $x_1(x_1+1)(x_2+1 \geq 8)$, $x_2(x_1+1)(x_2+1) \geq 24$. Considering the function $U(x,n) = f(x) - r \sum_{i=1}^{k} \ln g_1(x)$, and the system

$$\sigma U / \sigma x_1 = 0, \ \sigma U / \sigma x_2 = 0 \tag{36}$$

in [7] it is showed that if the solution $x_1(r), x_2(r)$ can't be explained from the system we can make $r \to 0$. Then the system becomes $x_1(x_1+1)(x_2+1) = 8$, $x_2(x_1+1)(x_2+1) = 24$ with the (real) solution $x_1 = 1, x_2 = 3$.

So we have $\min\{m / 3^4 \cdot 5^{12} \leq \rho(m)\} = m_0 = 3 \cdot 5^3$.

Indeed $\rho(m_0) = m_0^{\tau(m_0)/2} = m_0^4 = 3^4 \cdot 5^{12} = n$.

2. For $n = 3^2 \cdot 567$, from the system (39) it results for x_2 the equation $2x_2^3 + 9x_2^2 + 7x_2 - 98 = 0$, with th real solution $x_2 \in (2,3)$. It results $x_1 \in (4/6, 5/7)$. Considering $x_1 = 1$, we observe that for $x_2 = 2$ the pair (x_1, x_2) is not an admissible solution of the problem, but $x_2 = 3$ give $\Theta(3^2 \cdot 5^7) = 3^4 \cdot 5^{12}$.

3. Generaly for $n = p_1^{\alpha_1} \cdot p_2^{\alpha_3}$, from the system (39) it results the equation

$$\alpha_1 x_2^3 + (\alpha_1 + \alpha_2) \cdot x_2^2 + \alpha_2 x_2 - 2\alpha_2^2 = 0$$

with solutions given by Cartan's formula.

Of course, using "the method of the triplets", as for the Smarandache function, many other functions may be associated to Θ.

For the function ν given by (18) it is also possible to generate a class of function by means of such triplets.

In the sequel we'll focus the attention on the analogous of the Smarandache function and on his dual in this case.

Proposition 5.1. *If n has the decomposition into primes given by (1) then*

(i) $\nu(n) = \max_{i=1,t} p_i^{\alpha_i}$

(ii) $\nu(n_1 \overset{d}{\bigvee} n_2) = \nu(n_1) \bigvee \nu(n_2)$

Proof.

(i) Let $\max p_i^{\alpha_i} = p_u^{\alpha_u}$. Then $p_i^{\alpha_i} \leq p_u^{\alpha_u}$ for all $\overline{1,t}$, so $p_i^{\alpha_i} \leq_d [1, 2, \ldots p_u^{\alpha_u}]$. But $(p_i^{\alpha_i}, p_j^{\alpha_j}) = 1$ for $i \neq j$ and then $n \leq_d [1, 2, \ldots p_u^{\alpha_u}]$.

Now if for some $m < p_u^{\alpha_u}$ we have $n \leq_d [1, 2, \ldots, m]$, it results the contradiction $p_u^{\alpha_u} \leq_d [1, 2, \ldots, m]$.

(ii) If $n_1 = \prod p^{\alpha_p}$, $n_2 = \prod p^{\beta_p}$ then $n_1 \overset{d}{\vee} n_2 = \prod p^{\max(\alpha_p \beta_p)}$ so

$$\nu(n_1 \overset{d}{\bigvee} n_2) = \max p^{\max(\alpha_p, \beta_p)} = \max(\max p^{\alpha_p}, \max p^{\beta_p}).$$

The function $\nu_1 = \nu$ is defined by means of the triplet $(\vee, \in, \Re_{[d]})$ where $\mathbf{R}_{[d]} = \{m/n \leq_d [1, 2, \ldots, m]\}$. His dual, in the sense of above section, is the function defined by the triplet $(\vee, \in, \mathcal{L}_{[d]})$. Let us note ν_4 this function

$$\nu_4(n) = \bigvee\{m | [1, 2, \ldots, m] \leq_d n\}$$

That is $\nu_4(n)$ is the greatest natural number with the property that all $m \leq \nu_4(n)$ divide n.

Let us observe that necessary and sufficient condition to have $\nu_4(n) > 1$ is to exist $m > 1$ so that every prime $p \leq m$ devides n. From the definition of ν_4 it also results that $\nu_4(n) = m$ if and only if n is divisible by every $i \leq n$ and not by $m + 1$.

Proposition 5.2. *The function ν_4 satisfies*

$$\nu_4(n_1 \overset{d}{\bigvee} n_2) = \nu_4(n_1) \bigwedge \nu_4(n_2)$$

Proof. Let us note $n = n_1 \overset{d}{\wedge} n_2$, $\nu_4(n) = m$, $\nu_4(n_i) = m_i$ for $i = 1, 2$. If $m_1 = m_1 \wedge m_2$ than we prove that $m = m_1$. From the definition of ν_4 it results

$$\nu_4(n_i) = m_i \leftrightarrow [\forall i \leq m_i \rightarrow n \text{ is divisible by } i \text{ but not by } m + 1]$$

If $m < m_1$ then $m + 1 \leq m_1 \leq m$ so $m + 1$ divides n_1 and n_2. That is $m + 1$ divides n.

If $m > m_1$ then $m_1 + 1 \leq n$, so $m_1 + 1$ divides n. But n divides n_1, so $m_1 + 1$ divides n_1.

If $t_0 = \max\{i | j \leq i \Rightarrow n \text{ divides } n\}$ then $\nu_4(n)$ may be obtained solving the integer linear programming problem

$$(\max) f = \sum_{i=1}^{t_0} x_i \ln p$$
$$x_i \leq \alpha_i \; for \; i = \overline{1, t_0} \tag{37}$$

$$\sum_{i=1}^{t_0} x_i \ln p_i \leq \ln p_{t_0+1}.$$

If f_0 is the maximal value of f for above problem, then $\nu_4(n) = e^{f_0}$.

For instance $\nu_4(2^3 \cdot 3^2 \cdot 5 \cdot 11) = 6$.

Of course, the function ν may be extinded to the rational numbers in the same way as Smarandache function.

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