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# Integer Solutions of Linear Equations

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# INTEGER SOLUTIONS OF LINEAR EQUATIONS

**Definitions and properties of the integer solutions of linear equations.**

Consider the following linear equation:

(1) $$\sum_{i=1}^{n} a_i x_i = b ,$$

with all $a_i \neq 0$ and $b$ in $\mathbb{Z}$.

Again, let $h \in \mathbb{N}$, and $f_i : \mathbb{Z}^h \to \mathbb{Z}$, $i = \overline{1,n}$. ($\overline{1,n}$ means: all integers from 1 to $n$).

**Definition 1.**

$x_i = x_i^0$, $i = \overline{1,n}$, is a particular integer solution of equation (1), if all $x_i^0 \in \mathbb{Z}$ and $\sum_{i=1}^{n} a_i x_i^0 = b$.

**Definition 2.**

$x_i = f_i(k_1,...,k_h)$, $i = \overline{1,n}$, is the general integer solution of equation (1) if:

a) $\sum_{i=1}^{n} a_i f_i(k_1,...,k_h) = b; \ \ \forall (k_1,...,k_h) \in \mathbb{Z}^h$,

b) For any particular integer solution of equation (1), $x_i = x_i^0$, $i = \overline{1,n}$, there exist $(k_1^0,...,k_h^0) \in \mathbb{Z}^h$ such that $x_i^0 = f_i(k_1^0,...,k_h^0)$ for all $i = \overline{1,n}$ {i. e. any particular integer solution can be extracted from the general integer solution by parameterization}.

We will further see that the general integer solution can be expressed by linear functions.

For $1 \leq i \leq n$ we consider the functions $f_i = \sum_{j=1}^{h} c_{ij} k_j + d_i$ with all $c_{ij},\ d_i \in \mathbb{Z}$.

**Definition 3.**

$A = (c_{ij})_{i,j}$ is the matrix associated with the general solution of equation (1).

**Definition 4.**

The integers $k_1,...,k_s,\ 1 \leq s \leq h$ are independent if all the corresponding column vectors of matrix $A$ are linearly independent.

**Definition 5.**

An integer solution is $s$-times undetermined if the maximal number of independent parameters is $s$.

**Theorem 1.** The general integer solution of equation (1) is $(n-1)$-times undetermined.

*Proof:*

We suppose that the particular integer solution is of the form:

(2) $$x_i = \sum_{e=1}^{r} u_{ie} P_e + v_i, \quad i = \overline{1,n}$$, with all $u_{ie},\ v_i \in \mathbb{Z}$,

$P_e$ are parameters of $\mathbb{Z}$, while $a \le r < n-1$.

Let $(x_1^0, ..., x_n^0)$ be a general integer solution of equation (1) (we are not interested in the case when the equation does not have an integer solution). The solution:

$$\begin{cases} x_j = a_n k_j + x_j^0, & j = \overline{1,n-1} \\ x_n = -\left( \sum_{j=1}^{n-1} a_j k_j - x_n^0 \right) \end{cases}$$

is undetermined $(n-1)$-times (it can be easily checked that the order of the associated matrix is $n-1$). Hence, there are $n-1$ undetermined solutions. Let's consider, in the general case, a solution be undetermined $(n-1)$-times:

$$x_i = \sum_{j=1}^{n-1} c_{ij} k_j + d_i, \quad i = \overline{1,n} \text{ with all } c_{ij},\ d_i \in \mathbb{Z}.$$

Consider the case when $b = 0$.

Then

$$\sum_{i=1}^{n} a_i x_i = 0.$$

It follows:

$$\sum_{i=1}^{n} a_i x_i = \sum_{i=1}^{n} a_i \left( \sum_{j=1}^{n-1} c_{ij} k_j + d_i \right) = \sum_{i=1}^{n} a_i \sum_{j=1}^{n-1} c_{ij} k_j + \sum_{i=1}^{n} a_i d_i = 0.$$

For $k_j = 0$, $j = \overline{1,n-1}$ it follows that $\sum_{i=1}^{n} a_i d_i = 0$.

For $k_{j_0} = 1$ and $k_j = 0,\ j \ne j_0$, it follows that $\sum_{i=1}^{n} a_i c_{ij_0} = 0$.

Let's consider the homogenous linear system of $n$ equations with $n$ unknowns:

$$\begin{cases} \sum_{i=1}^{n} x_i c_{ij} = 0, & j = \overline{1,n-1} \\ \sum_{i=1}^{n} x_i d_i = 0 \end{cases}$$

which, obviously, has the solution $x_i = a_i$, $i = \overline{1,n}$ different from the trivial one. Hence the determinant of the system is zero, i.e., the vectors $c_j = (c_{1j}, ..., c_{nj})^t$, $j = \overline{1,n-1}$, $D = (d_1, ..., d_n)^t$ are linearly dependent.

But the solution being $(n-1)$-times undetermined it shows that $c_j,\ j=\overline{1,n-1}$ are linearly independent. Then $(c_1,...,c_{n-1})$ determines a free sub-module $\mathbb{Z}$ of order $n-1$ in $\mathbb{Z}_n$ of solutions for the given equation.

Let's see what can we obtain from (2). We have:

$$0=\sum_{i=1}^{n}a_i x_i=\sum_{i=1}^{n}a_i\left(\sum_{e=1}^{r}u_{ie}P_e+v_i\right).$$

As above, we obtain:

$$\sum_{i=1}^{n}a_i v_i=0 \text{ and } \sum_{e=1}^{r}a_i u_{ie_0}=0$$

similarly, the vectors $U_h=(u_{1h},...,u_{nh})$ are linearly independent, $h=\overline{1,r}$, $U_h$, $h=\overline{1,r}$ are $V=(v_1,...,v_n)$ particular integer solutions of the homogenous linear equation.

**Sub-case (a1)**

$U,h=\overline{1,r}$ are linearly dependent. This gives $\{U_1,...,U_r\}$= the free sub-module of order $r$ in $\mathbb{Z}^n$ of solutions of the equation. Hence, there are solutions from $\{V_1,...,V_{n-1}\}$ which are not from $\{U_1,...,U_r\}$; this contradicts the fact that (2) is the general integer solution.

**Sub-case (a2)**

$U_h,\ h=\overline{1,r},\ V$ are linearly independent. Then $\{U_1,...,U_r\}+V$ is a linear variety of the dimension $<n-1=\dim\{V_1,...,V_{n-1}\}$ and the conclusion can be similarly drawn.

Consider the case when $b\neq 0$. So, $\sum_{i=1}^{n}a_i x_i=b$.

Then:

$$\sum_{i=1}^{n}a_i\left(\sum_{j=1}^{n-1}c_{ij}k_j+d_i\right)=\sum_{j=1}^{n-1}\left(\sum_{i=1}^{n}a_i c_{ij}\right)k_j+\sum_{i=1}^{n}a_i d_i=b;\ \ \forall(k_1,...,k_{n-1})\in\mathbb{Z}^{n-1}.$$

As in the previous case, we obtain $\sum_{i=1}^{n}a_i d_i=b$ and $\sum_{i=1}^{n}a_i c_{ij}=0,\qquad \forall\, j=\overline{1,n-1}$.

The vectors $c_j=(c_{ij},...,c_{nj})^t$, $j=\overline{1,n-1}$, are linearly independent because the solution is undetermined $(n-1)$-times.

Conversely, if $c_1,...,c_{n-1},\ D$ (where $D=(d_1,...,d_n)^t$) were linearly dependent, it would mean that $D=\sum_{j=1}^{n-1}s_j c_j$ with all $s_j$ scalar; it would also mean that

$$b=\sum_{i=1}^{n}a_i d_i=\sum_{i=1}^{n}a_i\left(\sum_{j=1}^{n-1}s_j c_{ij}\right)=\sum_{j=1}^{n-1}s_j\left(\sum_{i=1}^{n}a_i c_{ij}\right)=0.$$

This is impossible.

(3) Then $\{c_1,...,c_{n-1}\}+D$ is a linear variety.

Let us see what we can obtain from (2). We have:

$$b = \sum_{i=1}^{n} a_i x_i = \sum_{i=1}^{n} a_i \left( \sum_{e=1}^{r} u_{ie} P_e + v_i \right) = \sum_{e=1}^{r} \left( \sum_{i=1}^{n} a_i u_{ie} \right) P_e + \sum_{i=1}^{n} a_i v_i$$

and, similarly: $\sum_{i=1}^{n} a_i v_i = b$ and $\sum_{i=1}^{n} a_i u_{ie} = 0, \quad \forall\ e = \overline{1, r}$, respectively. The vectors $U_e = \left(u_{1e}, ..., u_{ne}\right)^t$, $e = \overline{1, r}$ are linearly independent because the solution is undetermined $r$-times.

A procedure like that applied in (3) shows that $U_1, ..., U_r, V$ are linearly independent, where $V = \left(v_1, ..., v_n\right)^t$. Then $\left\{U_1, ..., U_r\right\} + V$ = a linear variety = free sub-module of order $r < n-1$. That is, we can find vectors from $\left\{c_1, ..., c_{n-1}\right\} + D$ which are not from $\left\{U_1, ..., U_r\right\} + V$, contradicting the "general" characteristic of the integer number solution. Hence, the general integer solution is undetermined $\left(n-1\right)$-times.

**Theorem 2.** The general integer solution of the homogeneous linear equation $\sum_{i=1}^{n} a_i x_i = 0$ (all $a_i \in \mathbb{Z} \setminus \left\{0\right\}$) can be written under the form:

$$(4) \qquad x_i = \sum_{j=1}^{n-1} c_{ij} k_j, \quad i = \overline{1, n}$$

(with $d_1 = ... = d_n = 0$).

**Definition 6.** This is called the standard form of the general integer solution of a homogeneous linear equation.

*Proof:*

We consider the general integer solution under the form:

$$x_i = \sum_{j=1}^{n-1} c_{ij} P_j + d_i, \quad i = \overline{1, n}$$

with not all $d_i = 0$. We'll show that it can be written under the form (4). The homogeneous equation has the trivial solution $x_i = 0,\ i = \overline{1, n}$. There is $\left(p_1^0, ..., p_{n-1}^0\right) \in \mathbb{Z}^{n-1}$ such that $\sum_{j=1}^{n-1} c_{ij} p_j^0 + d_i = 0, \quad \forall i = \overline{1, n}$.

Substituting: $P_j = k_j + p_j,\ j = \overline{1, n-1}$ in the form shown at the beginning of the demonstration, we will obtain form (4). We have to mention that the substitution does not diminish the degree of generality as $P_j \in \mathbb{Z} \Leftrightarrow k_j \in \mathbb{Z}$ because $j = \overline{1, n-1}$.

**Theorem 3.** The general integer solution of a non-homogeneous linear equation is equal to the general integer solution of its associated homogeneous linear equation plus any particular integer solution of the non-homogeneous linear equation.

*Proof:*

Let's consider that $x_i = \sum_{j=1}^{n-1} c_{ij} k_j, \quad i = \overline{1,n}$, is the general integer solution of the associated homogeneous linear equation and, again, let $x_i = v_i,\ i = \overline{1,n}$, be a particular integer solution of the non-homogeneous linear equation. Then $x_i = \sum_{j=1}^{n-1} c_{ij} k_j + v_i, \quad i = \overline{1,n}$, is the general integer solution of the non-homogeneous linear equation.

Actually, $\sum_{i=1}^{n} a_i x_i = \sum_{i=1}^{n} a_i \left( \sum_{j=1}^{n-1} c_{ij} k_j + v_i \right) = \sum_{i=1}^{n} a_i \left( \sum_{j=1}^{n-1} c_{ij} k_j \right) + \sum_{i=1}^{n} a_i v_i = b$;

if $x_i = x_i^0,\ i = \overline{1,n}$, is a particular integer solution of the non-homogeneous linear equation, then $x_i = x_i - v_i,\ i = \overline{1,n}$, is a particular integer solution of the homogeneous linear equation: hence, there is $(k_1^0, ..., k_{n-1}^0) \in \mathbb{Z}^{n-1}$ such that

$$\sum_{j=1}^{n-1} c_{ij} k_j^0 = x_i^0 - v_i,\ \forall\ i = \overline{1,n},$$

i.e.:

$$\sum_{j=1}^{n-1} c_{ij} k_j^0 + v_i = x_i^0,\ \forall\ i = \overline{1,n},$$

which was to be proven.

**Theorem 4.** If $x_i = \sum_{j=1}^{n-1} c_{ij} k_j,\ i = \overline{1,n}$ is the general integer solution of a homogeneous linear equation $\left(c_{ij}, ..., c_{nj}\right) \sim 1\ \ \forall\ j = \overline{1, n-1}$.

The demonstration is done by reduction ad absurdum. If $\exists j_0,\ 1 \le j_0 \le n-1$ such that $\left(c_{ij_0}, ..., c_{nj_0}\right) \sim d_{j_0} \neq \pm 1$, then $c_{ij_0} = c'_{ij_0} d_{ij_0}$ with $\left(c'_{ij_0}, ..., c'_{nj_0}\right) \sim 1, \quad \forall\ i = \overline{1,n}$.

But $x_i = c'_{ij_0}, \quad i = \overline{1,n}$, represents a particular integer solution as

$$\sum_{i=1}^{n} a_i x_i = \sum_{i=1}^{n} a_i c'_{ij_0} = 1/d_{j_0} \cdot \sum_{i=1}^{n} a_i c_{ij_0} = 0$$

(because $x_i = c_{ij_0},\ i = \overline{1,n}$ is a particular integer solution from the general integer solution by introducing $k_{j_0} = 1$ and $k_j = 0,\ j \neq j_0$. But the particular integer solution $x_i = c'_{ij_0},\ i = \overline{1,n}$, cannot be obtained by introducing integer number parameters (as it should) from the general integer solution, as from the linear system of $n$ equations and $n-1$ unknowns, which is compatible. We obtain:

$$x_i = \sum_{\substack{j=1 \\ j \neq j_0}}^{n} c_{ij} k_j + c'_{ij_0} d_{j_0} k_{j_0} = c'_{ij_0}, \quad i = \overline{1,n}.$$

Leaving aside the last equation – which is a linear combination of other $n-1$ equations – a Kramerian system is obtained, as follows:

$$k_{j_0} = \frac{\begin{vmatrix} c_{11}\ldots\ldots c'_{ij_0}\ldots\ldots. c_{1,n-1} \\ \vdots \\ c_{n-1,1}\ldots c'_{n-1j_0}\ldots c_{n-1n-1} \end{vmatrix}}{\begin{vmatrix} c_{11}\ldots\ldots. c'_{ij_0} d_{j_0}\ldots\ldots c_{1,n-1} \\ \vdots \\ c_{n-1,1}\ldots c'_{n-1j_0} d_{j_0}\ldots c_{n-1n-1} \end{vmatrix}} = \frac{1}{d_{j_0}} \notin \mathbb{Z}$$

Therefore the assumption is false (end of demonstration).

**Theorem 5.** Considering the equation (1) with $(a_1,...,a_n) \sim 1,\ \ b = 0$ and the general integer solution $x_i = \sum_{j=1}^{n-1} c_{ij} k_j,\ \ i = \overline{1,n}$, then

$$(a_1,...,a_{i-1},a_{i+1},...,a_n) \sim (c_{i1},...,c_{in-1}),\ \ \forall i = \overline{1,n}.$$

*Proof:*

The demonstration is done by double divisibility.

Let's consider $i_0,\ 1 \le i_0 \le n$ arbitrary but fixed. $x_{i_0} = \sum_{j=1}^{n-1} c_{i_0 j} k_j$. Consider the equation $\sum_{i \ne i_0} a_i x_i = -a_{i_0} x_{i_0}$. We have shown that $x_i = c_{ij},\ \ i = \overline{1,n}$ is a particular integer solution irrespective of $j,\ \ a \le j \le n-1$.

The equation $\sum_{i \ne i_0} a_i x_i = -a_{i_0} c_{i_0 j}$ obviously, has the integer solution $x_i = c_{ij},\ \ i \ne i_0$. Then $(a_1,...,a_{i_0-1},a_{i_0+1},...,a_n)$ divides $-a_{i_0} c_{i_0 j}$ as we have assumed, it follows that $(a_1,...,a_n) \sim 1$, and it follows that $(a_1,...,a_{i_0-1},a_{i_0+1},...,a_n) \mid c_{i_0 j}$ irrespective of $j$. Hence $(a_1,...,a_{i_0-1},a_{i_0+1},...,a_n) \mid (c_{i_0 1},...,c_{i_0 n-1}),\ \ \forall i = \overline{1,n}$, and the divisibility in one sense was proven.

Inverse divisibility:

Let us suppose the contrary and consider that $\exists i_1 \in \overline{1,n}$ for which $(a_1,...,a_{i_1-1},a_{i_1+1},...,a_n) \sim d_{i_1 1} \ne d_{i_1 2} \sim (c_{i_1 1},...,c_{i_1 n-1})$; we have considered $d_{i_1 1}$ and $d_{i_1 2}$ without restricting the generality. $d_{i_1 1} \mid d_{i_1 2}$ according to the first part of the demonstration. Hence, $\exists d \in \mathbb{Z}$ such that $d_{i_1 2} = d \cdot d_{i_1 1},\ \ |d| \ne 1$.

$$x_{i_1} = \sum_{j=1}^{n-1} c_{i_1 j} k_j = d \cdot d_{i_1 1} \sum_{j=1}^{n-1} c'_{i_1 j} k_j\ ;$$

$$\sum_{i=1}^{n} a_i x_i = 0 \Rightarrow \sum_{i \ne i_1}^{n} a_i x_i = -a_{i_1} x_{i_1} \sum_{i \ne i_1} a_i x_i = -a_{i_1} d \cdot d_{i_1 1} \sum_{j=1}^{n-1} c'_{i_1 j} k_j\ ,$$

where $\left(c_{i_1 1},...,c_{i_1 n-1}\right)\sim 1$.

The non-homogeneous linear equation $\sum_{i\neq i_1} a_i x_i = -a_{i_1} d_{i_1 1}$ has the integer solution because $a_{i_1} d_{i_1 1}$ is divisible by $\left(a_1,...,a_{i_1-1},a_{i_1+1},...,a_n\right)$. Let's consider that $x_i = x_i^0,\ i\neq i_1$, is its particular integer solution. It follows that the equation $\sum_{i=1}^{n} a_i x_i = 0$ the particular solution $x_i = x_i^0,\ i\neq i_1,\ \ x_{i_1} = d_{i_1}$, which is written as (5). We'll show that (5) cannot be obtained from the general solution by integer number parameters:

$$(6)\qquad \begin{cases} \sum_{j=1}^{n-1} c_{ij} k_j = x_i^0, & i\neq i_1 \\ d\cdot d_{i_1 1} \sum_{j=1}^{n-1} c_{ij} k_j = d_{i_1 1} \end{cases}$$

But the equation (6) does not have an integer solution because $d\cdot d_{i_1 1} \nmid d_{i_1 1}$ thus, contradicting, the "general" characteristic of the integer solution.

As a conclusion we can write:

**Theorem 6**. Let's consider the homogeneous linear equation $\sum_{i=1}^{n} a_i x_i = 0$, with all $a_i \in \mathbb{Z}\setminus\{0\}$ and $\left(a_1,...,a_n\right)\sim 1$.

Let $x_i = \sum_{j=1}^{h} c_{ij} k_j,\ i=\overline{1,n}$, with all $c_{ij}\in\mathbb{Z}$, all $k_j$ integer parameters and let's consider $h\in\mathbb{N}$ be a general integer solution of the equation. Then,

1) the solution is undetermined $(n-1)$-times;
2) $\forall\ j=\overline{1,n-1}$ we have $\left(c_{1j},...,c_{nj}\right)\ \sim 1$;
3) $\forall\ i=\overline{1,n}$ we have $\left(c_{i1},...,c_{in-1}\right)\ \sim \left(a_1,...,a_{i-1},a_{i+1},...,a_n\right)$.

The proof results from theorems 1,4 and 5.

**Note 1**. The only equation of the form (1) that is undetermined $n$-times is the trivial equation $0\cdot x_1 + ... + 0\cdot x_n = 0$.

**Note 2**. The converse of theorem 6 is not true.

Counterexample:

$$(7)\qquad \begin{cases} x_1 = -k_1 + \ \ k_2 \\ x_2 = 5k_1 + 3k_2 \\ x_3 = 7k_1 - \ k_2; \quad k_1,k_2\in\mathbb{Z} \end{cases}$$

is not the general integer solution of the equation

$$(8)\qquad -13x_1 + 3x_2 - 4x_3 = 0$$

although the solution (7) verifies the points 1), 2) and 3) of theorem 6. (1, 7, 2) is the particular integer solution of (8) but cannot be obtained by introducing integer number parameters in (7) because from

$$\begin{cases} -k_1 + k_2 = 1 \\ 5k_1 + 3k_2 = 7 \\ 7k_1 - k_2 = 2 \end{cases}$$

it follows that $k = \frac{1}{2} \notin \mathbb{Z}$ and $k = \frac{3}{2} \notin \mathbb{Z}$ (unique roots).

**REFERENCE**


[1] Smarandache, Florentin – Whole number solution of linear equations and systems – diploma thesis work, 1979, University of Craiova (under the supervision of Assoc. Prof. Dr. Alexandru Dincă)