FLORENTIN SMARANDACHE

An Integer Number Algorithm To Solve Linear Equations

In Florentin Smarandache: "Collected Papers", vol. I (second edition). Ann Arbor (USA): InfoLearnQuest, 2007.

AN INTEGER NUMBER ALGORITHM TO SOLVE LINEAR EQUATIONS

An algorithm is given that ascertains whether a linear equation has integer number solutions or not; if it does, the general integer solution is determined.

Input

A linear equation $a_1x_1 + ... + a_nx_n = b$, with $a_i, b \in \mathbb{Z}$, x_i being integer number unknowns, $i = \overline{1,n}$, and not all $a_i = 0$.

Output

Decision on the integer solution of this equation; and if the equation has solutions in $\mathbb{Z}$, its general solution is obtained.

Method

Step 1. Calculate $d = (a_1, ..., a_n)$.

Step 2. If d / b then "the equation has integer solution"; go on to Step 3. If $d \nmid b$ then "the equation does not have integer solution"; stop.

Step 3. Consider $h := 1$. If $|d| \neq 1$, divide the equation by d; consider $a_i := a_i / d,\ i = \overline{1,n},\ b := b / d$.

Step 4. Calculate $a = \min\limits_{a_s \neq 0} |a_s|$ and determine an i such that $a_i = a$.

Step 5. If $a \neq 1$ then go to Step 7.

Step 6. If $a = 1$, then:

(A) $x_i = -(a_1x_1 + ... + a_{i-1}x_{i-1} + a_{i+1}x_{i+1} + ... + a_nx_n - b) \cdot a_i$

(B) Substitute the value of x_i in the values of the other determined unknowns.

(C) Substitute integer number parameters for all the variables of the unknown values in the right term: $k_1, k_2, ..., k_{n-2}$, and k_{n-1} respectively.

(D) Write, for your records, the general solution thus determined; stop.

Step7. Write down all $a_j,\ j \neq i$ and under the form:

$$a_j = a_iq_j + r_j$$

$$b = a_iq + r \text{ where } q_j = \left[\frac{a_j}{a_i}\right],\ q = \left[\frac{b}{a_i}\right].$$

Step 8. Write $x_i = -q_1x_1 - ... - q_{i-1}x_{i-1} - q_{i+1}x_{i+1} - ... - q_nx_n + q - t_h$. Substitute the value of x_i in the values of the other determined unknowns.

Step 9. Consider

$$\begin{cases} a_1 := r_1 \\ \vdots \\ a_{i-1} := r_{i-1} \\ a_{i+1} := r_{i+1} \\ \vdots \\ a_n := r_n \end{cases} \quad \text{and} \quad \begin{cases} a_i := -a_i \\ b := r \\ x_i := t_h \\ h := h+1 \end{cases}$$

and go back to Step 4.

Lemma 1. The previous algorithm is finite.

Proof:

Let's $a_1x_1 + ... + a_nx_n = b$ be the initial linear equation, with not all $a_i = 0$; check for $\min_{a_s \neq 0} |a_s| = a_1 \neq 1$ (if not, it is renumbered). Following the algorithm, once we pass from this initial equation to a new equation: $a_1^{'}x_1 + a_2^{'}x_2 + ... + a_n^{'}x_n = b'$, with $|a_1^{'}| < |a_i|$ for $i = \overline{2,n}$, $|b'| < |b|$ and $a_1^{'} = -a_1$.

It follows that $\min_{a_s^{'} \neq 0} |a_s^{'}| < \min_{a_s \neq 1} |a_s|$. We continue similarly and after a finite number of steps we obtain, at Step 4, $a := 1$ (the actual a is always smaller than the previous a, according to the previous note) and in this case the algorithm terminates.

Lemma 2. Let the linear equation be:

(25) $a_1x_1 + a_2x_2 + ... + a_nx_n = b$, with $\min_{a_s \neq 0} |a_s| = a_1$ and the equation

(26) $-a_1t_1 + r_2x_2 + ... + r_nx_n = r$, with $t_1 = -x_1 - q_2x_2 - ... - q_nx_n + q$, where $r_i = a_i - a_iq_i$, $i = \overline{2,n}$, $r = b - a_1q$ while $q_i = \left[\frac{a_i}{a}\right]$, $r = \left[\frac{b}{a_1}\right]$. Then $x_1 = x_1^0$, $x_2 = x_2^0, \dots, x_n = x_n^0$ is a particular solution of equation (25) if and only if $t_1 = t_1^0 = -x_1 - q_2x_2^0 - ... - q_nx_n^0 + q$, $x_2, ..., x_n = x_n^0$ is a particular solution of equation (26).

Proof:

$x_1 = x_1^0$, $x_2 = x_2^0, \dots, x_n = x_n^0$, is a particular solution of equation (25) $\Leftrightarrow$
$a_1x_1^0 + a_2x_2^0 + ... + a_nx_n^0 = b \Leftrightarrow a_1x_1^0 + (r_2 + a_1q_2)x_2^0 + ... + (r_n + a_1q_n)x_n^0 = a_1q + r \Leftrightarrow$
$r_2x_2^0 + ... + r_nx_n^0 - a_1(-x_1^0 - q_2x_2^0 - ... - q_nx_n^0 + q) = r \Leftrightarrow -a_1t_1^0 + r_2x_2^0 + ... + r_nx_n^0 = r \Leftrightarrow$
$\Leftrightarrow t_1 = t_1^0, x_2 = x_2^0, ..., x_n = x_n^0$ is a particular solution of equation (26).

Lemma 3. $x_i = c_{i1}k_1 + ... + c_{in-1}k_{n-1} + d_i$, $i = \overline{1,n}$, is the general solution of equation (25) if and only if

(28) $t_1 = -(c_{11} + q_2c_{21} + ... + q_nc_{n1})k_1 - ... - (c_{1n-1} + q_2c_{2n-1} + ... + q_nc_{nn-1})k_n -$
$-(d_1 + q_2d_2 + ... + q_nd_n) + q$,
$x_j = c_{1j1}k_1 + ... + c_{jn-1}k_{n-1} + d_j$, $j = \overline{2,n}$

is a general solution for equation (26).

Proof:

$t_1 = t_1^0 = -x_1^0 - q_2 x_2^0 - ... - q_n x_n^0 + q,\ x_2 = x_2^0, ..., x_n = x_n^0$ is a particular solution of the equation (25) $\Leftrightarrow$ $x_1 = x_1^0,\ x_2 = x_2^0, ..., x_n = x_n^0$ is a particular solution of equation (26) $\Leftrightarrow$ $\exists k_1 = k_1^0 \in \mathbb{Z}, ..., k_n = k_n^0 \in \mathbb{Z}$ such that

$$x_i = c_{i1} k_1^0 + ... + c_{in-1} k_{n-1}^0 + d_i = x_i^0,\ i = \overline{1,n} \Leftrightarrow \exists k_1 = k_1^0 \in \mathbb{Z}, ..., k_n = k_n^0 \in \mathbb{Z},$$

such that

$$x_i = c_{i1} k_1^0 + ... + c_{in-1} k_{n-1}^0 + d_i = x_i^0,\ \ i = \overline{2,n},$$

and

$$t_1 = -(c_{11} + q_2 c_{21} + ... + q_n c_{n1}) k_1^0 - ... - (c_{1n-1} + q_2 c_{2n-1} + ... + q_n c_{nn-1}) k_{n-1}^0 -$$
$$(d_1 + q_2 d_2 + ... + q_n d_n) + q = -x_1^0 - q_2 x_2^0 - ... - q_n x_n^0 + q = t_1^0$$

Lemma 4. The linear equation

(29) $a_1 x_1 + a_2 x_2 + ... + a_n x_n = b$ with $|a_1| = 1$ has the general solution:

(30) $$\begin{cases} x_1 = -(a_2 k_2 + ... + a_n k_n - b) a_1 \\ x_i = k_i \in \mathbb{Z} \\ i = \overline{2,n} \end{cases}$$

Proof:

Let's consider $x_1 = x_1^0,\ x_2 = x_2^0, \ldots, x_n = x_n^0$, a particular solution of equation (29). $\exists k_2 = x_2^0,\ k_n = x_n^0$, such that $x_1 = \left(-a_2 x_2^0 + ... + a_n x_n^0 - b\right) a_1 = x_1^0,\ \ x_2 = x_2^0, ..., x_n = x_n^0$.

Lemma 5. Let's consider the linear equation $a_1 x_1 + a_2 x_2 + ... + a_n x_n = b$, with $\min\limits_{a_s \neq 0} |a_s| = a_1$ and $a_i = a_1 q_i,\ \ i = \overline{2,n}$.

Then, the general solution of the equation is:

$$\begin{cases} x_1 = -(q_2 k_2 + ... + q_n k_n - q) \\ x_i = k_i \in \mathbb{Z} \\ i = \overline{2,n} \end{cases}$$

Proof:

Dividing the equation by a_1 the conditions of Lemma 4 are met.

Theorem of Correctness. The preceding algorithm calculates correctly the general solution of the linear equation $a_1 x_1 + ... + a_n x_n = b$, with not all $a_i = 0$.

Proof:

The algorithm is finite according to Lemma 1. The correctness of steps 1, 2, and 3 is obvious. At step 4 there is always $\min\limits_{a_s \neq 0} |a_s|$ as not all $a_i = 0$. The correctness of sub-step 6 A) results from Lemmas 4 and 5, respectively. This algorithm represents a method of obtaining the general solution of the initial equation by means of the general solutions of the linear equation obtained after the algorithm was followed several times (according to Lemmas 2 and 3); from Lemma 3, it follows that to obtain the general solution of the initial linear equation is equivalent to calculate the general solution of an equation at step 6 A), equation whose general solution is given in algorithm (according to Lemmas 4 and 5). The Theorem of correctness has been fully proven.

Note. At step 4 of the algorithm we consider $a := \min_{a_s \neq 0} |a_s|$ such that the number of iterations is as small as possible. The algorithm works if we consider $a := |a_i| \neq \max_{s=\overline{1,n}} |a_s|$ but it takes longer. The algorithm can be introduced into a computer program.

Application

Calculate the integer solution of the equation:

$$6x_1 - 12x_2 - 8x_3 + 22x_4 = 14 .$$

Solution

The previous algorithm is applied.

1. $(6,-12,-8,22) = 2$
2. $2 \mid 14$ therefore the solution of the equation is in $\mathbb{Z}$.
3. $h := 1$; $|2| \neq 1$; dividing the equation by 2 we obtain:

$3x_1 = 6x_2 - 4x_3 + 11x_4 = 7$.

4. $a := \min\{|3|,\ |-6|,\ |-4|,\ |11|\} = 3,\ i = 1$
5. $a \neq 1$
7. $-6 = 3 \cdot (-2) + 0$

$-4 = 3 \cdot (-2) + 2$

$11 = 3 \cdot 3 + 2$

$7 = 3 \cdot 2 + 1$

8. $x_1 = 2x_2 + 2x_3 - 3x_4 + 2 - t_1$
9.

$a_2 := 0 \qquad a_1 := -3$

$a_3 := 2 \qquad b := 1$

$a_4 := 2 \qquad x_1 := t_1$

$h := 2$

4. We have a new equation:

$-3t_1 - 0 \cdot x_2 + 2x_3 + 2x_4 = 1$

$a := \min\{|-3|, |2|, |2|\}$ and

$i = 3$

5. $a \neq 1$
7. $-3 = 2 \cdot (-2) + 1$

$0 = 2 \cdot 0 + 0$

$2 = 2 \cdot 1 + 0$

$1 = 2 \cdot 0 + 0$

8. $x_3 = 2t_1 + 0 \cdot x_2 - x_4 + 0 - t_2$. Substituting the value of x_3 in the value determined for x_1 we obtain: $x_1 = 2x_2 - 5x_4 + 3t_1 - 2t_2 + 2$

9. $a_1 := 1$ $\quad a_3 := -2$

$a_2 := 0$ $\quad b := 1$

$a_4 := 0$ $\quad x_3 := t_2$

$h := 3$

4. We have obtained the equation:

$1 \cdot t_2 + 0 \cdot x_2 - 2 \cdot t_2 + 0 \cdot x_4 = 1$, $a = 1$, and $i = 1$

6. (A) $t_1 = -(0 \cdot x_2 - 2t_2 + 0 \cdot x_4 - 1) \cdot 1 = 2t_2 + 1$

(B) Substituting the value of t_1 in the values of x_1 and x_3 previously determined, we obtain:

$x_1 = 2x_2 - 5x_4 + 4t_2 + 5$ and

$x_3 = -x_4 + 3t_2 + 2$

(C) $x_2 := k_1$, $x_4 := k_2$, $t_2 := k_3$, $k_1, k_2, k_3 \in \mathbb{Z}$

(D) The general solution of the initial equation is:

$x_1 = 2k_1 - 5k_2 + 4k_3 + 5$

$x_2 = k_1$

$x_3 = -k_2 + 3k_3 + 2$

$x_4 = k_2$

k_1, k_2, k_3 are parameters $\in \mathbb{Z}$

REFERENCE

[1] Smarandache, Florentin – Whole number solution of equations and systems of equations – part of the diploma thesis, University of Craiova, 1979.