Massive Scalar Field Theory on Discrete n-Scales

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N-scales are a generalization of time-scales that has been put forward to unify continuous and discrete analyses to higher dimensions. In this paper we investigate massive scalar field theory on n-scales. In a specific case of a regular 2-scale, we find that the IR energy spectrum is almost unmodified when there are enough spatial points. This is regarded as a good sign because the model reproduces the known results in the continuum approximation. Then we give field equation on a general n-scale. It has been seen that the field equation can only be solved via computer simulations. Lastly, we propose that n-scales might be a good way to model singularities encountered in the general theory of relativity.

I. INTRODUCTION

In the Planck scale, it is believed that the spacetime has a granular structure. In order to explain the Planck scale physics, theoreticians put forward various theories. For example the causal dynamical triangulations approach triangulates the spacetime with filled-in cells, in the loop quantum gravity approach the spacetime itself is discrete.

Apart from the discussions of quantum gravity, mathematicians have been working on the concept of "time-scale." A time-scale is an arbitrary closed subset of $\mathbb{R}$ in the usual topology. For example the sets $[0, 1]$, $\mathbb{Z}$, $\mathbb{N}$ or $[0, 1] \cup \mathbb{Z}$ are all time scales. Time scale was developed in [1–3]. Time-scale calculus unify the discrete and continuous analyses. For a general overview one may see [7]. However a time-scale is one dimensional and its multi-dimensional counterparts are in the form of product spaces [4]. This inadequacy in covering the real world applications, which may require non-product spaces, the concept of n-cale has been developed [5]. The definition of an n-scale resembles that of a time scale: an n-Scale is an arbitrary closed subset of $\mathbb{R}^n$.

Here is the organization of the paper. In Section II we give the Lagrangian and Euler-Lagrange equations for the field, in Section III we give an analytical solution of field equation for massive scalar field theory on a regular 2-scale, in Section IV we give the general theory of massive scalar field theory on n-scales, and in SectionV we conclude the paper.

II. SCALAR FIELD THEORY ON AN N-SCALE

In this section we give the Lagrangian density and derive the Euler-Lagrange equation using the n-scale calculus. The Lagrangian density for a massive scalar field reads as follows:

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\Delta_\mu\phi\Delta_\nu\phi - \frac{1}{2}m^2\phi^2, \tag{1}$$

where $\Delta_\mu = \partial/\Delta x^\mu$ is the partial Δ-derivative with respect to x^μ and the inverse metric is $\eta^{\mu\nu} = (+, -, -, \cdots)$. The action is then given by the following Δ-integral:

$$S = \int \prod_{i=0}^{n-1} \Delta x_i \mathcal{L}. \tag{2}$$

For a definiton of the integral see [5]. The Euler-Lagrange equation is obtained by extremizing the action with respect to variations of the field and its Δ-derivatives. Using the integration by pars technique one can obtain the equation of motion (supposing that $\delta\phi$ vanishes on the boundary or at infinity):

$$\frac{\partial\mathcal{L}}{\partial\phi} - \Delta_\mu \frac{\partial\mathcal{L}}{\partial\Delta_\mu\phi} = 0. \tag{3}$$

In the specific case of a massive scalar field theory, equation 3 is as follows:

$$\Delta_\mu\Delta^\mu\phi + m^2\phi = 0. \tag{4}$$

In Cartesian coordinates, the modes are given by the exponential function on n-scales:

$$\prod_{j=0}^{n-1} e_{ik^j}(x^j), \tag{5}$$

with the condition that $\omega^2 - \vec{k}^2 = m^2$ where $\omega = k^0$. For definition of the function $e_\cdot(\cdot)$ see [7]. This type of solution is valid for an n-scale in the product form of $\mathbb{T}_1 \otimes \mathbb{T}_2 \otimes \cdots \mathbb{T}_n$ where each $\mathbb{T}_i$ are unbounded 1-scales. If one or many of the 1-scales are bounded, either from below or above or both, then boundary conditions should be imposed to find the mode solutions as superposition of Equation 5.

All of the n-scale may not be covered by a single coordinate chart. Then, solutions in each chart are found then glued together on the boundaries of each region.

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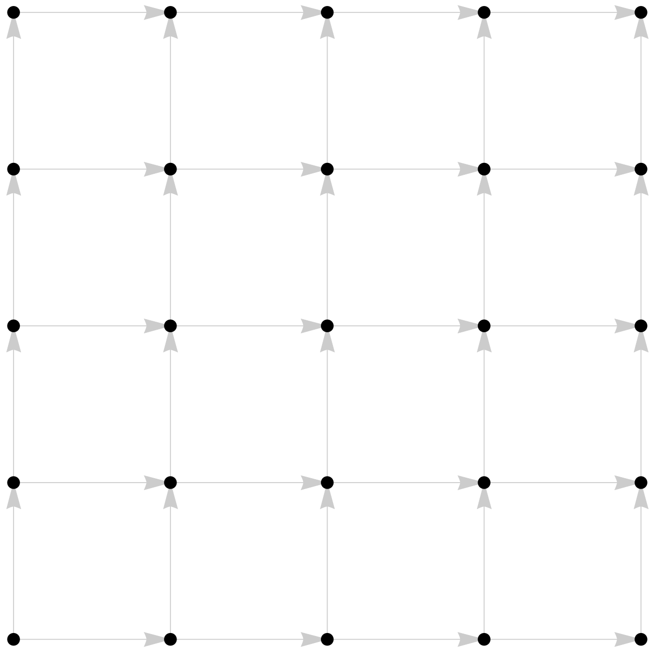

FIG. 1. Part of a 2-scale in the form of $a\mathbb{Z} \otimes b\mathbb{Z}$ is depicted. Lattice spacing in the horiziontal direction is b, whereas it is a in the vertical direction. Arrows show the neighborhood structure of the 2-scale.

III. MASSIVE SCALAR FIELD THEORY ON A REGULAR 2-SCALE

In this section we consider a specific 2-Scale, $a\mathbb{Z} \otimes b\mathbb{Z}$ where a denotes the time-spacing and b denotes the space-spacing. See Figure 1. Now, we can easily make a change of variable $t = ap, x = bq$ where p, q are integers. In this case the mode solution given in Cartesian coordinates (see equation 5) is $e_{i\omega}(t)e_{ik}(x)$. When the 2-scale exponential functions are evaluated, mode solutions are:

$$(1 + i\omega a)^p (1 + ikb)^q. \tag{6}$$

When the space is finite, say it consists of $n+1$ points, we choose a boundary condition on the field that one must have $\phi(x = 0) = \phi(x = nb) = 0$. Therefore the spatial part of the field must be $(1 + ikb)^q - (1 - ikb)^q$ for the mode to vanish at $x = 0$. Imposing the boundary condition on $x = nb$ we obtain:

$$(1 + ikb)^n - (1 - ikb)^n = 0. \tag{7}$$

The solution of this polynomial equation are found [6] as:

$$k_r b = \tan(r\pi/n), \quad r = 0, 1, \ldots, n-1 \tag{8}$$

As is seen there are finitely many solutions (When n is even we disregard $k_{n/2} = \infty$ because the polynomial equation becomes degree of $n - 1$). This makes sense because it should not have infinitely many solutions as in the case of quantum well because UV wavelengths has no meaning in this model. However, as a caution one must note that this model exhibits wavelengths shorter than b. However these modes do not have arbitrarily short wavelentghs. The structure of the 2-scale provides a barrier for UV modes.

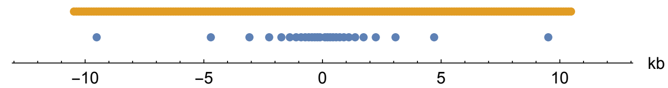

FIG. 2. For $n = 30$, k values obtained from the model and the square well are compared. Blue points denote the the values for discrete space model, whereas the orange points stand for the steadily increasing values of k in quantum square well.

The wavenumbers of this model almost matches with that of the quantum well (which is $k_r b = r\pi/n$) in the limit $n \gg 1$. This is a good sign showing that the discrete model gives the energy spectrum of quantum well of the same size when there are many lattice points.

There is another point to consider, which is that the maximum of kb increases with increasing n. If $n \to \infty$, this maximum approaches infinity as well. Therefore the 2-scale structure of spacetime in this model manifests itself best when n is smaller.

In Figure 2 we compare the k values of discrete model and quantum well. As one sees there is a maximum value for k in the discrete model corresponding to the maximum energy.

As the last step of this section, let us normalize the mode solutions. We have

$$\psi = \frac{A(1 + i\omega a)^p}{|1 + i\omega a|^p} \left[(1 + ikb)^q - (1 - ikb)^q\right], \tag{9}$$

for some $A \in \mathbb{R}$. We requre $\int_0^{nb} \Delta x |\psi|^2 = 1$. First of all the temporal part can be taken out of the integral, so only the phase of the temporal part will remain in the normalized ψ. It is

$$\exp(p \log(1 + i\omega a)). \tag{10}$$

The spatial part requires more care:

$$|A|^2 \int_0^{nb} \Delta x \left|(1 + ikb)^{x/b} - (1 - ikb)^{x/b}\right|^2. \tag{11}$$

This integral is equal to the following sum:

$$|A|^2 \sum_{q=0}^{n-1} \left|(1 + ikb)^q - (1 - ikb)^q\right|^2. \tag{12}$$

All in all the mode solutions are found as:

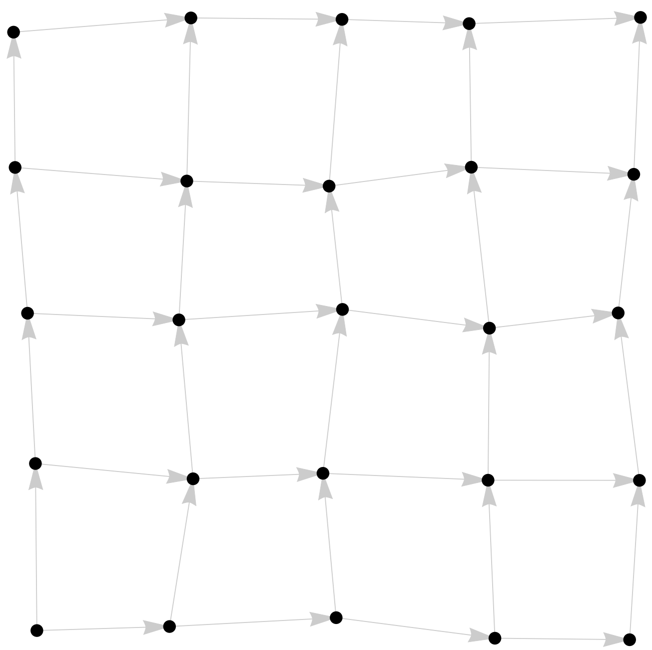

FIG. 3. Part of a general 2-scale is depicted. Note that the neighborhood structure is that of a rectangular lattice.

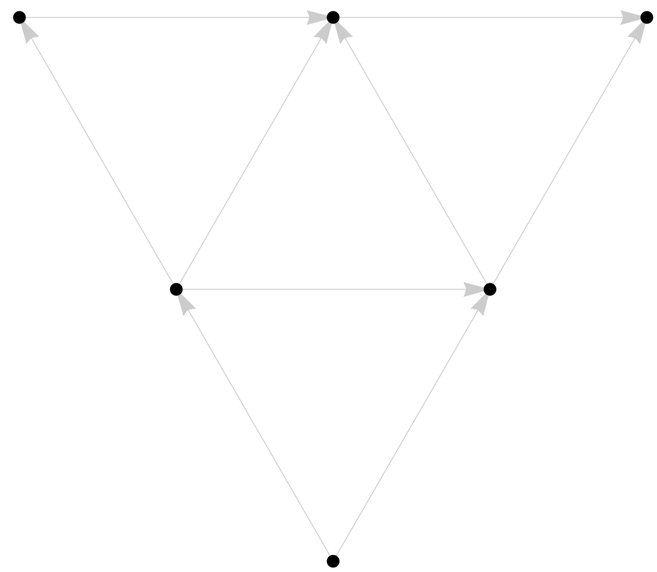

FIG. 4. A simple 2-scale is given. It shows that n-scales can be used to handle the big bang singularity or any other type of singularity. The single point at the bottom is the big bang singularity. In this case, it seems that space and time are the same thing, all coming into existence at once.

$$\psi = A\exp(p\log(1+i\omega a))\left[(1+ikb)^q - (1-ikb)^q\right], \quad (13)$$

where $1/A^2 = \sum_{q=0}^{n-1} |(1+ikb)^q - (1-ikb)^q|^2$.

Notice that the in the limit $a \to 0$ the temporal part in Equation 13 approaches $\exp(i\omega t)$. In the limit $b \to 0$, the spatial part becomes proportional to $\sin(kx)$ as required.

IV. MASSIVE SCALAR FIELD THEORY ON N-SCALE

When considering n-scales in physics applications, we think of neighborhood structure of a rectangular lattice. This makes the metric well defined. If the symmetry requirements of the system requires non-rectangular neighborhood structure, one can embed n-scale in a higher dimensional Minkowski spacetime. In Figure 3 part of a general 2-scale is shown.

The metric tensor on an n-scale is calculated similar to its counterpart on manifolds, using the basis vectors. However, in this case the basis vectors are found using the neighborhood structure and the forward jump operators. At the point p, the metric tensor is given as follows:

$$g_{\mu\nu} = [\sigma_\mu(p) - p] \cdot [\sigma_\nu(p) - p], \quad (14)$$

where the forward jump operator $\sigma_\mu(p)$ is the μ'th neighbor of the point p where it can be reached in the direction of a suitable arrow emanating from p. The metric tensor is not defined for points not lying on the n-scale.

On the other hand, the Δ-derivative at the point p becomes:

$$\Delta_\mu\phi|_p = \frac{\partial\phi}{\Delta x^\mu}|_p = \frac{\phi(\sigma_\mu(p)) - \phi(p)}{|\sigma_\mu(p) - p|}, \quad (15)$$

where norms are evaluated using the Minkowski metric and the coordinate x^μ increases along the points connected with the same neighborhood value of μ.

In general the field equation (4) is quite involved. Only under specific symmetry conditions can it be solved analytically. In general the solutions should be found via computer simulations.

Despite the hardness of finding analytical solutions, n-scales are very useful is that they can model singularities in spacetime. In Figure 4 a simple 2-scale is given that shows how to handle singularities in the n-scale approach.

V. CONCLUSION

In this paper we investigated massive scalar field theory on n-scales. In a specific case of a regular 2-scale, we found that the IR energy spectrum was almost unmodified when there are enough spatial points. This is regarded as a good sign because the model reproduced the known results in the continuum approximation.

Then field equation on a general n-scale were given. It has been seen that the field equation can only be solved via computer simulations.

Although we have considered n-scales with rectangular neighborhood structures, other neighborhood structures such as triangular or hexagonal or irregular structures can be considered as well. However those cases are harder to handle because both they require more complex solutions and it is mode likely that the field equation will be over-determined. Lastly, we propose that n-scales might be a good way to model singularities encountered in the general theory of relativity.

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