Note on Uniqueness Solutions of Navier-Stokes Equations

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Abstract: *First date: remembering the need of impose the boundary condition $u(x,t)=0$ at infinity to ensure uniqueness solutions to the Navier-Stokes equations. Second date: verifying that for potential and incompressible flows there is no uniqueness solutions when the velocity is equal to zero at infinity. More than this, when the velocity is equal to zero at infinity for all $t \geq 0$ there is no uniqueness solutions, in general case. Exceptions when $u^0 = 0$. The first date is historical only. Third date: non-uniqueness in time for incompressible and potential flows.*

Recently I wrote a paper named "A Naive Solution for Navier-Stokes Equations"[1] where I concluded that it is possible does not exist the uniqueness of solutions in these equations for $n=3$, even with all terms and for any $t>0$.

This conclusion inhibited me to publish officially my other article "Three Examples of Unbounded Energy for $t>0$"[2], also a very important paper.

This distressful and no way out situation disappears when we impose the boundary condition $\lim_{|x|\to\infty} u(x,t)=0$, which guarantees the desired uniqueness of solutions at least in a finite and not null time interval $[0,T]$. Possibly others boundary conditions also arrive at the uniqueness, but null velocity at infinite may imply a minimum volume of $|u|^2$ and the respective total kinetic energy.

Thus is necessary do some changes in the expressions of external forces, pressures and velocities used in [2] to establish again the breakdown solution in [3], due to occurrence of unbounded energy $\int_{\mathbb{R}^3} |u|^2 dx \to \infty$ in $t>0$. In special, a general example, for $1 \leq i \leq 3$ and $\nabla \cdot u = \nabla \cdot u^0 = \nabla \cdot v = 0$, is

$$u_i(x,t) = u_i^0(x)e^{-t} + v_i(x)e^{-t}(1-e^{-t}),\ u, u^0, v, x \in \mathbb{R}^3,$$

$$u_i^0(x) \in S(\mathbb{R}^3),\ v_i(x) \in C^\infty(\mathbb{R}^3),\ v \notin L^2(\mathbb{R}^3),\ \lim_{|x|\to\infty} v(x) = 0,$$

$$p \in C^\infty\big(\mathbb{R}^3 \times [0,\infty)\big),$$

$$f_i = \left(\frac{\partial p}{\partial x_i} + \frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} - \nu\, \nabla^2 u_i\right) \in S\big(\mathbb{R}^3 \times [0,\infty)\big).$$

The conditions (4) for initial velocity and (5) for external force, conforming description given in [3],

(4) $\quad |\partial_x^\alpha u^0(x)| \leq C_{\alpha K}(1+|x|)^{-K}\colon \mathbb{R}^3,\ \forall \alpha, K$

(5) $|\partial_x^\alpha \partial_t^m f(x,t)| \leq C_{\alpha m K}(1+|x|+t)^{-K}: \mathbb{R}^3 \times [0,\infty)\,, \forall \alpha, m, K$

is a kind of *straitjacket,* and for me do not seem good conditions to make possible physically reasonable solutions, rather only restricts the solutions to a very limited and very artificial set of possibilities. If it were possible to the external force be in the set $C^\infty(\mathbb{R}^3 \times [0,\infty))$, such as the velocity and pressure in $t > 0$, even being only limited functions and equals zero as $|x| \to \infty$, instead Schwartz Space, the possible solutions will be much more interesting and realistic.

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As we know, when $\nabla \times u = 0$ exist a potential function ϕ such that $u = \nabla\phi$. When $\nabla \times u = 0$ and $\nabla \cdot u = 0$ then $\nabla^2\phi = 0$ and $\nabla^2 u = 0$, therefore the Navier-Stokes equations are reduced to Euler's equations and the solutions for velocity are given by Laplace's equation, they are harmonic functions, i.e.,

$$\nabla^2 u = \nabla(\nabla \cdot u) - \nabla \times (\nabla \times u) = (\nabla^2 u_1, \nabla^2 u_2, \nabla^2 u_3) = 0$$

and

$$u = \nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right), \nabla \cdot u = 0 \Longrightarrow \left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right) = 0.$$

It is clear that there is no uniqueness solutions in all cases, in special when the velocity is both irrotational and incompressible, even if the velocity vanishes at infinity. Defining $\phi(x,t) = \phi^0(x)T(t)$, $T(0) = 1$, then we have $u = \nabla\phi = T(t)\nabla\phi^0 = T(t)u^0(x)$ and so there are endless possibilities for constructing u given u^0, because there are endless possibilities for constructing $T(t)$ with $T(0) = 1$, even if $\lim_{|x|\to\infty} u = T(t) \lim_{|x|\to\infty} u^0 = 0$.

According proof in my other paper [4], if $u(x,y,z,0) = u^0(x,y,z)$ is the initial velocity of the system, valid solution in $t = 0$, then $u(x,y,z,t) = u^0(x+t, y+t, z+t)$ is a solution for velocity in $t \geq 0$. Similarly, $p(x,y,z,t) = p^0(x+t,y+t,z+t)$ is the correspondent solution for pressure in $t \geq 0$, being $p^0(x,y,z)$ the initial condition for pressure. More than this, the velocities $u^0(x+t,y,z)$, $u^0(x,y+t,z)$ and $u^0(x,y,z+t)$ are also solutions, and respectively also the pressures $p^0(x+t,y,z)$, $p^0(x,y+t,z)$ and $p^0(x,y,z+t)$. That is, when the velocity is equal to zero at infinity for all $t \geq 0$ there is no uniqueness solutions, in general case. Apparently, an additional complication if the uniqueness condition is required.

Exception to the two previous paragraphs when $u^0 = 0$.

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In line with previous date, if $\nabla\cdot u=0$ and $\nabla\times u=0$ then $\nabla^2 u=0$. For $u=(u_1,u_2,u_3)$ and $w=(w_1,w_2,w_3)$, defining $w_i=A(t)u_i+B_i(t), 1\leq i\leq 3$, we will have $\nabla\cdot w=0$, $\nabla\times w=0$ and $\nabla^2 w=0$.

If $u=\nabla\phi$ solves the Navier-Stokes equations then

$$\nabla p+\frac{\partial u}{\partial t}+(u\cdot\nabla)u=\nu\nabla^2 u$$

$$\nabla p+\nabla\left(\frac{\partial\phi}{\partial t}\right)+(\nabla\times u)\times u+\frac{1}{2}\nabla|u|^2=$$
$$=\nu\big(\nabla(\nabla\cdot u)-\nabla\times(\nabla\times u)\big)$$

$$\nabla p+\nabla\left(\frac{\partial\phi}{\partial t}\right)+\nabla\left(\frac{1}{2}|u|^2\right)=0$$

$$\nabla\left(p+\frac{\partial\phi}{\partial t}+\frac{1}{2}|u|^2\right)=0$$

$$p+\frac{\partial\phi}{\partial t}+\frac{1}{2}|u|^2=\theta(t),$$

which is the Bernouilli's law without external force.

With a gradient external force $f=\nabla U$ we will have

$$p+\frac{\partial\phi}{\partial t}+\frac{1}{2}|u|^2=U+\theta(t).$$

For w defined as above, substituting $u\mapsto w$ in the Navier-Stokes equations comes

$$p+\frac{\partial\varphi}{\partial t}+\frac{1}{2}|w|^2=U+\theta(t),$$

where $\varphi=A(t)\phi$, and p is the new pressure for the velocity w.

If $A(0)=1$ and $B_i(0)=0$ then u and w obey the same initial condition and both solve the Navier-Stokes equations and conditions of incompressible and potential flows. In this case, there is no uniqueness solution.

Imposing the boundary condition at infinity $u|_{r\to\infty}=0, r=\sqrt{x^2+y^2+z^2}$, the velocity $w=A(t)u$ obey the same boundary condition, for $A(t)$ finite for all $t\geq 0$, i.e. $w(x,y,z,t)=A(t)u(x,y,z,t)$ and $u(x,y,z,t)$ obey the same initial and boundary conditions, so there is no uniqueness solutions for Navier-Stokes (and Euler) equations, in this case of incompressible and potential flows with velocity zero at infinity. Exception for initial velocity $u^0=0$, when $u=0$ is unique solution.

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References

[1] Godoi, Valdir M.S., *A Naive Solution for Navier-Stokes Equations*, in http://vixra.org/abs/1604.0107 (2016).

[2] Godoi, Valdir M.S., *Three Examples of Unbounded Energy for* $t > 0$, in http://vixra.org/abs/1602.0246 (2016).

[3] Fefferman, Charles L., *Existence and Smoothness of the Navier-Stokes Equation*, in http://www.claymath.org/sites/default/files/navierstokes.pdf (2000).

[4] Godoi, Valdir M.S., *Two Theorems on Solutions in Eulerian Description*, available in http://www.vixra.org/abs/1605.0236 (2016).