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A Generalization of the Inequality of Hölder

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One generalizes the inequality of Hödler thanks to a reasoning by recurrence. As particular cases, one obtains a generalization of the inequality of Cauchy-Buniakovski-Scwartz, and some interesting applications.

Theorem: If $a_i^{(k)} \in R_+$ and $p_k \in]1,+\infty[$, $i \in \{1,2,...,n\}$, $k \in \{1,2,...,m\}$, such that:, $\frac{1}{p_1}+\frac{1}{p_2}+...+\frac{1}{p_m}=1$, then:

$$\sum_{i=1}^{n}\prod_{k=1}^{m} a_i^{(k)} \le \prod_{k=1}^{m} \left(\sum_{i=1}^{n}\left(a_i^{(k)}\right)^{p_k}\right)^{\frac{1}{p_k}} \text{ with } m \ge 2.$$

Proof:

For $m=2$ one obtains exactly the inequality of Hödler, which is true. One supposes that the inequality is true for the values which are strictly smaller than a certain m .

Then:,

$$\sum_{i=1}^{n}\prod_{k=1}^{m} a_i^{(k)} = \sum_{i=1}^{n}\left(\left(\prod_{k=1}^{m-2} a_i^{k}\right)\cdot\left(a_i^{(m-1)}\cdot a_i^{(m)}\right)\right) \le \left(\prod_{k=1}^{m-2}\left(\sum_{i=1}^{n}\left(a_i^{(k)}\right)^{p_k}\right)^{\frac{1}{p_k}}\right)\cdot\left(\sum_{i=1}^{n}\left(a_i^{(m-1)}\cdot a_i^{(m)}\right)^{p}\right)^{\frac{1}{p}}$$

where $\frac{1}{p_1}+\frac{1}{p_2}+...+\frac{1}{p_{m-2}}+\frac{1}{p}=1$ and $p_h>1$, $1\le h\le m-2$, $p\ >1$;

but

$$\sum_{i=1}^{n}\left(a_i^{(m-1)}\right)^{p}\cdot\left(a_i^{(m)}\right)^{p} \le \left(\sum_{i=1}^{n}\left(\left(a_i^{(m-1)}\right)^{p}\right)^{t_1}\right)^{\frac{1}{t_1}}\cdot\left(\sum_{i=1}^{n}\left(\left(a_i^{(m)}\right)^{p}\right)^{t_2}\right)^{\frac{1}{t_2}}$$

where $\frac{1}{t_1}+\frac{1}{t_2}=1$ and $t_1>1$, $t_2>2$.

From it results that:

$$\sum_{i=1}^{n}\left(a_i^{(m-1)}\right)^{p}\cdot\left(a_i^{(m)}\right)^{p} \le \left(\sum_{i=1}^{n}\left(a_i^{(m-1)}\right)^{pt_1}\right)^{\frac{1}{pt_1}}\cdot\left(\sum_{i=1}^{n}\left(a_i^{(m)}\right)^{pt_2}\right)^{\frac{1}{pt_2}}$$

with $\frac{1}{pt_1}+\frac{1}{pt_2}=\frac{1}{p}$.

Let us note $pt_1 = p_{m-1}$ and $pt_2 = p_m$. Then $\frac{1}{p_1} + \frac{1}{p_2} + ... + \frac{1}{p_m} = 1$ is true and one has $p_j > 1$ for $1 \le j \le m$ and it results the inequality from the theorem.

Note: If one poses $p_j = m$ for $1 \le j \le m$ and if one raises to the power m this inequality, one obtains a generalization of the inequality of Cauchy-Buniakovski-Scwartz:

$$\left(\sum_{i=1}^{n} \prod_{k=1}^{m} a_i^{(k)} \right)^m \le \prod_{k=1}^{m} \sum_{i=1}^{n} \left(a_i^{(k)} \right)^m .$$

Application:

Let $a_1, a_2, b_1, b_2, c_1, c_2$ be positive real numbers.

Show that:

$$(a_1 b_1 c_1 + a_2 b_2 c_2)^6 \le 8(a_1^6 + a_2^6)(b_1^6 + b_2^6)(c_1^6 + c_2^6)$$

Solution:

We will use the previous theorem. Let us choose $p_1 = 2$, $p_2 = 3$, $p_3 = 6$; we will obtain the following:

$$a_1 b_1 c_1 + a_2 b_2 c_2 \le (a_1^2 + a_2^2)^{\frac{1}{2}} (b_1^3 + b_2^3)^{\frac{1}{3}} (c_1^6 + c_2^6)^{\frac{1}{6}},$$

or more:

$$(a_1 b_1 c_1 + a_2 b_2 c_2)^6 \le (a_1^2 + a_2^2)^3 (b_1^3 + b_2^3)^2 (c_1^6 + c_2^6),$$

and knowing that

$$(b_1^3 + b_2^3)^2 \le 2(b_1^6 + b_2^6)$$

and that

$$(a_1^2 + a_2^2)^3 = a_1^6 + a_2^6 + 3(a_1^4 a_2^2 + a_1^2 a_2^4) \le 4(a_1^6 + a_2^6)$$

since

$$a_1^4 a_2^2 + a_1^2 a_2^4 \le a_1^6 + a_2^6 \quad \text{(because: } -\left(a_2^2 - a_1^2\right)^2 \left(a_1^2 + a_2^2\right) \le 0 \text{)}$$

it results the exercise which was proposed.