Lost in an Isosceles Triangle

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Abstract: Sixty years ago Richard Bellman issued a difficult challenge to his fellow mathematicians. If a rambler is lost in a forest of known shape and size, how can she find the best path to follow in order to escape as quickly as possible? So far solutions are only known for a handful of simple cases and the general problem has therefore been described as "unapproachable." In this work a computational "random paths" method to search for optimal escape paths inside convex polygonal forests is described. In particular likely solutions covering all cases of isosceles triangles are given. Each conjectured solution provides a potentisl upper-bound for Moser's worm problem. Surprisingly there are two cases of triangles which would provide improvements on the best known proven upper bounds.

Introduction

In the field of discrete metric geometry there are a number of problems that are easily stated but notoriously difficult to solve. Some of these, such as the sphere packing problems, have progressed towards full solutions for some cases. Others, such as the covering problems of Lebesgue and Moser, still look utterly insoluble. Henri Lebesgue posed the problem in 1914 of finding the plane convex shape of smallest area which includes (as a subset) an isometric copy of any point set of diameter one [1]. Similarly in 1966 Leo Moser asked for the shape of smallest area which can cover any curve of length one, allowing it to be translated and rotated into place [2]. Moser's problem has become famous as a problem in recreational mathematics due to its interpretation as the search for the smallest blanket to cover a worm in any position.

Both of these problems have been attacked by a similar strategy of improving on lower and upper bounds for the areas sought. Lower bounds can be established by finding the smallest area that covers some small subset of selected shapes [3,4]. Upper bounds can be proven by defining a specific cover and describing a general strategy for fitting any of the required shapes inside [5,6]. Impressive as these results are, there appears to be little hope of bridging the gap to bring the bounds together and form a complete solution.

However, there is another approach to these problems which could offer hope of success. Moser's worm problem turns out to be closely related to another problem which had been posed by Richard Bellman ten years earlier in 1956. Bellman asked for the shortest escape path from a forest of known shape and size. It is assumed that you are lost inside the forest not knowing your position or the direction you are facing, but you are able to navigate along a path of designated shape by dead reckoning. The object is to ascertain the shape of the shortest path that is sure to arrive at the edge of the forest at some point [7]. In mathematical terms, you are given a planar shape F and your task is to find the shortest length for a continuous curve C which cannot be placed inside the shape using translations and rotations without touching the edges. This implies that any curve of the same length as C or shorter can be covered by F. Therefore any solution of Bellman's problem for a given forest shape provides an upper bound for Moser's worm problem. Furthermore, Moser's problem can be redefined as the problem of finding the forest shape of largest area such that Bellman's shortest escape path is of length one.

Unfortunately Bellman's "Lost in a Forest" problem is regarded as being just as difficult as Moser's worm problem. In simple cases such as a square or circular shaped forest the shortest escape path is simply a straight line equal in length to the diameter of the forest, but this does not provide a very useful upper bound to Moser's problem. There are only two cases where a non-straight solution path has been found. When the forest is in the shape of a long rectangle the shortest escape path is a calliper shaped curve with maximal diameter for a fixed length. This special case that Bellman highlighted was first solved by Zalgaller in 1961 [8]. A solution for an equilateral triangle was conjectured by Besicovitch in 1966 [9]. This took the form of a zig-zag path composed of three straight sections of equal length. In 2006 Coulton and Movshovich provided the proof that this is the correct shortest escape path for the equilateral triangle and a range of isosceles triangles up to the right angle case [10,11]. If the techniques used can be extended to a wider range of cases it will be regarded as a significant breakthrough in Bellman's problem. This might be possible if a further set of shortest escape paths can first be conjectured for other cases of triangular shaped forests. In this spirit a series of conjectures for the escape paths for the full range of isosceles triangles will be described in the next part of this paper. These have been found with the help of computer Monte Carlo computations.

Richard Bellman was an applied mathematician who wrote 619 papers and 39 influential books on a range of applied subjects. He is perhaps best known for his works on dynamic programming, computer algorithms and artificial intelligence. His "Lost in a Forest" problem is a problem of interest to pure mathematicians in its own right, but Bellman probably also saw it as a problem that could help develop geometric optimisation techniques which could be of considerable practical importance. Scott Williams agreed when he listed the problem as one of 11 whose method of resolution would be worth a million dollars to mathematics [12]. Although no actual prize money has ever been offered, the value of a solution with to future applications in robotics for example could easily have a much higher commercial value, especially if the methods used can be generalised to similar problems including those in three dimensions.

For reviews and references related to these problems see [13,14,15]

## Conjectured escape paths for isosceles triangles.

Guessing the form of the shortest escape path by hand for a given shape of forest F can be extremely difficult. In order to overcome this, a computational monte-carlo technique can be employed. Initially a large number of random paths C is generated, in the form of polylines of length one. The polylines consist of up to about 8 points joined by straight line segments. For each path a calculation finds the smallest scale factor for the forest shape F such that the path cannot fit inside the scaled shape in any orientation. The paths that provide the largest resulting scale factors are approximations to the shortest escape paths.

The program which is written in Java can accomplish this for any forest in the shape of a convex polygon. Best results can be obtained for shapes with bilateral symmetry since then the best escape path is likely to be have either bilateral symmetry or rotational symmetry and the space of possible paths can be covered in more depth.

An isosceles triangle can be specified by the angle T , $0 < T < 90$ between the base and side so that the apex angle is $180 - 2T$ in degrees. In the following sections the resulting conjectures for the full range are given.

Of course this algorithm is not good enough to count as a solution to Bellman's problem since it gives only a crude approximation and is not efficient enough for any practical purpose.

Phase 1: $0° < T < 28.0°$, Zalgaller's Caliper
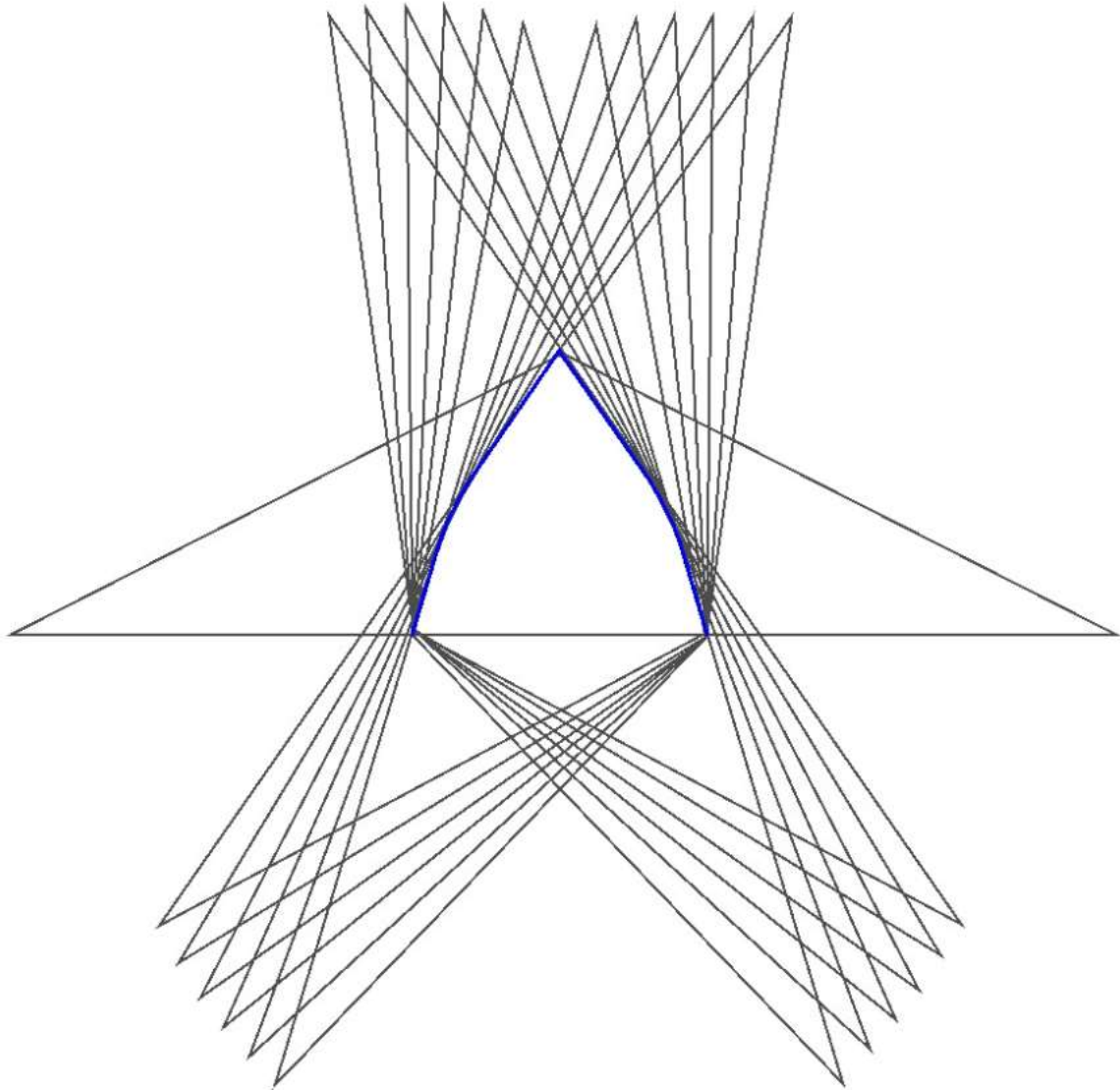
Zalgaller's Caliper (also known as the broadworm) is the solution to Bellman's problem for a long thin strip. It has two arms each consisting of two straight lines joined smoothly in the middle by a circular arc. The two arms meet at a sharp apex at an angle of nearly 70 degrees. The caliper also fits comfortably inside an isosceles triangle for small base angles T, so it also works as an escape path, but for what angles is it the shortest escape path?

The numerical simulations indicate that the caliper is the shortest escape path for base angles up to about 28 degrees. The shape of the shortest escape path does not change over this phase of the isosceles triangle shape, unlike all other phases

Phase 2: $28.0° < T < 30.8°$, Truncated Caliper
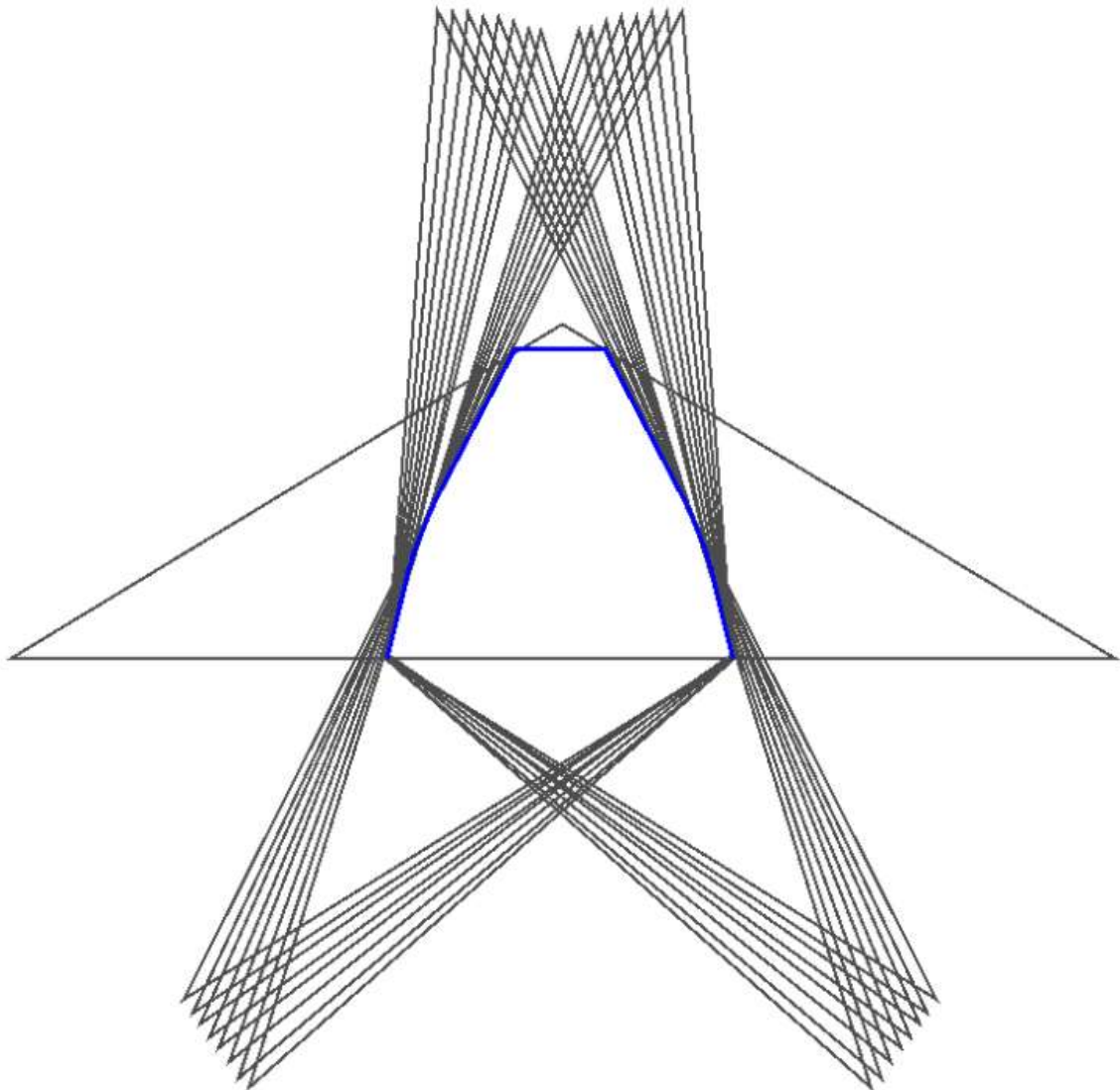
For the subsequent range of angles there an advantage can be gained by taking a short cut across the apex of the triangle. The shortest escape path is therefore a truncated and widened version of the caliper shape. The circular arcs in the longer arms are formed in the same way as they are for Zalgaller's caliper, by rotating the triangle with the apex pinned to the end points of the path.

Phase 3: $30.8° < T < 38.1°$, Tunnel
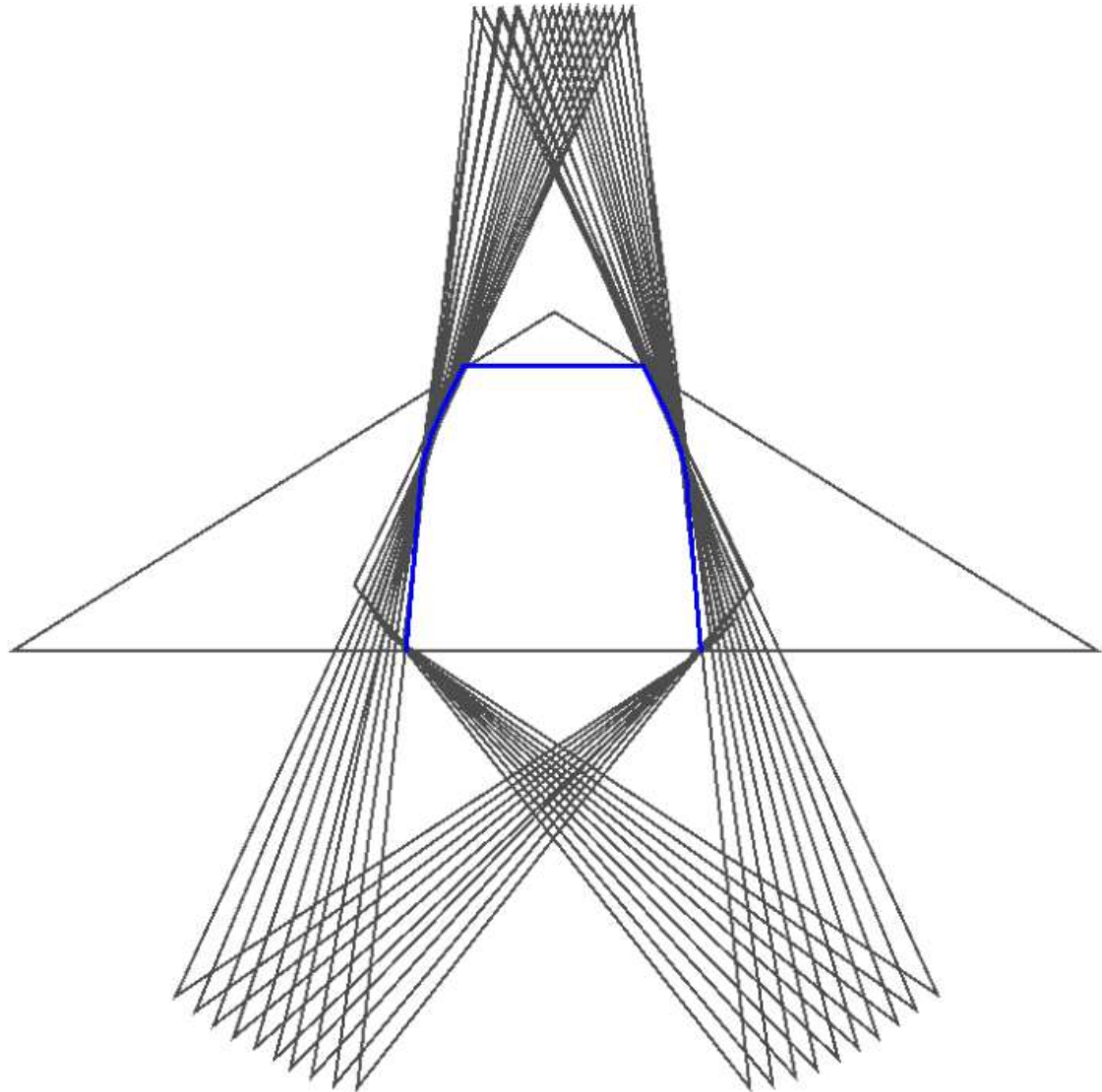
For the next phase the path is constructed from similar segments to the truncated caliper including circular arcs between two straight lines on each arm. However, the apex of the triangle is no longer pinned to the endpoint of the curve to form these arcs. Instead the triangle moves so that the end point slides along one of the edges of the triangle while the other two edges slide round the arcs.

From the theory of envelopes it can be shown that the radius R of the arc is the given by $\frac{1}{R} = \frac{1}{h_A} + \frac{1}{h_B}$ where h_A, h_B are the altitudes of the triangle [16]. The two centres of the arcs and either endpoint of the path form a triangle similar to the shape of the forest boundary.

Phase 4: $38.1° < T < 42.3°$, Trapezium
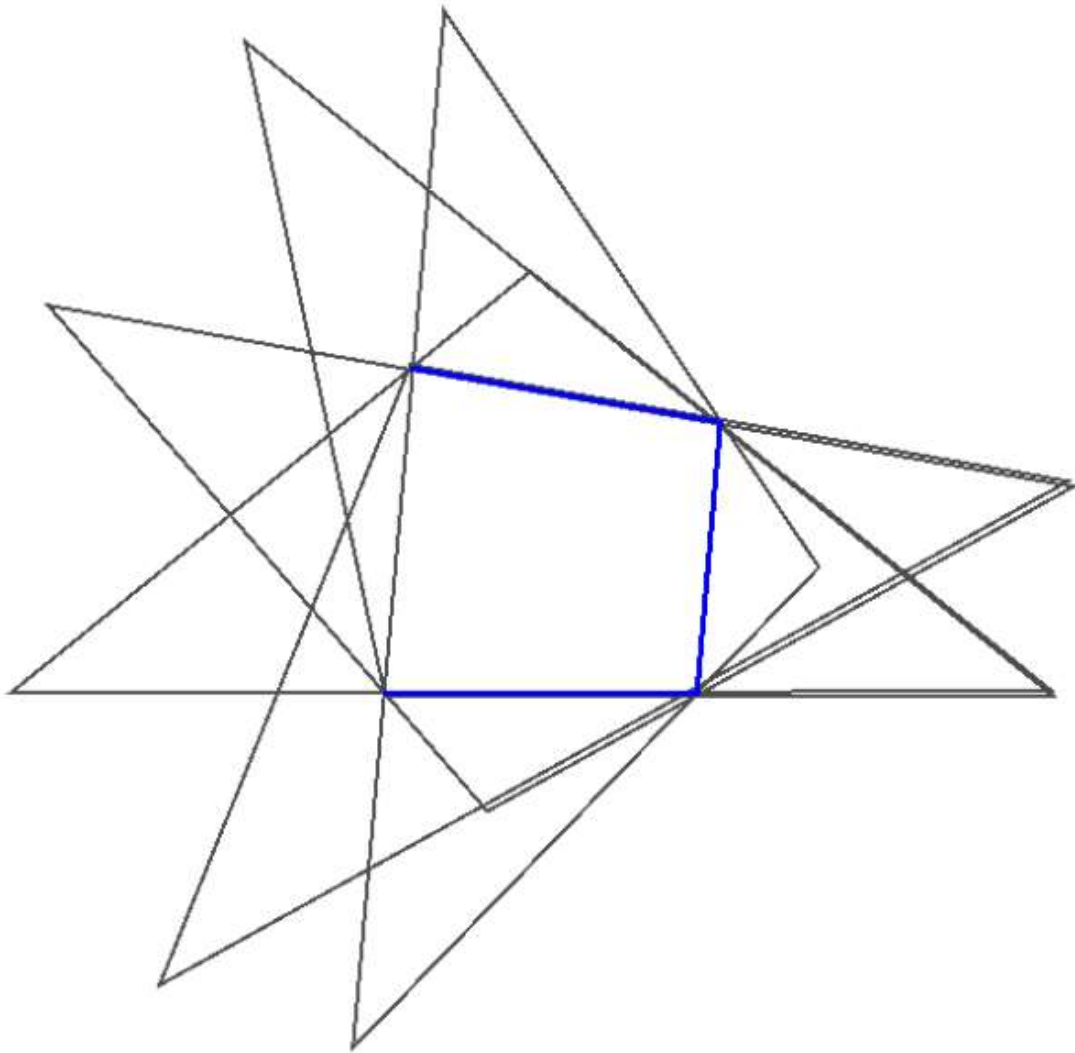
For the next range of angles the form of the shortest escape path takes the much simpler form of a trapezium. The lengths of the sides are fully determined by the requirement that it fits into the triangle perfectly in five different positions.
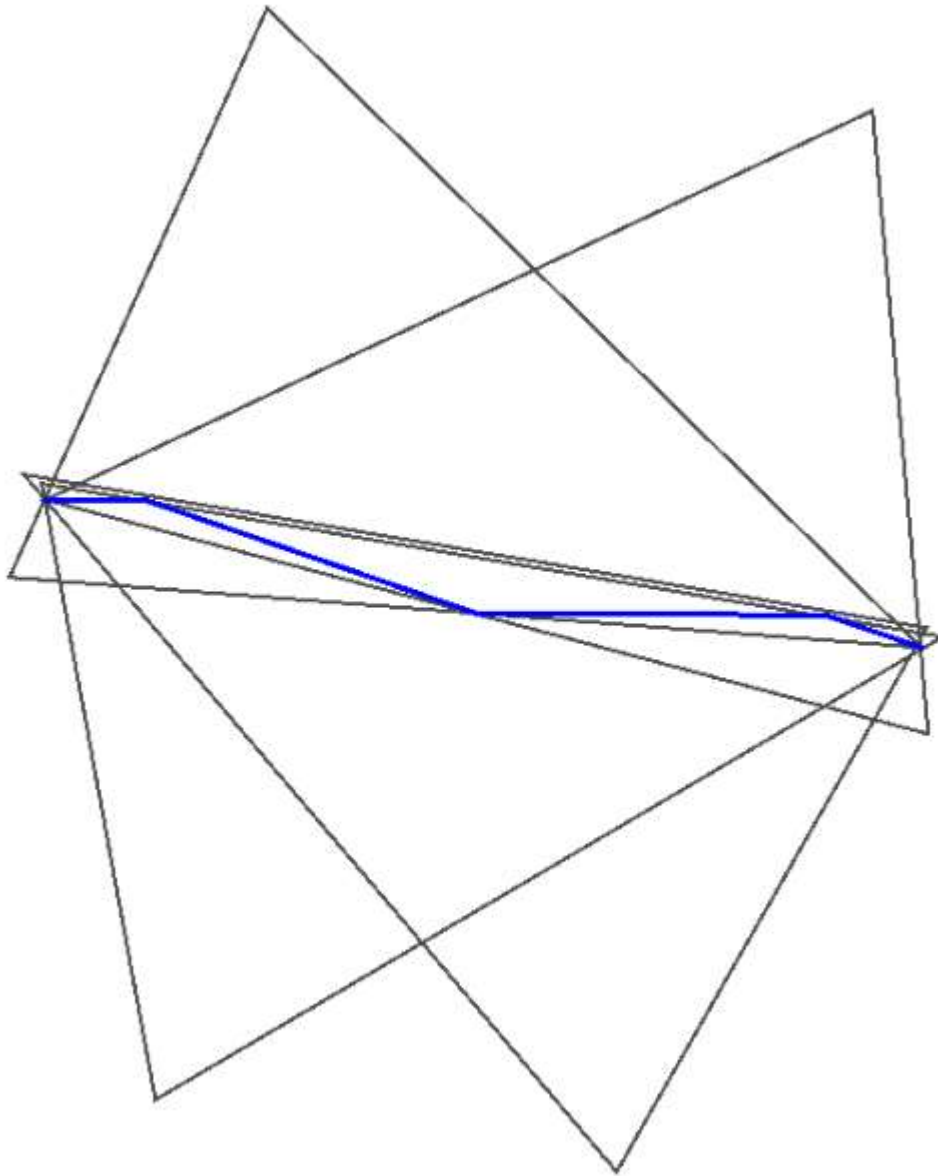
Phase 5: $42.3° < T < 60.6°$, Besicovitch
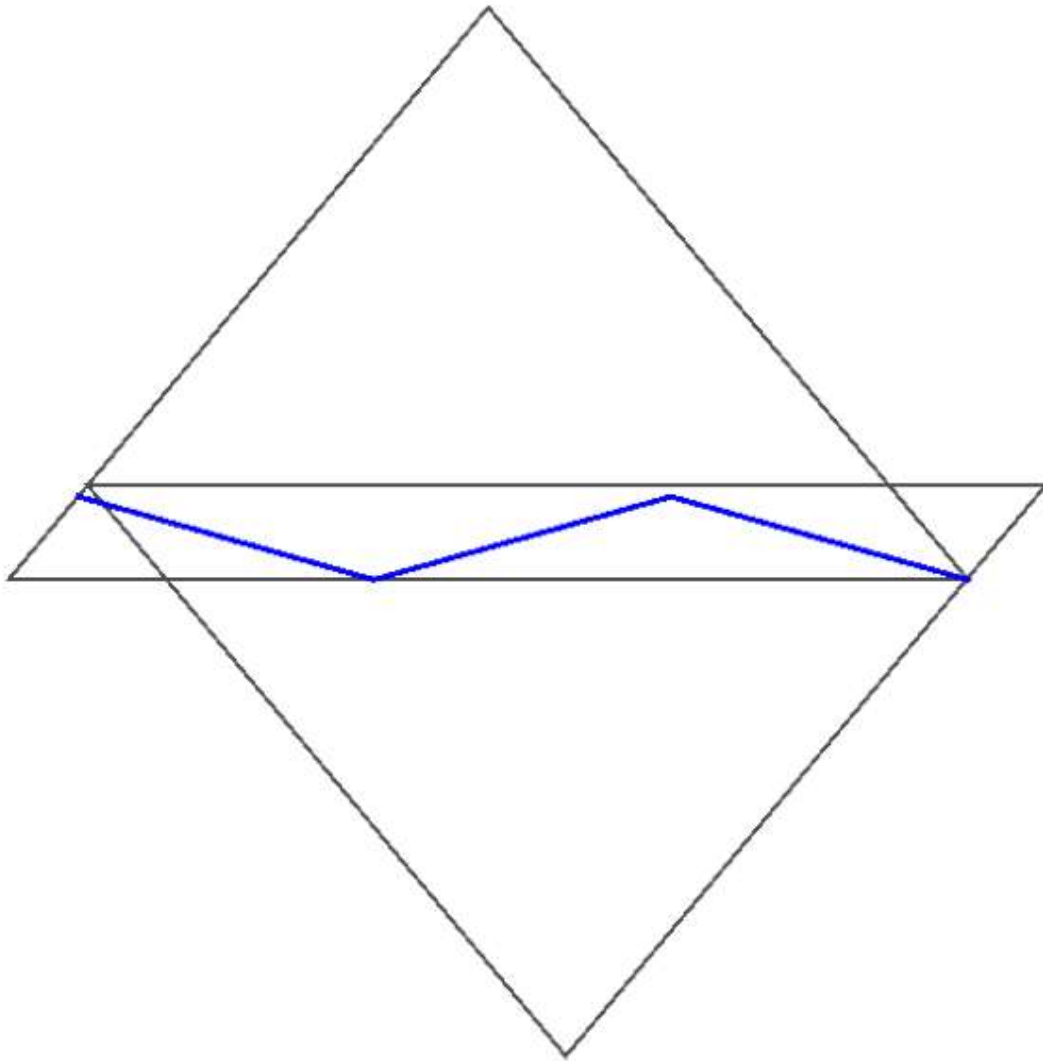
The next range up to the equilateral case and slightly beyond is the best known. The zig-zag solution was conjectured by Besicovitch [9] and proven recently for most of the range [10,11].
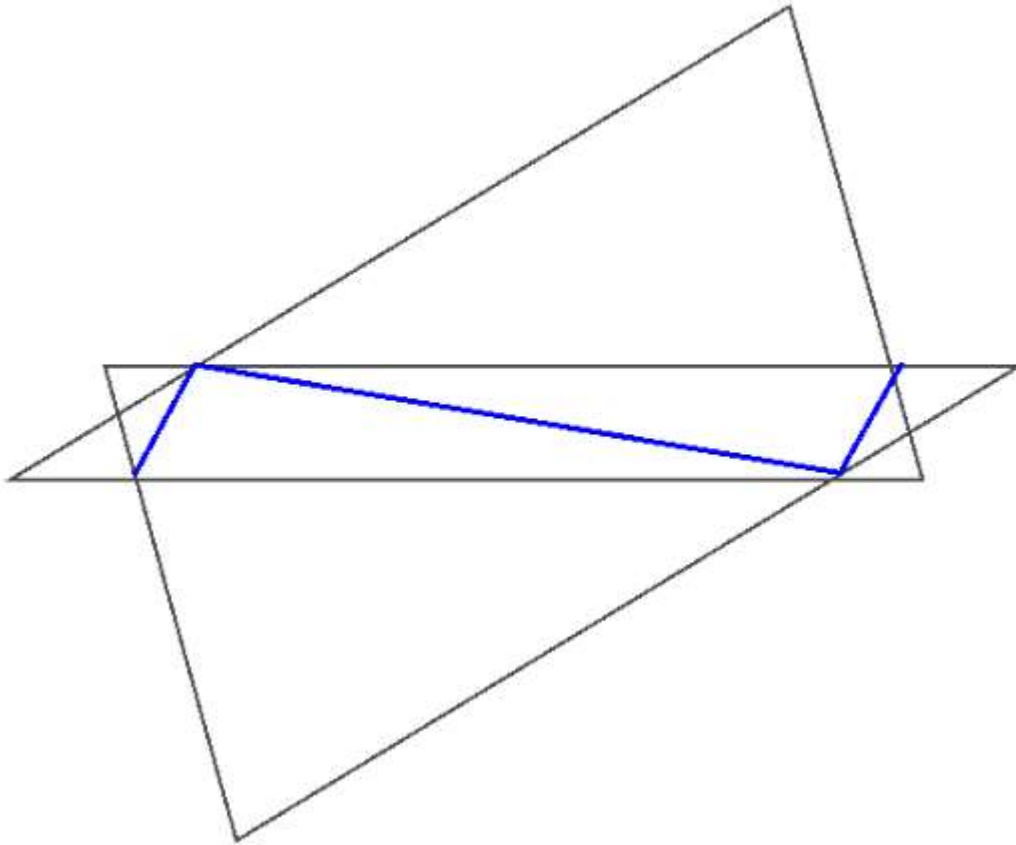
Phase 6: $60.6° < T < 74.4°$, Seagull.

As the apex becomes more acute the shortest escape path switches from the Besicovitch "Z" with three line segments to a "W" shape with four segments.

Phase 7: $74.4° < T < 75.6°$, Z-path.

The shortest escape path then returns to a three segment Z-shape, but one that is distinct from the Besicovitch solution. This time the path does not touch any corner of the triangle and its three sections are not of equal length.

Phase 8: $75.6° < T < 78.5°$, U-path.
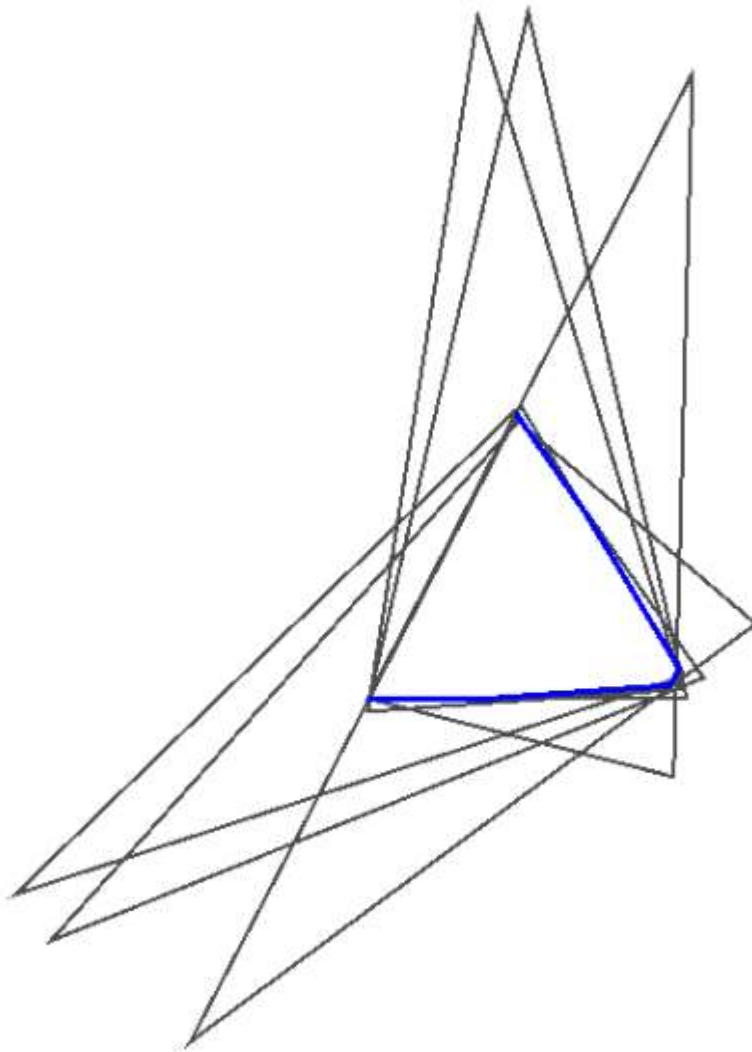
The shortest escape path during this phase is similar to phase 3 in some ways. It has a truncated top and two arms consisting of straight sections joined by circular arcs.

Phase 9: $78.5° < T < 80.0°$, V-path.
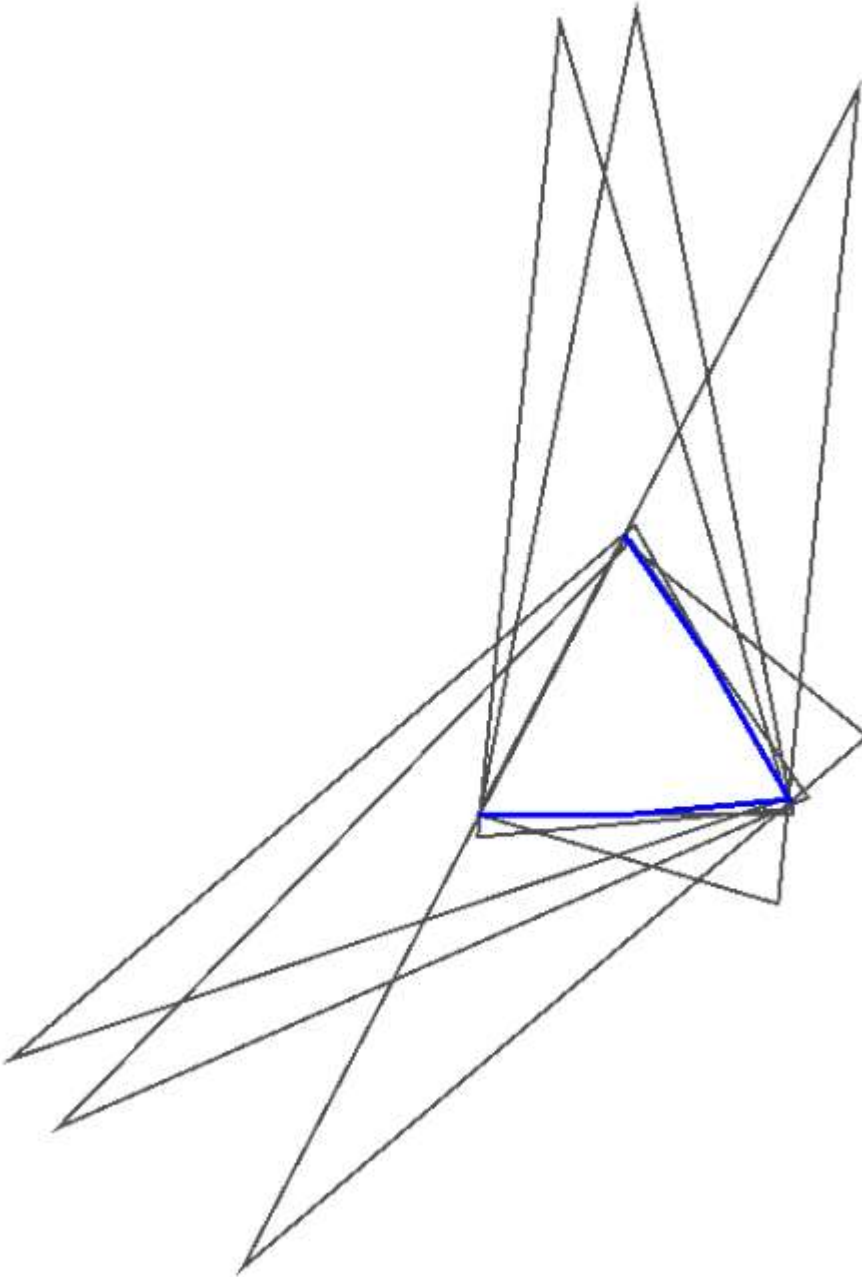
The only difference between this phase and the previous is that there is no truncated top to the shortest escape path. Instead it comes to a sharp point.

Phase 10: $T = 80.0$, Equilateral Triangle
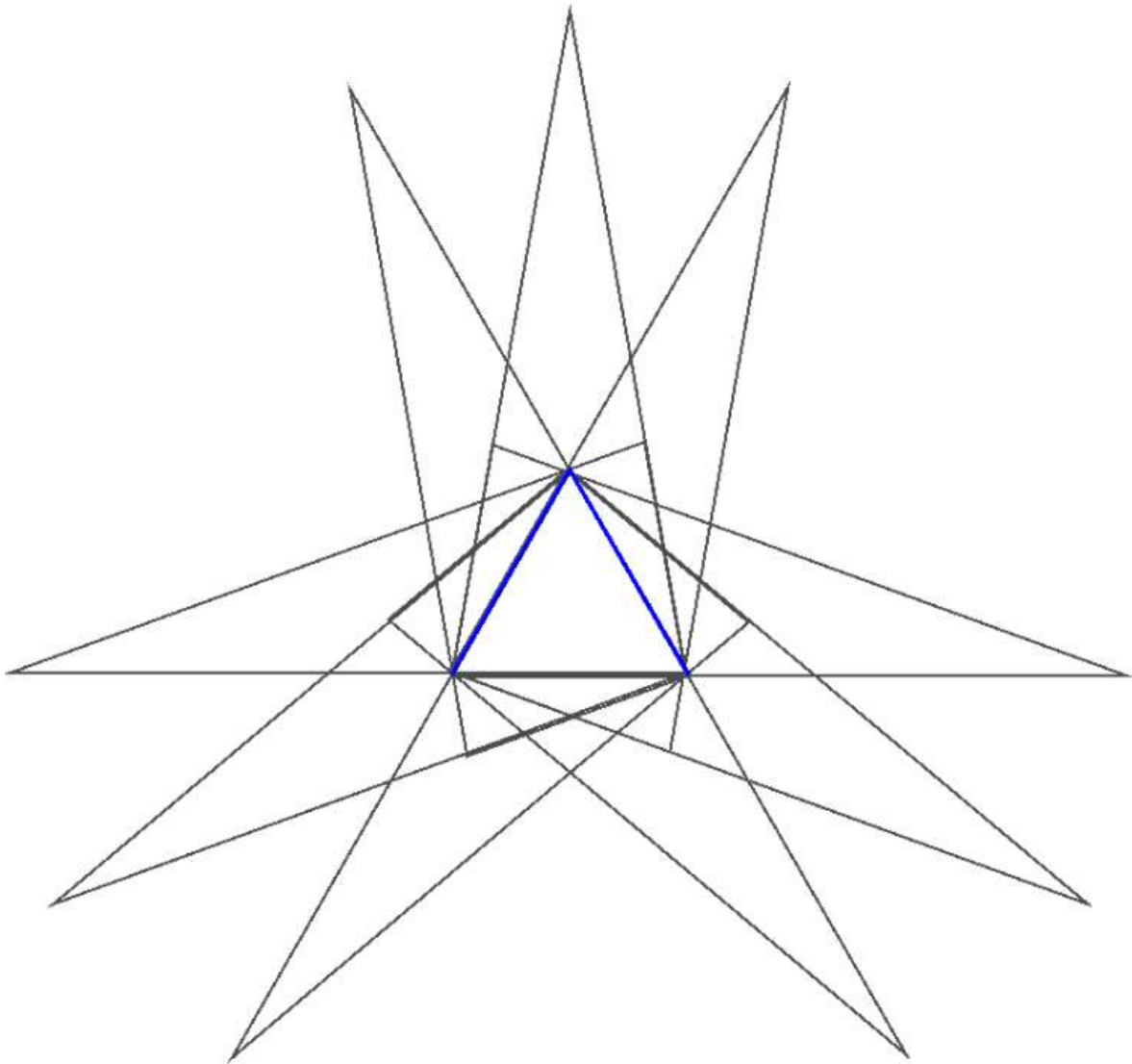
A unique case arises for a triangle where the base angle is exactly 80 degrees. In this situation the shortest escape path takes the simplest form of just two equal length straight segments at an angle of 60 degrees. In other words it is two sides of an equilateral triangle.

The escape path can then be positioned into the triangle in nine different ways as shown above.

Phase 11: $80.0° < T < 90.0°$, V-path.
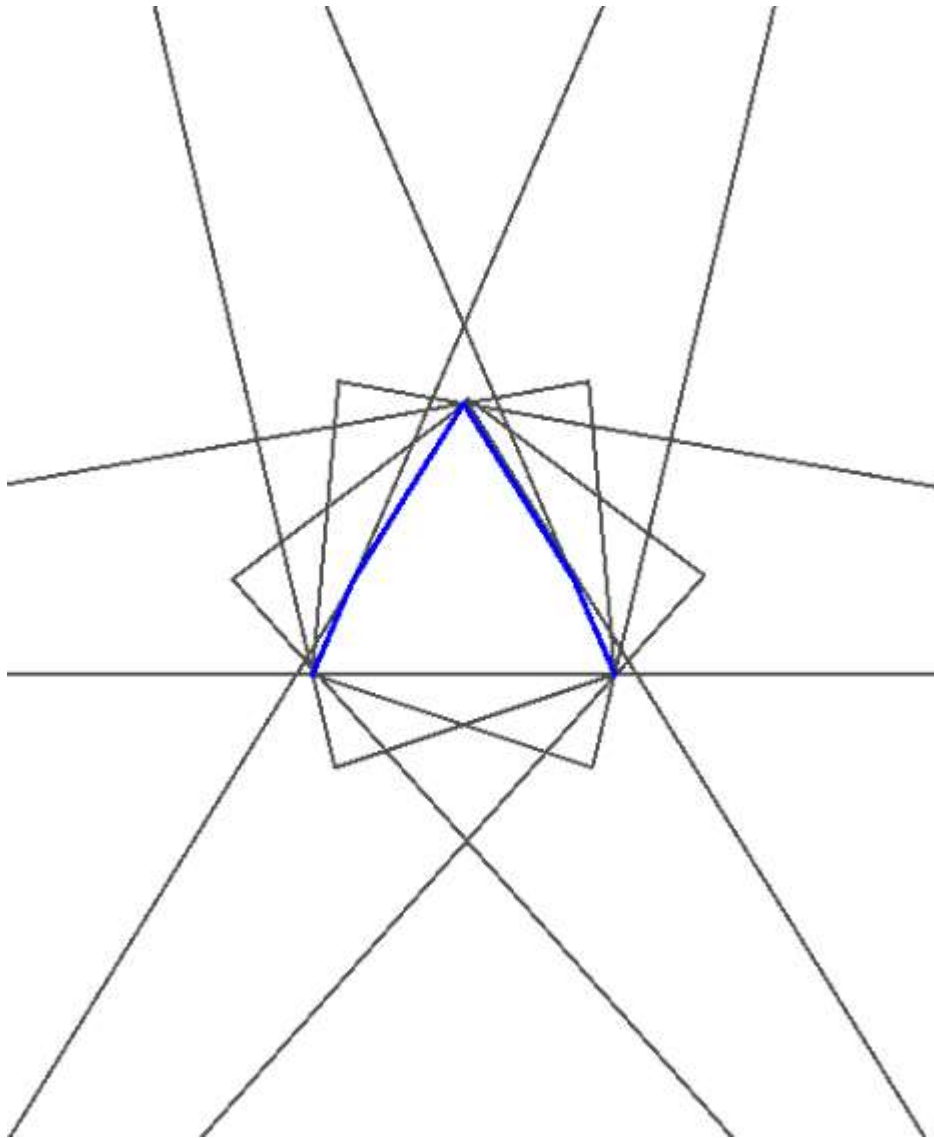
In the final phase the shortest escape path is similar to Zalgaller's Caliper as in the first phase, but here it is just approximately so. As the angle tends towards a right angle, the exact shape of the Caliper is approached in the limit. For slightly smaller angles it is squeezed into a slightly narrower shape but maintains the structure of two symmetric arms each consisting of two straight sections and a circular arc.

Implications for Moser's Worm

A solution to Bellman's problem provides for any convex shape of Forest provides an upper limit for the Moser worm problem. Which isosceles

triangular shapes would provide the best upper bound and how would that compare with the best known upper bound. To answer this it is sufficient to calculate the area of each triangle when the shortest escape path has unit length and find the minimum. This can be done using the monte carlo program. The answer obtained is approximate and conjectural but knowing the expected form of solution may make a rigorous determination possible in the future.

The area plot against base angle is as shown below.
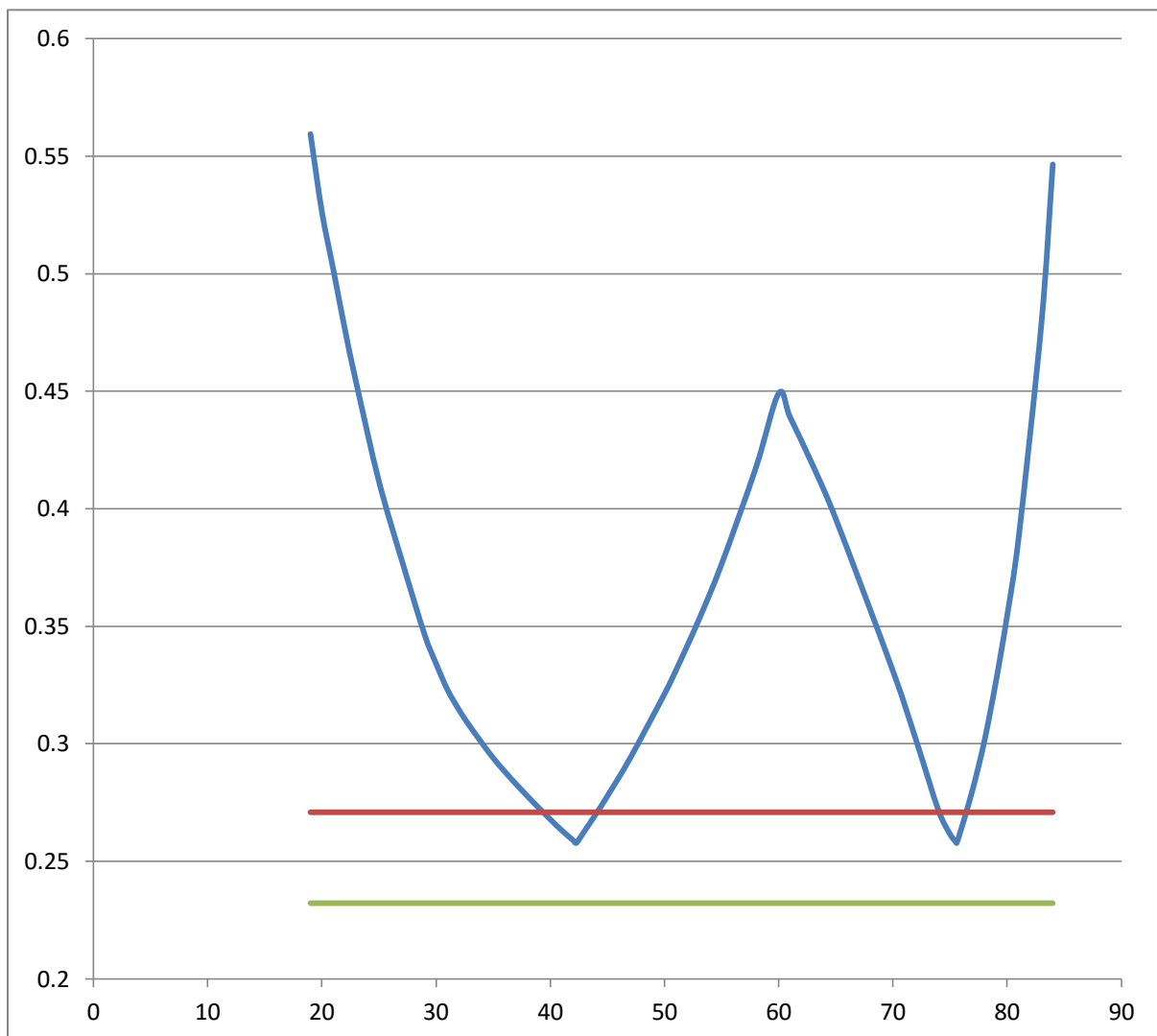
The upper and lower bound have also been shown as red and green lines. Surprisingly the area goes lower than the upper bound at two points. One at the boundary of phases 4 and 5, and the other at the boundary of phases 7 and 8. This means that a simple triangle can provide a better upper bound for Moser's worm problem than the best known bound to date, but only if the shortest escape paths can be proven to be the best. Since these points fall at a boundary

between two phases and the shortest escape path changes shape discontinuously, it follows that there will be two different shortest escape paths at the point of lowest area (three if reflections are counted as distinct)

Which of the two points gives the lowest area and therefore the best upper bound? For no known reason, the areas at these two points turn out to be very close in value. The first minimum is found at a base angle of approximately 42.2815 degrees where the area is 0.257882. The second minimum is at 75.581 degrees where the area is 0.257856. The second therefore appears to be the better but it is hard to estimate errors so there is some room for doubt. Despite how close these areas are there is no obvious reason to think that they would be equal.

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