FLORENTIN SMARANDACHE

Amusing Problems

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AMUSING PROBLEMS

1. Calculate the volume of a square.

(Solution: Volume = Area of the Base x Height = Side^2 x 0 = 0! We look at the square as an extreme case of parallelepiped with the height null.)

2. ? x 7 = 2?

(Solution: of course $\frac{2}{7} \times 7 = 2$!)

3. Ten birds are on a fence. A hunter shoots three of them. How many birds remain?

(Answer: **none**, because the three dead birds fell down from the fence and the other seven flew away!)

4. Ten birds are in a meadow. A hunter shoots three of them. How many birds remain?

(Answer: **three birds**, the dead birds, because the others flew away!)

5. Ten birds are in a cage. A hunter shoots three of them. How many birds remain?

(Answer: **ten birds**, dead and alive, because none could get out!)

6. Ten birds are up in the sky. A hunter shoots three of them. How many birds remain?

(Answer: **seven birds**, at last, those who are still flying and those that fell down!)

7. Prove that the equation $X = X + 2$ has two distinct solutions.

(Answer: $X = \pm\infty$!)

8. (Solving Fermat's last theorem) Prove that for any non-null integer n, the equation $X^n + Y^n = Z^n$, $XYZ \neq$, has at least one integer solution!

(Answer: (a) $n \geq 1$. Let $X_k = Y_k = Z_k = 2^k$, $k = 1, 2, 3, \ldots$ All $X_k \in N$, $K \geq 1$. $L = \lim_{k \to \infty} X_{k \in N}$. But $L = \infty \in N$, that is the integer infinite, and $\infty^n + \infty^n = \infty^n$! If n is even, the equation has eight distinct integer solutions: $X = Y = Z = \pm\infty$! Similarly, we take the negative infinite integer: - ∞∈Z]

(b) $n \leq -1$. Clearly there are at last eight distinct integer solutions: $X = Y = Z = \pm\infty$!)

WHERE IS THE ERROR IN THE BELOW DIOPHANTINE EQUATIONS ?

Statement:

(1) To solve in $\mathbb{Z}$ the equation: $14x + 26y = -20$.

"Resolution": The integer general solution is:

$$\begin{cases} x = -26k + 6 \\ y = 14k - 4 \end{cases} \quad (k \in \mathbb{Z})$$

(2) To solve in $\mathbb{Z}$ the equation: $15x - 37y + 12z = 0$.

"Resolution" The integer general solution is:

$$\begin{cases} x = k+4 \\ y = 15k \qquad (k \in \mathbb{Z}) \\ z = 45k-5 \end{cases}$$

(3) To solve in $\mathbb{Z}$ the equation: $3x-6y+5z-10w=0$.

"Resolution" the equation is written: $3(x-2y)+5z-10w=0$.

Since x, y, z, w are integer variables, it results that 3 divides z and that 3 divides w. I. e: $z=3t_1$ $(t_1 \in \mathbb{Z})$ and $w=3t_2$ $(t_2 \in \mathbb{Z})$.

Thus $3(x-2y)+3(5t_1-10t_2)=0$ where $x-2y+5t_1-10t_2=0$.

Then: $\begin{cases} x = 2k_1+5k_2-10k_3 \\ y = k_1 \\ z = 3k_2 \\ w = 3k_3 \end{cases}$ with $(k_1,k_2,k_3 \in \mathbb{Z}^3)$,

constitute the integer general solution of the equation.

Find the error of each "resolution".

SOLUTIONS.

(1) $x=-26k+6$ and $y=14k-4$ ($k \in \mathbb{Z}$) is an integer solution for the equation (because it verifies it), but it is not the general solution, because $x=-7$ and $y=3$ verify the equation, they are a particular integer solution, but:

$\begin{cases} -26k+6=-7 \\ 14k-4=3 \end{cases}$ implies that $k=\frac{1}{2}$ (does not belong to $\mathbb{Z}$).

Thus one cannot obtain this particular from the previous general solution.

The true general solution is: $\begin{cases} x=-13k+6 \\ y=7k-4 \end{cases}$ $(k \in \mathbb{Z})$. (from [1])

(2) In the same way, $x=5$, $y=3$, $z=3$ is a particular solution of the equation, but which cannot be obtained from the "general solution" because:

$$\begin{cases} k+4=5 \Rightarrow k=-1 \\ 15k=3 \Rightarrow k=\frac{1}{5} \\ 45k-5=3 \Rightarrow k=\frac{8}{45} \end{cases},$$

contradictions.

The integer general solution is: $\begin{cases} x=k_1 \\ y=3k_1+12k_2 \\ z=8k_1+37k_2 \end{cases}$ (with $(k_1,k_2) \in \mathbb{Z}^2$, cf. [1]).

(3) The error is that: "3 divides ($5z-10w$)" does not imply that "3 divides z and 3 divides w". If one believes that one loses solutions, then this is true because

$(x,y,z,w)=(-5,0,5,1)$ constitutes a particular integer solution, which cannot be obtained from the "solution" of the statement.

The correct resolution is: $3(x-2y)+5(z-2w)=0$, that is $3p_1+5p_2=0$, with $p_1=x-2y$ in $\mathbb{Z}$, and $p_2=z-2w$ in $\mathbb{Z}$.

It results that: $\begin{cases} p_1=-5k=x-2y \\ p_2=3k=z-2w \end{cases}$ in $\mathbb{Z}$.

From which one obtains the integer general solution:

$$\begin{cases} x=2k_1-5k_2 \\ y=\;\;k_1 \\ z=\qquad 3k_2+2k_3 \\ w=\qquad\qquad k_3 \end{cases} \quad \text{with } (k_1,k_2,k_3)\in\mathbb{Z}^3$$

[1] One can find these solutions using: Florentin SMARANDACHE - "Un algorithme de résolution dans l'ensemble des numbers entiers pour les équations linéaires".

WHERE IS THE ERROR ON THE BELOW INTEGRALS ?

Let the function $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=2\sin x\,\cos x$.

Let us calculate its primitive:

(1) First method.

$\int 2\sin x\,\cos x\,dx = 2\int u\,du = 2\frac{u^2}{2}=u^2=\sin^2 x$, with $u=\sin x$.

One thus has $F_1(x)=\sin^2 x$.

(2) Second method:

$\int 2\sin x\,\cos x\,dx = -2\int \cos x(-\sin x)dx = -2\int v\,dv = -v^2$,

thus $F_2(x)=-\cos^2 x$

(3) Third method:

$\int 2\sin x\,\cos x\,dx = \int \sin 2x\,dx = \frac{1}{2}\int(\sin 2x)\,2dx = \frac{1}{2}\int \sin t\,dt = -\frac{1}{2}\cos t$

thus $F_3(x)=-\frac{1}{2}\cos 2x$.

One thus obtained 3 different primitives of the same function.

How is this possible?

Answer: There is no error! It is known that a function admits an infinity of primitives (if it admits one), which differ only by one constant.

In our example we have:

$F_2(x)=F_1(x)-1$ for any real x, and $F_3(x)=F_1(x)-\frac{1}{2}$ for any real x.

WHERE IS THE ERROR IN THE BELOW REASONING BY RECURRENCE ?

At an admission contest at an University, was given the following problem:

"Find the polynomials $P(x)$ with real coefficients such that $xP(x-1)=(x-3)P(x)$, for all x real."

Some candidates believed that they would be able to show by recurrence that the polynomials of the statement are those which verify the following property: $P(x)=0$ for all natural values.

In fact, they said, if one puts $x=0$ in this relation, it results that $0\cdot P(-1)=-3\cdot P(0)$, therefore $P(0)=0$.

Likewise, with $x=1$, one has: $1\cdot P(0)=-2\cdot P(1)$, therefore $P(1)=0$, etc.

Let's suppose that the property is true for ($n-1$), therefore $P(n-1)=0$, and we are looking to prove it for n:

One has: $n\cdot P(n-1)=(n-3)\cdot P(n)$, and since $P(n-1)=0$, it results that $P(n)=0$.

Where the proof failed?

Answer: If the candidates would have checked for the rank $n=3$, they would have found that: $3\cdot P(2)=0\cdot P(3)$ thus $0=0\cdot P(3)$, which does not imply that $P(3)$, is null: in fact this equality is true for any real $P(3)$.

The error, therefore, is created by the fact that the implication: "$(n-3)\cdot P(n)=n\cdot P(n-1)=0 \Rightarrow P(n)=0$" is not true.

One can find easily that $P(x)=x(x-1)(x-2)k$, $k\in\mathbb{R}$.

WHERE IS THE ERROR?

Given the functions $f,g:\mathbb{R}\to\mathbb{R}$, defined as follows:

$$f(x)=\begin{cases} e^x, & x\le 3\\ e^{-x}, & x>3\end{cases} \quad \text{and} \quad g(x)=\begin{cases} x^2, & x\le 0\\ -2x+7, & x>3\end{cases}$$

Compute $f\circ g$.

"*Solution*": We can write:

$$f(x)=\begin{cases} e^x, & x\le 0\\ e^x, & 0<x\le 3\\ e^{-x}, & x>0\end{cases} \quad \text{and} \quad g(x)=\begin{cases} x^2, & x\le 0\\ -2x+7, & 0<x\le 3\\ -2x+7, & x>3\end{cases}$$

from where

$$(f\circ g)(x)=f(g(x))=\begin{cases} e^{x^2}, & x\le 0\\ e^{-2x+7}, & 0<x\le 3\\ e^{2x-7}, & x>3\end{cases}$$

and $f \circ g : \mathbb{R} \to \mathbb{R}$.

Correct solution:

$$f \circ g = f(g(x)) = \begin{cases} e^{g(x)}, & \text{if } g(x) \le 3 \\ e^{-g(x)}, & \text{if } g(x) > 3 \end{cases} \qquad f \circ g : \mathbb{R} \to \mathbb{R}$$

$$g(x) \le 3 \Rightarrow \begin{cases} x^2 \le 3 \quad \Rightarrow x \in [-\sqrt{3}, 0] \\ or \\ -2x + 7 \le 3 \Rightarrow x \in [2, +\infty) \end{cases}$$

$$g(x) > 3 \Rightarrow \begin{cases} x^2 > 3 \quad \Rightarrow x \in (-\infty, -\sqrt{3}) \\ or \\ -2x + 7 > 3 \Rightarrow x \in (0, 2) \end{cases}$$

Therefore

$$f \circ g)(x) = \begin{cases} e^{-x^2}, & x \in (-\infty, -\sqrt{3}) \\ e^{x^2}, & x \in [-\sqrt{3}, 0) \\ e^{2x-7}, & x \in (0, 2) \\ e^{2x-7}, & x \in [2, +\infty) \end{cases}$$

[Published in “Gazeta matematică”, nr.7/1981, Anul LXXXVI, pp. 282-283.]

WHERE IS THE ERROR IN THE BELOW SYSTEM OF INEQUALITIES ?

Solve the following inequalities system:

$$\begin{cases} x \ge 0 & (1) \\ y \ge 0 & (2) \\ x - 2y + 3z \ge 0 & (3) \\ -3x - y + 4z \ge 4 & (4) \end{cases}$$

“Solution”: Multiply the third inequality by 3 and add it to the fourth inequality. The sense will be conserved. It results:

$-7y + 13z \ge 4$, or $z \ge \frac{1}{13}(7y + 4)$.

Therefore, $x \ge 0$ and $y \ge 0$ (from the inequalities (1) and (2))

and $z \ge \frac{1}{13}(7y + 4)$ (*).

But $x = 13 \geq 0$, $y = 0 \geq 0$, and $z = 2 \geq \frac{4}{13} = \frac{1}{13}(7 \cdot 0 + 4)$ verifies (*). But we observe that it does not verify the inequalities system, because substituting in the fourth inequality we obtain: $-3 \cdot 13 - 0 + 4 \cdot 2 \geq 4$ which is not true.

Where is the contradiction?

Solution.

The previous solution is incomplete. We didn't intersect all four inequalities. Giving a geometrical interpretation in $\mathbb{R}^3$, and writing the inequalities as equations, we have, in fact, four planes, each dividing the space in semi spaces. Therefore, the system's solution will be formed by the points which belong to the intersection of those four semi spaces, (each inequality determines a semi space). The inequality obtained by adding the third inequality with the fourth represents, is, in fact, another semi space that includes the system's solution, and it does not simplify the system (in the sense that we cannot eliminate any of the system's inequalities).

Therefore $x = 0$, $y = 3$, $z = \frac{5}{13}$ verifies (*) but it does not verify, this time, the third inequality (although the fourth one is verified).

THE ILLOGICAL MATHEMATICS!

Find a "logic" for the following statements:

(1) $4 - 5 \approx 5!$

(2) 8 divided by two is equal to zero!

(3) 10 minus 1 equals 0.

(4) $\int f(x)\ dx = f(x)!$

(5) 8+8=8!

Solutions:

These mathematical fantasies are entertainments, amusing problems; they disregard current logic, but having their own "logic", fantasist logic: thus

(1) can be explained if one does not consider "4 - 5" as the writing of "4 minus 5" but that of "from 4 to 5"; from which a reading of the statement "$4 - 5 \approx 5$" should be: "between 4 and 5, but closer to 5".

(2) 8 can be divided by two … in the following way:…, i. e. it will be cut into two equal parts, which are equal to "0" above and below the cutting line!

(3) "10 minus 1" can be treated as: the two typographical characters 1, 0 minus the 1, which justifies that there remains the character 0.

(4) The sign will be considered as the opposite function of the integral.

(5) The operation "$\infty + \infty = \infty$" is true: writing it vertically:

∞

$+$

∞

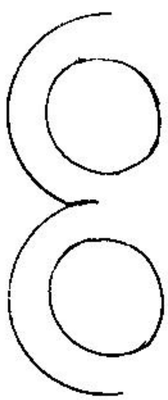
$=$

∞

which, transposed horizontally (by a mechanic rotation of the graphic signs) will give us the statement: “$8 + 8 = 8$”.

OPTICAL ILLUSION
(Mathematical Psychology)

What digit is it, 8 or 3?



[Answer: Both of them!]

1. EPMEK = Reverse of Kempe.

2. DEDE/KIND = DedeKind’s cut.

3.
B
R
O = Angle of Brocard.
C
A
R
D

4. •
BRIANCHON = Point of Brianchon.

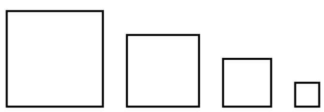
5. $\begin{vmatrix} SYL \\ VES \\ TER \end{vmatrix}$ = Determinant of Sylvester.

6. E A O T E E
r t sh n s = The Sieve of Eratosthenes.

7. $\begin{array}{cccccc} & & A & & & \\ & R & & C & & \\ & & T & & S & \\ D & & E & & & S \end{array}$ = Foliate curve of Descartes.

8. $\begin{pmatrix} M & R & X \\ R & A & I \\ X & I & T \end{pmatrix}$ = Symmetrical matrix.

9. $\overline{\text{SHEFFER}}$ = Bar of Sheffer.

10.  = Method of the smallest squares.

11. $\begin{pmatrix} J & 1 & 0 & 0 & 0 & 0 \\ 0 & \emptyset & 1 & 0 & 0 & 0 \\ 0 & 0 & R & 1 & 0 & 0 \\ 0 & 0 & 0 & D & 1 & 0 \\ 0 & 0 & 0 & 0 & A & 1 \\ 0 & 0 & 0 & 0 & 0 & N \end{pmatrix}$ = Matrix of Jordan.

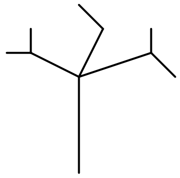
12. NOITCNUF = Inverse function.

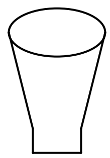
13. SERUGIF = Inverse figures.

14. $\begin{array}{cccccccc} & R & & V & & R & & V \\ M & & K & & M & & K & \\ & A & & O & & A & & O \end{array}$ = Markov Chains.

15. $\dfrac{\text{USA}}{\text{WEST EUROPE}}$ = Harmonious rapport.

16. $\dfrac{\text{USA}}{\text{USSR}}$ = Unharmonious rapport.

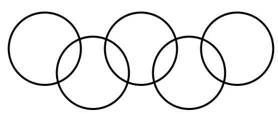
17.  = Tree.

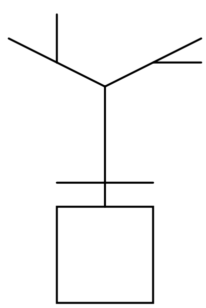
18.  = Convergent filter.

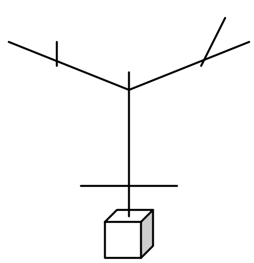
19.
A
P S

O U = Apollonius' circle.

L I
O N

20.  = Fascicles of circles.

21.  = Square root.

22.  = Cubic root

23. $X^{\infty} + Y^{\infty} = Z^{\infty}$ = Fermat's last theorem

24. I-W-A-S-A-W-A = Iwasawa's decomposition

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25.  R E            =   Latin square!
     O M
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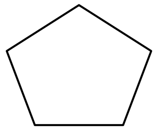
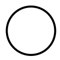
26.  = The Pentagon!

27. Ø = *Reductio ad absurdum.*

28.  = Ring.

```
29. F           N
     U        O      =   Convex function.
      N      I
        C T
```

```
30.  P    N    S     =   Non-collinear points.
       I    T
      O
```

```
31.    G
     R   P           =   Group of rotations.
      O U
```

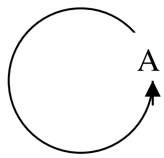
32. ELEMENTS = Non-disjoint elements.

```
33.      M
       X   A         =   Circular matrix.
       I  T
         R
```

```
         O L
34.    P      I      =   7-gon.
      N
         O G
```

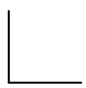
```
35.      SPA
          CE         =   Compact space.
```

```
36.       A
          L
          G
          E          =   Higher algebra
          B
          R
          A
```

37. 

= Vicious circle

38. A
R
I
T
H
M
E
T
I
C = The higher arithmetic.

39. 
= Square angle.

40. SYMBOL OF (LEOPOLD) KRONECKER = L.K.

41. KOLMOGOROV'S SPACE = USSR.

42. LANGUAGE OF CHOMSKY = American.

43. GRAMMAR OF KLEENE = English.

44. CATASTROPHIC POINT = Atom bomb.

45. MACHINE OF TURING = Motor car.

46. NUMBER OF GOLD = 79 (Chemically).

47. FLY OF LA HIRE = Insect.

48. MOMENT OF INERTIA = Apathy.

49. AXIOM OF SEPARATION = Divorce.

50. CLOSED SET = Prisoners.

51. RUSSIAN MULTIPLICATION = Conquest.

52. SLIPS OF MÖBUS = Bathing trunks.

53. SINGULAR CARDINAL	=	Mazarin (1602-1661, France).
54. CLAN OF LEBESGUE	=	His family.
55. SPHERE OF RIEMANN	=	Head.
56. MATHEMATICAL HOPE	=	Fields prize.
57. CRITICAL WAY	=	Slope.
58. BOTTLE OF KLEIN	=	Beer bottle.
59. CONSTANT OF EULER	=	Mathematics.
60. CONTRACTING FUNCTION	=	Frost.
61. BILINEAR COMBINATION	=	Concubine.
62. HARDY SPACE	=	England.
63. INTRODUCTION TO ALGEBRA!	=	AL.
64. INTRODUCTORY ECONOMETRICS	=	ECO.
65. BOREL BODY	=	Corpse.
66. CHOICE FUNCTION	=	Marriage.
67. GEOMETRICAL PLACES	=	ATHENA, ERLANGEN, etc.

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MATHEMATICAL LOGIC

How many propositions are true and which ones from the following:
1. There exists one false proposition amongst those n propositions.
2. There exist two false propositions amongst those n propositions.

…………………………………………………………………….

… There exist i false propositions amongst those n propositions.

…………………………………………………………………….

n. There exist n false propositions amongst those n propositions.

(This is a generalization of a problem proposed by professor FRANCISCO BELLOT ROSADO, in the journal NUMEROS, No. 9/1984, p. 69, Canary Island, Spain.)

Comments:

Let P_i be the proposition i, $1 \le i \le n$. If n is even, then the propositions $1, 2, ..., \frac{n}{2}$ are true and the rest are false. (We start our reasoning from the end; P_n cannot be true, therefore P_1 is true; then P_{n-1} cannot be true, then P_2 is true, etc.)

Remark: If n is odd we have a **paradox**, because if we follow the same solving method we find that P_n is false, which implies that P_1 is true; P_{n-1} false, implies that P_2 is true,…, $P_{\frac{n+1}{2}}$ false implies $P_{n+1-\frac{n+1}{2}}$ true, that is $P_{\frac{n+1}{2}}$ false implies $P_{\frac{n+1}{2}}$ true, which is absurd.

If $n = 1$, we obtain a variant of liar's paradox ("I lie" is true or false?)

1. There is a false proposition in this rectangle.

Which is obviously a paradox.

PARADOX OF RADICAL AXES

Property: The radical axes of n circles in the same plan, taken two by two, whose centers are not aligned, are convergent.

"Proof" by recurrence on $n \ge 3$.

For the case $n = 3$ it is known that 3 radical axes are concurrent in a point which is called the radical center. One supposes that the property is true for the values smaller or equal to a certain n.

To the n circles one adds the $(n+1)$-th circle.

One has (1): the radical axes of first n circles are concurrent in M.

Let us take 4 arbitrary circles, among which is the $(n+1)$-th.

Those have the radical axes convergent, in conformity with the recurrence hypothesis, in the point M (since the first 3 circles, which belong to n circles of the recurrence hypothesis, have their radical axes concurrent in M).

Thus the radical axes of $(n+1)$ circles are convergent, which shows that the property is true for all circles $n \ge 3$ of N.

AND YET, one can build the following counterexample:

Consider the parallelogram $ABCD$ which does not have any right angle.

Then one builds 4 circles of centers A, B, C and D respectively, and of the same radius. Then the radical axes of the circles $e(A)$ and $e(B)$, respectively $e(C)$ and $e(D)$, are two lines, which are medians of the segments AB and CD respectively.

Because (AB) and (CD) are parallel, and that the parallelogram does not have any right angle, it results that the two radical axes are parallel, i.e. they never intersect.

Can we explain this (apparent!) contradiction with the previous property?

Response: The "property "is true only for $n = 3$. However in the demonstration suggested one utilizes the premise (distorted) according to which for $m + 4$ the property would be true. To complete the proof by recurrence it would have been necessary to be able to prove that $P(3) \Rightarrow P(4)$, which is not possible since $P(3)$ is true but the counterexample proves that $P(4)$ is false.

A CLASS OF PARADOXES

Let A be an attribute and non-A its negation.

P1. ALL IS "A", THE "NON-A" TOO.

Examples:

E_{11}: All is possible, the impossible too.

E_{12}: All are present, the absentee too.

E_{13}: All is finite, the infinite too.

P2. ALL IS "NON-A", THE "A" TOO.

Examples:

E_{21}: All is impossible, the possible too.

E_{22}: All are absent, the present too.

E_{23}: All is infinite, the finite too.

P3. NOTHING IS "A" NOT EVEN THE "A".

Examples:

E_{31}: Nothing is perfect, not even the perfect.

E_{32}: Nothing is absolute, not even the absolute.

E_{33}: Nothing is finite, not even the finite.

Remark: $P1 \Leftrightarrow P2 \Leftrightarrow P3$.

More generally: ALL (verb) "A", the "NON-A" too.

Of course, from these appear unsuccessful paradoxes, but the proposed method obtains beautiful ones.

Look at a pun, which reminds you of Einstein:

All is relative, the (theory of) relativity too! So:

The shortest way between two pints is the meandering way!

The unexplainable is, however, explained by this word: "unexplainable"!