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Inequalities for The Integer Part Function

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Inequalities For The Integer Part Function[1]

In this paper we prove some inequalities for the integer part function and we give some applications in the number theory.

Theorem 1. For any x, y > 0 we have the inequality

(1) $[5x] + [5y] \geq [3x+y] + [3y+x]$, where $[\cdot]$ means the integer part function.

Proof. We use the notations $x_1 = [x]$, $y_1 = [y]$, $u = \{x\}$, $v = \{y\}$, $x_1, y_1 \in N$ and $u, v \in [0, 1)$. We can write the inequality (1) as

$x_1 + y_1 + [5u] + [5v] \geq [3u+v] + [3v+u]$. We distinguish the following cases:

α) Let $u \geq v$. If $u \leq 2v$, then $5v \geq 3v+u$ and $[5v] \geq [3v+u]$, analogously $5u \geq 3u+v$ and $[5u] \geq [3u+v]$, from where by addition we obtain (1). If $u > 2v$ and $5u = a+b$, $5v = c+d$, $a, c \in N$, $0 \leq b < 1$, $0 \leq d < 1$, then we have to prove the following inequality

$$a + c + x_1 + y_1 \geq \left[\frac{3a+c+3b+d}{5}\right] + \left[\frac{3c+a+3d+b}{5}\right] \quad (2).$$

But, considering that $1 > u > 2v$, we get $5 > 5u > 10v$, from where $5 > a+b > 2c+2d$, thus $a+b < 5$ and $a \leq 4$. If $a < 2c$, then $a \leq 2c - 1$ and $a + 1 - 2c \leq 0$, thus $a+b-2c < 0$; contradiction with $a+b-2c > 2d$, thus $4 \geq a$, $a \geq 2c$ and $3b+d < 4$, $3d+b < 4$. From $4 \geq a \geq 2c$ we have the cases from the table and in each of the nine cases is verified the inequality (2).

a	4	4	4	3	3	2	2	1	0
c	2	1	0	0	1	1	0	0	0

Application 1. For any m, n ∈ N, (5m)!(5n)! is divisible by m!n!(3m+n)!(3a+m)!.

Proof. If p is a prime number, the power exponent of p in decomposition of m! is $\left[\frac{m}{p}\right] + \left[\frac{m}{p^2}\right] + \ldots$ It is sufficient to prove that

$$\left[\frac{5m}{r}\right] + \left[\frac{5n}{r}\right] \geq \left[\frac{m}{r}\right] + \left[\frac{n}{r}\right] + \left[\frac{3m+n}{r}\right] + \left[\frac{3n+m}{r}\right]$$

for any $r \in N$, $r \geq 2$. If $m = rm_1 + x$, $n = rn_1 + y$, where $0 \leq x < r$, $0 \leq y < r$, $m, n \in Z$, is sufficient to prove that

$\left[\frac{5x}{r}\right] + \left[\frac{5y}{r}\right] \geq \left[\frac{3x+y}{r}\right] + \left[\frac{3y+x}{r}\right]$, but this inequality verifies the theorem 1.

Remark. If x, y > 0 , then we have the inequatity

$[5x] + [5y] \geq [x] + (y) + [3x+y] + [3y+x]$.

[1] Together with Mihály Bencze and Florin Popovici.

Theorem 2. (Szilárd András). If x, y, z ≥ 0, then we have the inequality $[3x]+[3y]+[3z] \geq [x]+(y]+[z]+[x+y]+[y+z]+[z+x]$.

Application 2. For any a,b,c ∈ N, (3a)!(3b)!(3c)! is divisible by a!b!c!(a+b)!(b+c)!(c+a)!.

Proof. Let k_1, k_2, k_3 be the biggest power for which $p^{k_1} \mid (3a)!$, $p^{k_2} \mid (3b)!$, $p^{k_3} \mid (3c)!$ respectively, and r_i ($i \in \{1, 2, 3\ 4, 5, 6\}$) the biggest power for which $p^{r_1} \mid a!$, $p^{r_2} \mid b!$, $p^{r_3} \mid c!$, $p^{r_4} \mid (a+b)!$, $p^{r_5} \mid (b+c)!$, $p^{r_6} \mid (c+a)!$ respectively, then

$$k_1+k_2+k_3 = \left(\left[\frac{3a}{p}\right]+\left[\frac{3a}{p^2}\right]+\ldots\right)+\left(\left[\frac{3b}{p}\right]+\left[\frac{3b}{p^2}\right]+\ldots\right)+\left(\left[\frac{3c}{p}\right]+\left[\frac{3c}{p^2}\right]+\ldots\right)$$

and

$$\sum_{i=1}^{6} r_i \left(\left[\frac{a}{p}\right]+\left[\frac{a}{p^2}\right]+\ldots\right)+\left(\left[\frac{b}{p}\right]+\left[\frac{b}{p^2}\right]+\ldots\right)+\left(\left[\frac{c}{p}\right]+\left[\frac{c}{p^2}\right]+\ldots\right)$$

$$+\left(\left[\frac{a+b}{p}\right]+\left[\frac{a+b}{p^2}\right]+\ldots\right)+\left(\left[\frac{b+c}{p}\right]+\left[\frac{b+c}{p^2}\right]+\ldots\right)+\left(\left[\frac{c+a}{p}\right]+\left[\frac{c+a}{p^2}\right]+\ldots\right).$$

We have to prove that $k_1+k_2+k_3 \geq \sum_{i=1}^{6} r_i$, but this inequality reduces to theorem 2.

Theorem 3. If x, y, z ≥0, then we have the inequality

$$[2x]+[2y]+[2z] \leq [x]+[y]+[z]+[x+y+z].$$

Application 3. If a,b,c ∈ N, then a!b!c!(a+b+c)! is divisible by (2a)!(2b)!(2c)!.

Theorem 4. If x, y ≥ 0 and n, k ∈ N so that n ≥ k ≥ 0, then we have the inequality $[nx] + [ny] \geq k[x] + k[y] + (n - k)[x + y]$.

Application 4. If a, b, n, k ∈ N and n ≥ k, then (na)!(nb)! is divisible by $(a!)^k(b!)^k((a+b)!)^{n-k}$.

Theorem 5. If $x_k \geq 0$ (k = 1, 2, .., n), then we have the inequality

$$2\sum_{k=1}^{n}[2x_k] \geq 2\sum_{k=1}^{n}[x_k] + [x_1+x_2] + [x_2+x_3] + \ldots + [x_n+x_1].$$

Application 5. If $a_k \in N$ (k = 1, 2, ..., n), then $\prod_{k=1}^{n}((2a_k)!)^2$ is divisible by $\prod_{k=1}^{n}(a_k!)^2 (a_1+a_2)!(a_2+a_3)! \ldots (a_n+a_1)!$.

Theorem 6. If $x_k \geq 0$ (k = 1, 2, ..., n), then we have the inequality

$$m\sum_{k=1}^{n}[2x_k] + n\sum_{p=1}^{m}[2x_p] \geq m\sum_{k=1}^{n}[x_k] + n\sum_{p=1}^{m}[x_p] + \sum_{k=1}^{n}\sum_{p=1}^{m}[x_k+x_p].$$

Application 6. If $a_k \in N$ (k = 1, 2, ..., n), then

$$\prod_{k=1}^{n}(2a_k!)^m \prod_{p=1}^{m}(2a_p!)^n \text{ is divisible by } \prod_{k=1}^{n}(a_k!)^m \prod_{p=1}^{m}(a_p!)^n \prod_{k=1}^{n}\prod_{p=1}^{m}((a_k+a_p)!).$$

Theorem 7. If x,y ≥ 1, then we have the inequality

$$\left[\sqrt{x}\right]+\left[\sqrt{y}\right]+\left[\sqrt{x+y}\right]\geq\left[\sqrt{2x}\right]+\left[\sqrt{2y}\right].$$

Proof. By the concavity of the square root function

$$\sqrt{x+y}=\sqrt{\frac{2x+2y}{2}}\geq\frac{1}{2}\sqrt{2x}+\frac{1}{2}\sqrt{2y}\geq\left[\frac{1}{2}\sqrt{2x}\right]+\left[\frac{1}{2}\sqrt{2y}\right],$$

it follows that $\left[\sqrt{x+y}\right]\geq\left[\frac{1}{2}\sqrt{2x}\right]+\left[\frac{1}{2}\sqrt{2y}\right]$.

Therefore it is sufficient to show that $\left[\sqrt{x}\right]+\left[\frac{1}{2}\sqrt{2x}\right]\geq\left[\sqrt{2x}\right]$ for x≥l. The identity $[x]+\left[x+\frac{1}{2}\right]$ has a straightforward proof. We use it to replace $\left[\frac{1}{2}\sqrt{2x}\right]$ with $\left[\sqrt{2x}\right]-\left[\frac{1}{2}\sqrt{2x}+\frac{1}{2}\right]$.

This yields $\left[\sqrt{x}\right]\geq\left[\frac{1}{2}\sqrt{2x}+\frac{1}{2}\right]$ for x≥1. This last inequality followed by notice that x ≥4 implies $\left(2-\sqrt{2}\right)\sqrt{x}\rangle 1$ or $\left[\sqrt{x}\right]\rangle\left[\frac{1}{2}\sqrt{2x}+\frac{1}{2}\right]$ and 1≤ x < 4 implies $\frac{1}{2}\sqrt{2x}+\frac{1}{2}\langle 2$.

Application 7. If a,b ∈ N, then $a!b!\left\lfloor\sqrt{a^2+b^2}\ !\right\rfloor$ is divisible by $\left\lfloor a\sqrt{2}\right\rfloor!\ \left\lfloor b\sqrt{2}\right\rfloor!$

["Octogon", Braşov, Vol. 5, No. 2, 60-2, 1997.]