Mathematical Formulas : Part 5

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Abstract

Some formulas related with the constant pi :

$\pi = 3.14159265358979\ldots$

Keywords: number pi

1. Introducción : notación

$$\mathbb{N} = \{1, 2, 3, \ldots\} \tag{1}$$

$$\mathbb{N}_0 = \mathbb{N} \bigcup \{0\} \tag{2}$$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \;,\; n \geq k \tag{3}$$

$$(a)_0 = 1 \;,\; (a)_n = a\,(a+1)\,(a+2) \ldots (a+n-1) \;,\; n \in \mathbb{N} \tag{4}$$

$$B_n = \left\{\frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \frac{1}{30}, \frac{5}{66}, \ldots\right\} \;,\; \text{números de Bernoulli} \tag{5}$$

Polinomios de Bernoulli:

$$\frac{t\,e^{t\,x}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\,\frac{t^n}{n!} \;,\; |t| < 2\,\pi \tag{6}$$

$$H_n = \sum_{k=1}^{n} \frac{1}{k} \;,\; H_{n,s} = \sum_{k=1}^{n} \frac{1}{k^s} \;,\; n, s \in \mathbb{N} \tag{7}$$

$$i = \sqrt{-1} \tag{8}$$

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} \;,\; |z| \leq 1 \tag{9}$$

$$\mathrm{Li}_3(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^3} \;,\; |z| \leq 1 \tag{10}$$

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = 2.71828\ldots \tag{11}$$

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2\,n+1)^2} \tag{12}$$

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n} \;,\; n > 1 \tag{13}$$

$$\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n\right) \tag{14}$$

2. Fórmulas

$$\pi = 8 \sum_{n=0}^{\infty} \frac{1}{2n+1} \left(\frac{a(p,q,n)}{(p^2 + 4q^2)^{2n+1}} + \frac{b(p,q,n)}{(4(p+q)^2 + (p-q)^2)^{2n+1}} \right) \;,\; p, q \in \mathbb{N},\; p < q \tag{15}$$

$$a(p,q,n) = \sum_{m=0}^{n} (-1)^m \binom{2n+1}{2m+1} p^{4n-2m+1} (2q)^{2m+1} \tag{16}$$

$$b(p,q,n) = \sum_{m=0}^{n} (-1)^{m-1} \binom{2n+1}{2m+1} (p-q)^{4n-4m} \left(2\left(p^2 - q^2\right)\right)^{2m+1} \tag{17}$$

$$\pi = 4 \sum_{n=1}^{\infty} \left(H_{2n} - \frac{1}{2} H_n\right) \left(\frac{1}{(4n-3) H_{2n-1}} - \frac{1}{(4n-1) H_{2n}} \right) - \sum_{n=1}^{\infty} \frac{H_{2n}}{2} \left(\frac{1}{(4n-1) H_{2n}} - \frac{1}{(4n+1) H_{2n+1}} \right) \;,\; n \in \mathbb{N} \tag{18}$$

$$\pi = 4 \sum_{n=1}^{\infty} A_n \left(\frac{1}{(2n-1)(a^n + b^n)} - \frac{1}{(2n+1)(a^{n+1} + b^{n+1})} \right) \;,\; a, b \in \mathbb{N} \tag{19}$$

$$A_n = \frac{a(1 - (-a)^n)}{1 + a} + \frac{b(1 - (-b)^n)}{1 + b} \;,\; n \in \mathbb{N} \;,\; A_n \in \mathbb{Z} \tag{20}$$

$$\pi = 4 \sum_{n=1}^{\infty} A_n \left(\frac{1}{(2n-1) x_n} - \frac{1}{(2n+1) x_{n+1}} \right) \tag{21}$$

$$\begin{cases} x_{n+2} = (a+b) x_{n+1} - a b x_n \;,\; x_1 = a + b \;,\; x_2 = a^2 + b^2 \;,\; a, b \in \mathbb{N} \\ A_{n+1} = A_n + (-1)^n (a^{n+1} + b^{n+1}) \;,\; A_1 = a + b \;,\; a, b \in \mathbb{N} \end{cases} \tag{22}$$

$$\pi = \lim_{n\to\infty} \frac{2^{n+1}}{3} \left(\underbrace{\sqrt{2 - \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}}}_{\longleftarrow\; n \text{ radicales} \;\longrightarrow} + \underbrace{\sqrt{2 - \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}}}_{\longleftarrow\; n+1 \text{ radicales} \;\longrightarrow} \right) \tag{23}$$

$$\pi = \lim_{n\to\infty} 2^{n+1} \left(\underbrace{\sqrt{2 - \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}}}_{\longleftarrow\; n \text{ radicales} \;\longrightarrow} - \underbrace{\sqrt{2 - \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}}}_{\longleftarrow\; n+1 \text{ radicales} \;\longrightarrow} \right) \tag{24}$$

$$\pi = \lim_{n\to\infty} \frac{2^{2n+4}}{3} \left(\underbrace{\sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}}}_{\longleftarrow\; n+1 \text{ radicales} \;\longrightarrow} - \underbrace{\sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}}}_{\longleftarrow\; n \text{ radicales} \;\longrightarrow} \right) \tag{25}$$

$$\frac{1}{\pi} = \frac{3}{2^n} \left(3 \underbrace{\sqrt{2 - \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}}}_{\longleftarrow\ -\ -\ -\ n\ \text{radicales}\ -\ -\ -\ \longrightarrow} - \left(\underbrace{\sqrt{2 - \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}}}_{\longleftarrow\ -\ -\ -\ n\ \text{radicales}\ -\ -\ -\ \longrightarrow} \right)^3 \right)^{-1} \prod_{k=1}^{\infty} \left(1 - \left(\frac{3}{2^{n+1}\, k} \right)^2 \right) \tag{26}$$

$$n \in \mathbb{N}$$

$$\pi + 3 \ln\left(\frac{\sqrt{3} + 1}{\sqrt{3} - 1} \right) = 12 \sqrt{3} \sum_{n=1}^{\infty} \frac{3^{-(2n-1)}}{4n - 3} \tag{27}$$

$$\pi - 3 \ln\left(\frac{\sqrt{3} + 1}{\sqrt{3} - 1} \right) = -12 \sqrt{3} \sum_{n=1}^{\infty} \frac{3^{-2n}}{4n - 1} \tag{28}$$

$$\pi + 2 \ln 2 + 2 \ln 3 = 8 \sum_{n=1}^{\infty} \frac{2^{-(4n-3)} + 3^{-(4n-3)}}{4n - 3} \tag{29}$$

$$\pi - 2 \ln 2 - 2 \ln 3 = -8 \sum_{n=1}^{\infty} \frac{2^{-(4n-1)} + 3^{-(4n-1)}}{4n - 1} \tag{30}$$

$$\pi = 6 \sum_{k=1}^{n-1} \frac{\operatorname{Im}\left(\left(1 + i \sqrt{3} \right)^k \right)}{2^{2k}\, k} + 6 \sum_{k=0}^{\infty} \frac{(-1)^k\, (n)_k\, \operatorname{Im}\left(i^{k+n} \right)}{k!\, (k + n) \left(\sqrt{3} \right)^{k+n}}\ ,\ n \in \mathbb{N} \tag{31}$$

$$\pi = 4 \sum_{k=1}^{n-1} \frac{1}{k} \left(\frac{\operatorname{Im}\left((1 + 2i)^k \right)}{5^k} + \frac{\operatorname{Im}\left((1 + 3i)^k \right)}{10^k} \right) + 4 \sum_{k=0}^{\infty} \frac{(-1)^k\, (n)_k \left(2^{-k-n} + 3^{-k-n} \right) \operatorname{Im}\left(i^{k+n} \right)}{k!\, (k + n)}\ ,\ n \in \mathbb{N} \tag{32}$$

$$\pi = 4 \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} (1 - x)^{n-k}\, x^k\, \operatorname{Im}\left(i^k \right)\ ,\ 0 < x < 1 \tag{33}$$

$$\pi = 4 \sum_{n=1}^{\infty} \frac{1}{n\, 2^n} \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} \operatorname{Im}\left(i^k \right)\ ,\ 0 < x < 1 \tag{34}$$

$$\pi = 4 \sum_{n=1}^{\infty} \frac{1}{n\, 3^n} \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} 2^{n-k} \operatorname{Im}\left(i^k \right)\ ,\ 0 < x < 1 \tag{35}$$

$$\pi = 4 \sqrt{3} + 3 \ln\left(\frac{\sqrt{3} - 1}{\sqrt{3} + 1} \right) + 4 \sqrt{3} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{3^{-2^{n+1} (2k+1)}}{2^{n+2} (2k + 1) + 1} \tag{36}$$

$$\pi = 4 \sqrt{3} + 3 \ln\left(\frac{\sqrt{3} - 1}{\sqrt{3} + 1} \right) + 4 \sqrt{3} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{3^{-2^{k+1} (2n-2k+1)}}{2^{k+2} (2n - 2k + 1) + 1} \tag{37}$$

$$\pi \sqrt{3}\ \zeta(s) = 18 \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{(2k - 1)^k} + 6 \sum_{n=1}^{\infty} \frac{(-1)^n\, H_{n,s}}{(2n + 1)\, 3^n}\ ,\ s > 1 \tag{38}$$

$$\pi = 4 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\, y_n \tag{39}$$

$$y_{n+2} = 2x\, y_{n+1} - \left(2x^2 + 2x + 1 \right) y_n\ ,\ y_1 = 1 + x\,,\ y_2 = 2x\,(2 + x)\,,\ -1 < x < 0 \tag{40}$$

$$\pi = 4 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\, q^n}\, y_n \tag{41}$$

$$y_{n+2} = -2\,p\,y_{n+1} - \left(p^2 + (q-p)^2\right) y_n \;,\; y_1 = q - p \;,\; y_2 = -2\,p\,(q-p) \;,\; p, q \in \mathbb{N},\; p < q \tag{42}$$

$$\pi = \frac{6\,(3\,x^2 - 1)}{x\,\sqrt{3}} \sum_{n=1}^{\infty} \left(\frac{1}{3\,x^2}\right)^n \sum_{k=1}^{n} \frac{(-1)^{k-1}\,x^{2\,k-1}}{2\,k-1} \;,\; \frac{1}{\sqrt{3}} < x \le 1 \tag{43}$$

$$\pi = 4 \sum_{n=1}^{\infty} (-1)^{n-1}\,((n+1)! - 1) \left(\frac{1}{(2\,n-1)\,n\,n!} + \frac{1}{(2\,n+1)\,(n+1)\,(n+1)!}\right) \tag{44}$$

$$\pi = 4\,\sqrt{\frac{2}{3}} \sum_{n=0}^{\infty} \binom{2\,n}{n} \frac{(-1)^n}{2^{2\,n}\,3^n} \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k} \binom{k}{j} \frac{(-1)^k\,3^{-k-j}}{2\,n + 2\,k + 2\,j + 1} \tag{45}$$

$$\pi = 6\,\sqrt{\frac{2}{7}} \sum_{n=0}^{\infty} \binom{2\,n}{n} \frac{(-1)^n}{2^{2\,n}\,7^n} \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k} \binom{k}{j} \frac{(-1)^k\,7^{-k-j}}{2\,n + 2\,k + 2\,j + 1} \tag{46}$$

$$\pi = 6\,a_0\!\left(\frac{1}{\sqrt{3}}\right) + 6 \sum_{n=1}^{\infty} (-1)^n\,c_n\,a_n\!\left(\frac{1}{\sqrt{3}}\right) \tag{47}$$

$$a_n(x) = \int_0^x J_n\!\left(t^2\right) dt \;,\; n \in \mathbb{N}_0 \tag{48}$$

$$c_n = \sum_{k=0}^{[n/2]} \frac{n\,(n-k-1)!\,2^{n-2\,k}}{k!} \;,\; n \in \mathbb{N} \tag{49}$$

En (48), $J_n(x)$ es la función de Bessel de orden n

$$\pi = 2 \ln \prod_{k=1}^{\infty} \left(\frac{k^2 + k + 2}{k^2 + k}\right) - 8 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{(4\,n-1)} \left(\frac{1}{k^2 + k + 1}\right)^{4\,n-1} \tag{50}$$

$$\pi = 3 \ln \prod_{k=1}^{\infty} \left(\frac{3\,k^2 + 3\,k + 1 + \sqrt{3}}{3\,k^2 + 3\,k + 1 - \sqrt{3}}\right) - 12 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{(4\,n-1)} \left(\frac{\sqrt{3}}{3\,k^2 + 3\,k + 1}\right)^{4\,n-1} \tag{51}$$

$$\pi = 3 \ln \prod_{k=1}^{\infty} \left(\frac{\left(\sqrt{3}\right)^{2\,k+1} + \left(\sqrt{3}\right)^{k+1} + \left(\sqrt{3}\right)^{k} + 1}{\left(\sqrt{3}\right)^{2\,k+1} - \left(\sqrt{3}\right)^{k+1} - \left(\sqrt{3}\right)^{k} + 1}\right) - 12 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{(4\,n-1)} \left(\frac{\left(\sqrt{3} + 1\right) 3^{k/2}}{3^k\,\sqrt{3} + 1}\right)^{4\,n-1} \tag{52}$$

$$\frac{\pi}{4\,\sqrt{3}} + \frac{3}{8} = \frac{1}{a} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{n+k}\,(a + b\,n)}{3^n\,(2\,n + 2\,k + 1)} \left(\frac{a-b}{3\,a}\right)^k \;,\; 0 < b < a \tag{53}$$

$$\frac{\pi}{12} + \frac{\sqrt{3}}{8} = \frac{1}{b-a} \sum_{n=0}^{\infty} (a + b\,n) \left(\frac{a}{a-b}\right)^n A_n \;,\; 0 < b < a \tag{54}$$

$$A_n = \sqrt{\frac{a-b}{a}}\,\tan^{-1}\!\left(\sqrt{\frac{a-b}{3\,a}}\right) + \sqrt{3} \sum_{k=1}^{n} \frac{(-1)^k}{2\,k-1} \left(\frac{a-b}{3\,a}\right)^k ,\; 0 < b < a \tag{55}$$

$$\frac{\pi}{12} + \frac{\sqrt{3}}{8} = \frac{1}{b-a} \sum_{n=0}^{\infty} (-1)^n\,(a + b\,n) \left(\frac{a}{a-b}\right)^n C_n \;,\; 0 < a < b \le 2\,a \tag{56}$$

$$C_n = \frac{1}{2} \sqrt{\frac{b-a}{a}}\,\ln\!\left(\frac{\sqrt{3\,a} + \sqrt{b-a}}{\sqrt{3\,a} - \sqrt{b-a}}\right) - \sqrt{3} \sum_{k=1}^{n} \frac{1}{2\,k-1} \left(\frac{b-a}{3\,a}\right)^k ,\; 0 < a < b \le 2\,a \tag{57}$$

$$\pi = 32 \sum_{n=1}^{\infty} \frac{10^{-n}}{n} \operatorname{sen}(n\,\alpha) \tag{58}$$

$$\operatorname{sen}\alpha = \frac{x}{1+x^2}\left(10 - \sqrt{1-99\,x^2}\right),\ \ x = \sqrt{\frac{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}\ ,\ \ \alpha = 1.2731\ \ldots \tag{59}$$

$$\pi = 12\sqrt{10} \sum_{n=1}^{\infty} \frac{10^{-n}}{2\,n-1} \operatorname{sen}((2\,n-1)\,\beta) \tag{60}$$

$$\operatorname{sen}\beta = \frac{9}{2\sqrt{30}}\ ,\ \ \beta = 0.9641\ \ldots \tag{61}$$

$$\pi = 12\sqrt{10} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\,10^{-n}}{2\,n-1} \cos((2\,n-1)\,\delta) \tag{62}$$

$$\operatorname{sen}\delta = \frac{1}{2}\sqrt{\frac{13}{10}}\ ,\ \ \delta = 0.6066\ \ldots \tag{63}$$

$$\pi = 8 \sum_{n=0}^{\infty} \frac{1}{2\,n+1}\left[\operatorname{Im}\left(\left(\frac{x-1+i\,y}{x+1+i\,y}\right)^{2\,n+1}\right) + \operatorname{Im}\left(\left(\frac{u-1+i\,v}{u+1+i\,v}\right)^{2\,n+1}\right)\right] \tag{64}$$

$$x,\ y,\ u,\ v \in \mathbb{N},\ 0 < y < x\ ,\ \ 0 < u < v\ ,\ \ x\,u + y\,v = x\,v - y\,u$$

$$\frac{\pi}{\sqrt{b^2-a^2}} = \sum_{n=0}^{\infty}(-1)^n \sum_{k=0}^{n}\binom{n}{k}\left(\frac{1+(-1)^k\,b^{2\,n-2\,k}(2\,a)^k\,A^{-2\,n+k-1}}{2\,n-k+1}\right) + 2\,A \sum_{n=1}^{\infty}\sum_{m=1}^{2^n-1} \frac{(-1)^{m+1}\,2^n}{2^{2\,n}\,b^2 + 2^{n+1}\,a\,A\,(2\,m-2^n) + A^2(2\,m-2^n)^2} \tag{65}$$

$$b > |a|\ ,\ \ A > |a| + \sqrt{a^2+b^2}$$

$$\frac{\pi}{4} = \sum_{k=1}^{\infty} \frac{1}{k} \operatorname{Im}\left(\left(\frac{n-1+i\,n}{n+i\,n}\right)^k\right),\ \ n \in \mathbb{N} \tag{66}$$

$$\frac{\pi}{4} = \sum_{k=1}^{\infty} \frac{1}{k\,(2\,n)^k} \operatorname{Im}\left((2\,n-1+i)^k\right),\ \ n \in \mathbb{N} \tag{67}$$

$$\frac{\pi}{4} = \sum_{k=1}^{\infty} \frac{1}{k\,(2\,n)^k} \sum_{m=0}^{[(k-1)/2]} (-1)^m \binom{k}{2\,m+1} (2\,n-1)^{k-2\,m-1}\ ,\ \ n \in \mathbb{N} \tag{68}$$

$$\pi\sqrt{3} = \sum_{k=1}^{\infty} \frac{1}{k\,2^{5\,k}} \sum_{m=0}^{[(2\,k-1)/2]} (-1)^m \binom{2\,k}{2\,m+1} 3^{3\,k-3\,m} \tag{69}$$

$$\pi = 4 \sum_{n=0}^{\infty} (-1)^n \left(\frac{a^n\sqrt{a} + b^n\sqrt{b}}{2\,n+1}\right)\left(\frac{\sqrt{a+b+6\sqrt{a\,b}} - \sqrt{a} - \sqrt{b}}{2\sqrt{a\,b}}\right)^{2\,n+1}\ ,\ \ a,\ b \in \mathbb{N} \tag{70}$$

$$\frac{\pi\sqrt{3}}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n\,F_n}{(2\,n+1)\,3^{n-1}} - \sum_{n=0}^{\infty} \frac{(-1)^n\,(F_{n+2}-1)}{(2\,n+5)\,3^{n+1}} \tag{71}$$

$$F_{n+2} = F_{n+1} + F_n\ ,\ \ F_0 = F_1 = 1 \tag{72}$$

$$\frac{\pi}{4}\sqrt{\frac{1+2m^2}{1+m^2}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)(1+2m^2)^n}\sum_{k=0}^{n}\binom{2k}{k}(-1)^k\left(\frac{m}{2}\right)^{2k} \ , \ m\in\mathbb{N} \tag{73}$$

$$\frac{\pi}{8} = \sum_{n=0}^{\infty}\frac{(-1)^n a^{4n+2}}{4n+2} + \sum_{n=0}^{\infty}(-1)^n\left(\frac{1-a}{1+a^4}\right)^{n+1}\sum_{m=0}^{n}\sum_{k=0}^{m}\sum_{j=0}^{k} f(n,m,k,j,a) \ , \ 0<a\leq 1 \tag{74}$$

$$f(n,m,k,j,a) = \binom{n}{m}\binom{m}{k}\binom{k}{j} * \frac{2^{2n-m+k-2j}\,3^{m-k}\,a^{3n-m-k-j}(1-a)^{m+k+j}\,(n+m+k+j+1+a)}{(n+m+k+j+1)(n+m+k+j+2)} \tag{75}$$

$$\frac{\pi}{8} = \sum_{n=0}^{\infty}\frac{(-1)^n a^{4n+2}}{4n+2} + \sum_{n=0}^{\infty}(-1)^n c_n\left(\frac{1-a}{2}\right)^{n+1}\frac{(na+1+a)}{(n+1)(n+2)} \ , \ 0.2347\ldots<a\leq 1 \tag{76}$$

$$c_n = -4c_{n-1} - 12c_{n-2} - 16c_{n-3} - 8c_{n-4} \ , \ c_0=1,\ c_1=-4,\ c_2=4,\ c_3=16 \tag{77}$$

$$\pi = 32\sum_{n=0}^{\infty}\frac{(2n+3+(-1)^n)\,a_n}{17^{n+1}(n+1)(n+2)} \tag{78}$$

$$a_n = -4a_{n-1} - 102a_{n-2} - 1156a_{n-3} - 4913a_{n-4} \ , \ a_0=1,\ a_1=-4,\ a_2=-86,\ a_3=-404 \tag{79}$$

$$\frac{\pi}{4}\sqrt{\frac{1+2m^2}{1+m^2}} = \sum_{n=0}^{\infty}\frac{(-1)^n 2^{-2n}}{(1+2m^2)^n}\binom{2n}{n} f(n,k,j,m) \ , \ m\in\mathbb{N} \tag{80}$$

$$f(n,k,j,m) = \sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k}\binom{k}{j}\frac{(-1)^j(2-m^2)^{n-k}(1-2m^2)^{k-j}m^{2j}}{(2n+2k+2j+1)(1+2m^2)^{k+j}} \tag{81}$$

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{(1+a)^{n+1}}\sum_{k=0}^{n}\binom{n}{k}\frac{(-a)^k}{(2n-2k+1)} \ , \ a>0 \tag{82}$$

$$\begin{aligned}\frac{\pi}{8} = 1-\ln 2 + \sum_{n=1}^{\infty}(-1)^n\sum_{k=0}^{n}\binom{n}{k}\Bigg(&\sum_{m=0}^{n-1}(-1)^{m+1}\binom{4n-3k+1}{m}\left(\frac{2^{m-n}-1}{m-n}\right) + \\ &\sum_{m=n+1}^{4n-3k+1}(-1)^{m+1}\binom{4n-3k+1}{m}\left(\frac{2^{m-n}-1}{m-n}\right) + (-1)^{n+1}\binom{4n-3k+1}{n}\ln 2\Bigg)\end{aligned} \tag{83}$$

$$\pi = 6\sqrt{\frac{1+\sqrt{5}}{2}}\sum_{m=0}^{\infty}\left(F_m\left(\frac{1+\sqrt{5}}{2}\right) - F_{m+1}\right)\left(\frac{1}{2m+1}\right)\left(\sqrt{\frac{1+\sqrt{5}}{6}}\right)^{2m+1} \tag{84}$$

$$F_{n+2} = F_{n+1} + F_n \ , \ F_0 = F_1 = 1 \tag{85}$$

$$\pi = 6\sqrt{3}\sum_{n=0}^{\infty}\frac{(\ln 2)^{2n+1}}{(2n+1)\,2^{2n+1}}c_n \tag{86}$$

$$\pi = 8\sqrt{3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)\,2^{2n+1}}\left(\ln\left(\frac{\sqrt{3}+\sqrt{2}-1}{\sqrt{3}-\sqrt{2}+1}\right)\right)^{2n+1} c_n \tag{87}$$

$$c_0 = 1 \ , \ c_n = -4\sum_{k=1}^{n}\frac{2^{2k-1}}{(2k)!}c_{n-k} \ , \ n\in\mathbb{N} \tag{88}$$

$$\begin{cases} u_{n+2} = \frac{1}{2}u_{n+1} + u_n + \frac{1}{2}u_{n+1}^{-1}(6(n+1)^{-2} - u_n^2) \ , \ n\in\mathbb{N} \\ u_1 = 3 \ , \ u_2 = 5/2 \ , \ u_n\to\pi \end{cases} \tag{89}$$

$$\begin{cases} u_{n+2} = \frac{1}{2}\, u_{n+1} + u_n + \frac{1}{2}\, u_{n+1}^{-1} (8\,(2\,n+1)^{-2} - u_n^2)\ ,\ n \in \mathbb{N} \\ u_1 = 3\ ,\ u_2 = 17/6\ ,\ u_n \to \pi \end{cases} \tag{90}$$

$$\begin{cases} u_{n+2} = \frac{1}{2}\, u_{n+1} + u_n + \frac{1}{2}\, u_{n+1}^{-1} \left(18\,(n+1)^{-2} \binom{2\,n+2}{n+1}^{-1} - u_n^2 \right)\ ,\ n \in \mathbb{N} \\ u_1 = 3\ ,\ u_2 = 3\ ,\ u_n \to \pi \end{cases} \tag{91}$$

$$\begin{cases} u_{n+2} = \frac{1}{2}\, u_{n+1} + u_n + \frac{1}{2}\, u_{n+1}^{-1} (12\,(-1)^n\,(n+1)^{-2} - u_n^2)\ ,\ n \in \mathbb{N} \\ u_1 = 3\ ,\ u_2 = 7/2\ ,\ u_n \to \pi \end{cases} \tag{92}$$

$$\begin{cases} u_{n+2} = \frac{1}{2}\, u_{n+1} + u_n + \frac{1}{2}\, u_{n+1}^{-1} \left((-1)^{n+1} \binom{2\,n+2}{n+1}^5 \left(\frac{20\,n^2+48\,n+29}{2^{12\,n+15}} \right) - u_n^2 \right)\ ,\ n \in \mathbb{N}_0 \\ u_0 = 1/3\ ,\ u_1 = 17/48\ ,\ u_n \to 1/\pi \end{cases} \tag{93}$$

$$\pi = 2\sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(1+i)^n} \sum_{k=0}^{n} \binom{n}{k} \frac{(-i)^k}{(2\,n-2\,k+1)\,3^{n-k}}\ ,\ i = \sqrt{-1} \tag{94}$$

$$\pi = 8\,\mathrm{sh}\left(\frac{1}{2}\right) \sum_{n=0}^{\infty} \frac{\mathrm{sh}\left(n+\frac{1}{2}\right)}{(\mathrm{ch}\,n)\,(\mathrm{ch}(n+1))} \sum_{k=0}^{n} \frac{(-1)^k\,\mathrm{ch}\,k}{2\,k+1} \tag{95}$$

$$\frac{343}{15\,\pi^3} = \prod_{n=1}^{\infty} \left(1 - \left(\frac{1}{14\,n}\right)^2\right)\left(1 - \left(\frac{3}{14\,n}\right)^2\right)\left(1 - \left(\frac{5}{14\,n}\right)^2\right) \tag{96}$$

$$\pi = \sum_{n=1}^{\infty} \sum_{m=1}^{2^n-1} \frac{(-1)^{m+1}\,2^{n+2}}{2^{2\,n} + m^2} \tag{97}$$

$$\pi = \sum_{n=1}^{\infty} \sum_{m=1}^{2^n-1} \frac{(-1)^{m+1}\sqrt{2^{2\,n} - m^2}}{2^{2\,n-2}} \tag{98}$$

$$\pi = \frac{1 - e^{-\pi\,x_1}}{x_1} - \sum_{n=1}^{\infty} \left(\frac{1 - e^{-\pi\,x_n}}{x_n} - \frac{1 - e^{-\pi\,x_{n+1}}}{x_{n+1}} \right) \tag{99}$$

$$n \in \mathbb{N}\,,\ 0 < x_n < x_{n+1}\ ,\ \lim_{n\to\infty} x_n = 0 \tag{100}$$

$$\pi = 1 - e^{-\pi} + \sum_{n=1}^{\infty} \left(1 - (n+1)\,e^{-\pi/(n+1)} + n\,e^{-\pi/n}\right) \tag{101}$$

$$\frac{\pi}{4} = \tan^{-1}\left(\frac{e^{x_1} - 1}{x_1}\right) - \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{x_{n+1}(e^{x_n} - 1) - x_n(e^{x_{n+1}} - 1)}{x_n\,x_{n+1} + (e^{x_n} - 1)\,(e^{x_{n+1}} - 1)} \right) \tag{102}$$

$$n \in \mathbb{N}\,,\ 0 < x_n < x_{n+1}\ ,\ \lim_{n\to\infty} x_n = 0 \tag{103}$$

$$\frac{\pi}{4} = \tan^{-1}\left(2\,e^{-1}\right) - \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{(n+1)^{2\,n+1} - (n^2+n)^n\,(n+2)}{e\,n^n(n+1)^n + e^{-1}(n+1)^n\,(n+2)^{n+1}} \right) \tag{104}$$

$$\frac{\pi}{4} = \tan^{-1}\left(\frac{\mathrm{sen}\,x_1}{x_1}\right) - \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{x_{n+1}\,\mathrm{sen}\,x_n - x_n\,\mathrm{sen}\,x_{n+1}}{x_n\,x_{n+1} + (\mathrm{sen}\,x_n)\,(\mathrm{sen}\,x_{n+1})} \right) \tag{105}$$

$$n \in \mathbb{N}\,,\ 0 < x_n < x_{n+1}\ ,\ \lim_{n\to\infty} x_n = 0 \tag{106}$$

$$\pi = 4 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\,2^{2\,n-1}\,(2^{2\,n} - 1)\,B_n\,\mathrm{Im}(z^{2\,n})}{n\,(2\,n)!} \tag{107}$$

$$z=\frac{1}{2}\ln\left(\frac{\tan 1+1}{\tan 1-1}\right)+i\ ,\ z=1+i\tan^{-1}\left(\frac{e^2+1}{e^2-1}\right),\ B_n \text{ números de Bernoulli} \tag{108}$$

$$\pi=4\sum_{n=1}^{\infty}\frac{2^{2n-1}(2^{2n}-1)B_n\operatorname{Im}(z^{2n})}{n(2n)!} \tag{109}$$

$$z=1+\frac{i}{2}\ln\left(\frac{\tan 1+1}{\tan 1-1}\right)+i\ ,\ z=i+\tan^{-1}\left(\frac{e^2+1}{e^2-1}\right),\ B_n \text{ números de Bernoulli} \tag{110}$$

$$\pi=3\sqrt{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(1+a)^{n+1}}\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}\frac{(-a)^{n-k}}{k+m+1}\ ,\ a>1/2 \tag{111}$$

$$\frac{\ln 2}{2}+i\frac{\pi}{4}=\sum_{n=0}^{\infty}\binom{2n}{n}\frac{(-1)^n(1+3i)^{2n+1}}{(2n+1)2^{6n+2}}\ ,\ i=\sqrt{-1} \tag{112}$$

$$\pi=2\sum_{n=0}^{\infty}\frac{(-1)^n\left(\sqrt{m^4+6m^2+1}-m^2-1\right)^{2n+1}(1+m^{-4n-2})}{(2n+1)2^{2n}}\ ,\ m\in\mathbb{N}-\{1\} \tag{113}$$

$$\pi=\sqrt{3}\sum_{n=0}^{\infty}\frac{(-1)^n\left(\sqrt{9m^4+30m^2+9}-3m^2-3\right)^{2n+1}(1+m^{-4n-2})}{(2n+1)12^n}\ ,\ m\in\mathbb{N}-\{1\} \tag{114}$$

$$\pi=8\sum_{n=0}^{\infty}\frac{1}{(2n+1)}(-1)^n\left(\left(\sqrt{2}+1\right)/2\right)^{2n+1}\left(\sqrt{m^4+14m^2+1-8m^2\sqrt{2}}-m^2-1\right)^{2n+1}\left(1+m^{-4n-2}\right)\ ,\ m\in\mathbb{N}-\{1\} \tag{115}$$

$$\pi=\sum_{k=0}^{n}\frac{2^{2k+1}}{\binom{2k}{k}(2k+1)}+\frac{2^{2n+4}}{\binom{2n+2}{n+1}}\int_0^1\frac{x^{2n+2}}{(1+x^2)^{n+2}}\,dx\ ,\ n\in\mathbb{N}_0 \tag{116}$$

$$\pi=\sum_{k=0}^{n}\frac{2^{2k+1}}{\binom{2k}{k}(2k+1)}+\frac{2^{2n+4}}{\binom{2n+2}{n+1}}\sum_{j=1}^{\infty}\sum_{m=1}^{2^j-1}\frac{(-1)^{m+1}m^{2n+2}2^j}{(2^{2j}+m^2)^{n+2}}\ ,\ n\in\mathbb{N}_0 \tag{117}$$

$$\pi=4\tan^{-1}\left(1-x^2\right)+4\sum_{m=0}^{\infty}\tan^{-1}\left(\frac{(1-x)x^{2^{m+1}}P(x,m)}{1+(1-x)^2\left(1+x^{2^{m+1}}\right)(P(x,m))^2}\right)\ ,\ |x|<1 \tag{118}$$

$$P(x,m)=\prod_{k=0}^{m}\left(1+x^{2^k}\right)\ ,\ |x|<1\ ,\ m\in\mathbb{N}_0 \tag{119}$$

$$\pi=4\tan^{-1}(3/4)+4\sum_{m=0}^{\infty}\tan^{-1}\left(\frac{2^{1-2^{m+1}}P(1/2,m)}{4+\left(1+2^{-2^{m+1}}\right)(P(1/2,m))^2}\right) \tag{120}$$

$$P(1/2,m)=\prod_{k=0}^{m}\left(1+2^{-2^k}\right)\ ,\ m\in\mathbb{N}_0 \tag{121}$$

$$\pi=4+4\sum_{m=1}^{\infty}\frac{(-1)^m H_{2m}}{h_{2m}}-8\sum_{m=1}^{\infty}\frac{(-1)^m m}{(2m+1)h_{2m}} \tag{122}$$

$$\pi^2=6\sum_{m=1}^{\infty}\frac{H_m}{m\,h_m}+6\sum_{m=1}^{\infty}\frac{H_m}{m^2h_m}-6\sum_{m=1}^{\infty}\frac{1}{m\,h_m} \tag{123}$$

$$h_m = \sum_{k=1}^{m} H_k \ , \ H_m = \sum_{k=1}^{m} \frac{1}{k} \ , \ m \in \mathbb{N} \tag{124}$$

$$\frac{16}{\pi} = \sum_{n=0}^{\infty} \frac{(42\,n+5)}{2^{12\,n}} \binom{2\,n}{n}^3 \phi^{-7\,n} - 3 \sum_{n=0}^{\infty} \frac{(2\,n+1)}{2^{12\,n}} \binom{2\,n}{n}^3 \phi^{-8\,n} \ , \ \phi = \frac{1+\sqrt{5}}{2} \tag{125}$$

$$\frac{32}{\pi} = \sum_{n=0}^{\infty} \binom{2\,n}{n}^3 \frac{(5\,(42\,n+5)\,C_n + (30\,n-1)\,A_n)}{2^{20\,n}} + \sqrt{5} \sum_{n=0}^{\infty} \binom{2\,n}{n}^3 \frac{((42\,n+5)\,A_n + (30\,n-1)\,C_n)}{2^{20\,n}} \tag{126}$$

$$\frac{32}{\pi} = \sum_{n=0}^{\infty} \binom{2\,n}{n}^3 \frac{(30\,(A_n + 7\,C_n)\,n + (-A_n + 25\,C_n)}{2^{20\,n}} + \sqrt{5} \sum_{n=0}^{\infty} \binom{2\,n}{n}^3 \frac{(6\,(7\,A_n + 5\,C_n) + (5\,A_n - C_n))}{2^{20\,n}} \tag{127}$$

$$A_{n+1} = 6016\,A_n - 13\,440\,C_n \ , \ C_{n+1} = -2688\,A_n + 6016\,C_n \ , \ A_0 = 1, \ C_0 = 0 \tag{128}$$

$$A_n = 2^{7\,n-1} \left(47 + 21\sqrt{5}\right)^{-n} \left(4^n + \left(47 + 21\sqrt{5}\right)^{2\,n}\right) \tag{129}$$

$$C_n = -\frac{2^{7\,n-1} \left(47 + 21\sqrt{5}\right)^{-n+1} \left(-4^n + \left(47 + 21\sqrt{5}\right)^{2\,n}\right)}{105 + 47\sqrt{5}} \tag{130}$$

$$\pi = 4\tan^{-1}(e^{-y}) + 2\,\mathrm{sh}\,y - \sum_{n=1}^{\infty} \frac{2^{2\,n}}{(2\,n)!}\, c(y,\ n) \ , \ y > 0 \tag{131}$$

$$c(y,\ n) = \int_0^y \frac{x^{2\,n}}{\mathrm{ch}\,x}\, dx \ , \ n \in \mathbb{N}, \ y > 0 \tag{132}$$

$$\pi = 4\tan^{-1}(e^{-y}) + 2 \sum_{n=0}^{\infty} \frac{y^{2\,n+1}}{(2\,n+1)\,(2\,n+1)!} - \sum_{n=1}^{\infty} \frac{2^{2\,n+1}}{(2\,n+1)!}\, c(y,\ n) \ , \ y > 0 \tag{133}$$

$c(y,\ n)$ definida como en (132)

$$\pi = 4\tan^{-1}(e^{-y}) + 2\,\mathrm{th}\,y + 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\,E_n}{(2\,n)!}\, c(y,\ n) \ , \ 0 < y < \frac{\pi}{2} \tag{134}$$

$c(y,\ n)$ definida como en (132) , $E_n = \{1,\ 5,\ 61,\ 1385,\ \ldots\}$ números de Euler

$$\mathrm{sh}\,y - \tan^{-1}(\mathrm{sh}\,y) = \sum_{n=1}^{\infty} \frac{2^{2\,n-1}}{(2\,n)!}\, c(y,\ n) \tag{135}$$

$y \geq 0$, $c(y,\ n)$ definida por (132)

$$\pi = 4\tan^{-1}\left(e^{-1}\right) + 2\left(\frac{e^2-1}{e^2+1}\right) + 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\,E_n}{(2\,n)!}\, c(1,\ n) \tag{136}$$

$$\frac{\pi}{4} = 1 - \sum_{n=1}^{\infty} \frac{2^{2\,n-1}}{(2\,n)!}\, c\left(\ln\left(1+\sqrt{2}\right),\ n\right) \tag{137}$$

$$\frac{\pi}{6} = \frac{1}{\sqrt{3}} - \sum_{n=1}^{\infty} \frac{2^{2\,n-1}}{(2\,n)!}\, c\left(\frac{\ln 3}{2},\ n\right) \tag{138}$$

$$\frac{\pi}{4} = \frac{5}{6} - \sum_{n=1}^{\infty} \frac{2^{2\,n-1}}{(2\,n)!} \left(c\left(\ln\left(\frac{1+\sqrt{5}}{2}\right),\ n\right) + c\left(\ln\left(\frac{1+\sqrt{10}}{3}\right),\ n\right)\right) \tag{139}$$

$$\frac{q^2}{p\sqrt{q^2-p^2}}\tan^{-1}\left(\frac{p\,\text{th}\,y}{\sqrt{q^2-p^2}}\right)=\sum_{n=0}^{\infty}\left(\frac{p}{q}\right)^{2n}a(y,n)\ ,\ \ y>0,\ 0<p<q \tag{140}$$

$$a(y,n)=\int_0^y\frac{1}{(\text{ch}\,x)^{2n+2}}\,dx\ ,\ \ n\in\mathbb{N}_0\ ,\ \ y>0 \tag{141}$$

$$a(y,n)=\frac{\text{sh}\,y}{(2n+1)(\text{ch}\,y)^{2n+1}}+\frac{2n}{2n+1}a(y,n-1)\ ,\ \ n\in\mathbb{N}\ ,\ \ a(y,0)=\text{th}\,y \tag{142}$$

$$p\,\text{th}\,y=\sqrt{q^2-p^2}\implies\frac{q^2\pi}{4p\sqrt{q^2-p^2}}=\sum_{n=0}^{\infty}\left(\frac{p}{q}\right)^{2n}a(y,n) \tag{143}$$

$$\frac{4\pi}{3\sqrt{7}}=\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^{2n}a\left(\frac{1}{2}\ln\left(\frac{3+\sqrt{7}}{3-\sqrt{7}}\right),n\right) \tag{144}$$

$$\tan^{-1}\left(e^{2b}\right)-\tan^{-1}\left(e^{2a}\right)=\sum_{n=0}^{\infty}2^{-n-1}c(a,b,n)\ ,\ \ a<b \tag{145}$$

$$c(a,b,n)\int_a^b\left(\frac{1}{\text{ch}\,x}\right)^{2n+2}dx\ ,\ \ n\in\mathbb{N}_0 \tag{146}$$

$$c(a,b,n)=\text{th}\,b-\text{th}\,a \tag{147}$$

$$c(a,b,n)=\frac{\text{sh}\,b}{(2n+1)(\text{ch}\,b)^{2n+1}}-\frac{\text{sh}\,a}{(2n+1)(\text{ch}\,a)^{2n+1}}+\frac{2n}{2n+1}c(a,b,n-1)\ ,\ \ n\in\mathbb{N} \tag{148}$$

$$\frac{\pi}{4}=\tan^{-1}\left(e^{-2}\right)+\sum_{n=0}^{\infty}2^{-n-1}c(0,1,n) \tag{149}$$

$$\frac{\pi}{2}=\sum_{n=0}^{\infty}2^{-n-1}c(-\infty,\infty,n) \tag{150}$$

$$\frac{\pi}{4}=\sum_{n=0}^{\infty}2^{-n-1}c(0,\infty,n) \tag{151}$$

$$\frac{\pi}{12}=\sum_{n=0}^{\infty}2^{-n-1}c\left(0,\frac{\ln 3}{2},n\right) \tag{152}$$

$$\frac{\pi}{4}=\sum_{n=0}^{\infty}\frac{e^{-(n-x+1)}\,\text{sen}(n-x+1)}{n-x+1}-\sum_{n=1}^{\infty}\frac{B_n(x)\,\text{Im}((1+i)^n)}{n\,n!} \tag{153}$$

$$0\le x<1\ ,\ \ B_n(x)\ \text{polinomios de Bernoulli} \tag{154}$$

$$\ln\left(\frac{4}{\pi}\right)=\sum_{n=1}^{\infty}\left(\sqrt{\ln\left(\frac{n\sqrt[n]{e}}{n+1}\right)}-\sqrt{\ln\left(\frac{(n+1)\sqrt[n+1]{e}}{n+2}\right)}\right)\sum_{k=1}^{n}(-1)^{k-1}\sqrt{\ln\left(\frac{k\sqrt[k]{e}}{k+1}\right)} \tag{155}$$

$$\pi^2=6\sum_{n=1}^{\infty}n^{-2}+12\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{1}{2n-2k+1}\left(\frac{e^{(k+m)^{-2}}-1}{e^{(k+m)^{-2}}+1}\right)^{2n-2k+1}\ ,\ \ m\in\mathbb{N}_0 \tag{156}$$

$$\frac{\tan^{-1}\left(\sqrt{x^2-1}\ \operatorname{th} y\right)}{\sqrt{x^2-1}} = \sum_{n=0}^{\infty} (-1)^n x^{2n} F(y, n) \tag{157}$$

$$x > 1\,,\ \ y > 0\,,\ \ x \operatorname{sh} y < 1 \tag{158}$$

$$F(y, n) = \int_0^y (\operatorname{sh} t)^{2n}\, dt\ ,\ \ n \in \mathbb{N}_0 \tag{159}$$

$$F(y, 0) = y \tag{160}$$

$$F(y, n) = \frac{(\operatorname{sh} y)^{2n-1} \operatorname{ch} y}{2n} - \frac{2n-1}{2n} F(y, n-1)\ ,\ \ n \in \mathbb{N} \tag{161}$$

$$F(y, n) = \int_0^{\operatorname{sh} y} \frac{t^{2n}}{\sqrt{1+t^2}}\, dt\ ,\ \ n \in \mathbb{N}_0 \tag{162}$$

$$\sqrt{x^2-1}\ \operatorname{th} y = z\,,\ \ |z| < 1 \Longrightarrow x > \sqrt{\frac{1+z^2}{1-z^2}} \tag{163}$$

$$\frac{\pi}{6\sqrt{3}} = \sum_{n=0}^{\infty} (-1)^n 2^{2n} F\left(\frac{\ln 2}{2}, n\right) \tag{164}$$

$$\frac{\pi}{12\sqrt{2}} = \sum_{n=0}^{\infty} (-1)^n 3^{2n} F\left(\frac{1}{2} \ln\left(\frac{2\sqrt{6}+1}{2\sqrt{6}-1}\right), n\right) \tag{165}$$

$$\frac{\pi}{6\sqrt{m^2-1}} = \sum_{n=0}^{\infty} (-1)^n m^{2n} F\left(\frac{1}{2} \ln\left(\frac{\sqrt{3(m^2-1)}+1}{\sqrt{3(m^2-1)}-1}\right), n\right)\ ,\ \ m \in \mathbb{N} - \{1\} \tag{166}$$

$$\pi = 4 \sum_{n=1}^{\infty} \frac{\operatorname{Im}\left((1+i)^{(4k+1)n}\right)}{n\left(2(-4)^k\right)^n}\ ,\ \ k \in \mathbb{N}_0 \tag{167}$$

$$\pi = 4 \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \operatorname{Im}\left((1+i)^{(4k+3)n}\right)}{n\left(4(-4)^k\right)^n}\ ,\ \ k \in \mathbb{N}_0 \tag{168}$$

$$\pi = 2 \sum_{n=1}^{\infty} 2^{-n} \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} \frac{(-1)^k}{2k+1} \tag{169}$$

$$\pi = 2 \sum_{n=0}^{\infty} 2^{-n} \sum_{k=0}^{[n/2]} \binom{n}{2k} \frac{(-1)^k}{2k+1} \tag{170}$$

$$\pi = 2 \sum_{n=1}^{\infty} \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} \frac{(-1)^k 2^{-2k}}{n-2k} \left(\left(\frac{3}{4}\right)^{n-2k} - \left(-\frac{1}{2}\right)^{n-2k}\right) \tag{171}$$

$$\pi = 2 \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \binom{n}{2k} (-1)^k 2^{-2k} \left(\ln 6 + \sum_{m=1}^{n-2k} \frac{(-1)^m}{m} \binom{n-2k}{m} \left(\left(\frac{3}{2}\right)^m - \left(\frac{1}{4}\right)^m\right)\right) \tag{172}$$

$$\pi = 4 \tan^{-1}\left(\frac{\sqrt{2}+\sqrt{\sqrt{5}-1}}{\sqrt{2}+\sqrt{\sqrt{5}+1}}\right) + 2 \sum_{n=1}^{\infty} \binom{2n}{n} \frac{(-1)^{n-1}}{2^{4n} n} \sum_{k=0}^{n-1} (-1)^k \binom{2n}{2k+1} \tag{173}$$

$$\pi = 6\tan^{-1}\left(\frac{\sqrt{2} + \sqrt{\sqrt{13} - 1}}{\sqrt{6} + \sqrt{\sqrt{13} + 1}}\right) - 3\sqrt{3}\sum_{n=1}^{\infty}\binom{2n}{n}\frac{(-1)^{n-1}}{2^{6n}\,n}\sum_{k=0}^{n-1}(-3)^k\binom{2n}{2k+1} \tag{174}$$

$$\pi = 2(2-a-b)\int_0^1 \frac{((1-a)(1-b)+x^2)}{((1-a)^2+x^2)((1-b)^2+x^2)}\,dx - 2\sum_{n=1}^{\infty}\frac{a^n+b^n}{n\,2^n}\sum_{k=0}^{[(n-1)/2]}(-1)^k\binom{n}{2k+1} \tag{175}$$

$$0 < a < 1\,,\ 0 < b < 1$$

$$\frac{\pi}{\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}_{\longleftarrow\ m\ \text{radicales}\ \longrightarrow}} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k E_k\, 2^{-m(2n-2k)}}{(2k)!\,(2n-2k)!\,(2n+1)} + \sum_{n=0}^{\infty}(-1)^n\left(\frac{e^{-(2n+1-2^{-m})}}{2n+1-2^{-m}} + \frac{e^{-(2n+1+2^{-m})}}{2n+1+2^{-m}}\right) \tag{176}$$

$$m \in \mathbb{N}\,,\ \{E_k : k \in \mathbb{N}_0\} = \{1,\ 1,\ 5,\ 61,\ 1385,\ \ldots\}\ \text{números de Euler}$$

$$\pi = 2\tan^{-1}(e^x) + 1 - \operatorname{th}(x/2) + 4\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^n(k+1)\,e^{-(2n-k+2)x}}{2n-k+2}\,,\ x > 0 \tag{177}$$

$$\pi = 2\tan^{-1}(e^x) - 1 + (\operatorname{th}(x/2))^{-1} - 4\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k(n-k+1)\,e^{-(n+k+2)x}}{n+k+2}\,,\ x > 0 \tag{178}$$

$$\pi = 2\tan^{-1}(\operatorname{th} x) + 2\tan^{-1}(\operatorname{th} y) + 2\tan^{-1}\left(\frac{\operatorname{ch}(x-y)}{\operatorname{sh}(x+y)}\right)\,,\ 0 < x < y \tag{179}$$

$$\pi = 4\sum_{n=1}^{m-1}\tan^{-1}(u_n) + 4\tan^{-1}\left(\frac{1}{p_m - 1}\right)\,,\ m \in \mathbb{N} \tag{180}$$

$$\pi = 4\sum_{n=1}^{m-1}\operatorname{sen}^{-1}\left(\frac{u_n}{\sqrt{1+u_n^2}}\right) + 4\operatorname{sen}^{-1}\left(\frac{1}{\sqrt{(p_m-1)^2+1}}\right)\,,\ m \in \mathbb{N} \tag{181}$$

$$\pi = 4\sum_{n=1}^{m-1}\cos^{-1}\left(\frac{1}{\sqrt{1+u_n^2}}\right) + 4\cos^{-1}\left(\frac{p_m - 1}{\sqrt{(p_m-1)^2+1}}\right)\,,\ m \in \mathbb{N} \tag{182}$$

$$u_n = \frac{p_{n+1} - p_n}{2 + p_n\,p_{n+1} - p_n - p_{n+1}}\,,\ n \in \mathbb{N} \tag{183}$$

$$p_n\ :\ n-\text{ésimo número primo}\,,\ \{p_n : n \in \mathbb{N}\} = \{2,\ 3,\ 5,\ 7,\ 11,\ \ldots\} \tag{184}$$

$$\{u_n : n \in \mathbb{N}\} = \left\{\frac{1}{3},\ \frac{2}{9},\ \frac{2}{25},\ \frac{4}{61},\ \frac{2}{121},\ \ldots\right\} \tag{185}$$

$$\pi = 4\tan^{-1}\left(\frac{1}{3}\right) + 4\tan^{-1}\left(\frac{2}{9}\right) + 4\tan^{-1}\left(\frac{1}{4}\right) \tag{186}$$

$$\pi = 4\tan^{-1}\left(\frac{1}{3}\right) + 4\tan^{-1}\left(\frac{2}{9}\right) + 4\tan^{-1}\left(\frac{2}{25}\right) + 4\tan^{-1}\left(\frac{1}{6}\right) \tag{187}$$

$$\pi = 4\operatorname{sen}^{-1}\left(\frac{1}{\sqrt{10}}\right) + 4\operatorname{sen}^{-1}\left(\frac{1}{\sqrt{5}}\right) \tag{188}$$

$$\pi = 4\,\mathrm{sen}^{-1}\left(\frac{1}{\sqrt{10}}\right) + 4\,\mathrm{sen}^{-1}\left(\frac{2}{\sqrt{85}}\right) + 4\,\mathrm{sen}^{-1}\left(\frac{1}{\sqrt{17}}\right) \tag{189}$$

$$\pi = 4\cos^{-1}\left(\frac{3}{\sqrt{10}}\right) + 4\cos^{-1}\left(\frac{2}{\sqrt{5}}\right) \tag{190}$$

$$\pi = 4\cos^{-1}\left(\frac{3}{\sqrt{10}}\right) + 4\cos^{-1}\left(\frac{9}{\sqrt{85}}\right) + 4\cos^{-1}\left(\frac{4}{\sqrt{17}}\right) \tag{191}$$

$$\pi = 4\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{p_{n+1}-p_n}{2+p_n\,p_{n+1}-p_n-p_{n+1}}\right) \tag{192}$$

$$\pi = 3\ln\left(\frac{1+\tan\left(\tan^{-1}(2/3)-\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{12}{144\,n^2-5}\right)\right)}{1-\tan\left(\tan^{-1}(2/3)-\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{12}{144\,n^2-5}\right)\right)}\right) \tag{193}$$

$$\sqrt[k]{\pi} = \lim_{m\to\infty} 2^{m+1+\frac{1}{k}}\,\sqrt[2k]{\underbrace{2-\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}_{\longleftarrow\ -\ m\,k+k-1\ \text{radicales}\ -\ \longrightarrow}} \tag{194}$$

$$\pi = 9\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^n\,(n-k+1)}{(2\,k+1)\,2^{n-k}} \tag{195}$$

$$\pi^2 = 3\cdot 2^{2\,k+1}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\frac{1}{\left(n+2^k\,m+2^k\right)^2}-\frac{1}{\left(n+2^k\,m+2^k+1\right)^2}\right),\ k\in\mathbb{N} \tag{196}$$

$$\pi = 6\sum_{j=1}^{m}(-1)^{j+1}\left(\frac{3^j}{\sqrt{3}}-\frac{\sqrt{3}}{3^j}\right)\sum_{n=j}^{m}\binom{2\,n-1}{n-j}\frac{1}{(2\,n-1)\,2^{2\,n-1}} + 6\sqrt{3}\sum_{n=m+1}^{\infty}\frac{(-1)^{n-1}}{(2\,n-1)\,3^n},\ m\in\mathbb{N} \tag{197}$$

$$\pi = 2\left(\sqrt{3}-1\right)\sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{3}}\right)^n\sum_{k=0}^{n}\frac{(-1)^k}{(2\,k+1)\left(\sqrt{3}\right)^k} \tag{198}$$

$$\pi = \sum_{n=0}^{\infty}2^{-n}\sum_{k=0}^{n}\frac{(-1)^k\,2^{-k}}{2\,k+1}+\frac{8}{9}\sum_{n=0}^{\infty}3^{-n}\sum_{k=0}^{n}\frac{(-1)^k\,3^{-k}}{2\,k+1} \tag{199}$$

$$\pi = \frac{16}{3}\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{(-1)^{n-k}\,\mu(k)}{k^2(2\,n-2\,k+1)^3} = \frac{16}{3}\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{(-1)^{k-1}\,\mu(n-k+1)}{(n-k+1)^2\,(2\,k-1)^3} \tag{200}$$

$\mu(k)$: función de Mobius

$$\pi = \frac{2^{2\,k}\,(k!)^2}{(2\,k)!}\sum_{n=1}^{\infty}\sum_{m=1}^{2^n-1}\frac{(-1)^{m+1}\,m^k}{2^{k\,m}\sqrt{m\,(2^n-m)}},\ k\in\mathbb{N} \tag{201}$$

$$\frac{\pi}{2^{k+1}\,\mathrm{sen}\left(\pi\,2^{-k-1}\right)} = 1+\sum_{n=0}^{\infty}(-1)^n\,e^{-2^{k+1}\,(n+1)}\left(\frac{e^u}{2^{k+1}\,(n+1)-1}-\frac{e^{-u}}{2^{k+1}\,(n+1)+1}\right)+\int_0^u\frac{e^x-e^{-x}}{e^{2^{k+1}\,x}+1}\,dx \tag{202}$$

$k\in\mathbb{N}_0\ ,\ u\geq 0$

$$\operatorname{sen}\left(\pi\, 2^{-k-1}\right) = \begin{cases} 1 & , k = 0 \\ \frac{1}{2}\underbrace{\sqrt{2 - \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}}}_{\longleftarrow\; k \text{ radicales}\; \longrightarrow} & , k \in \mathbb{N} \end{cases} \tag{203}$$

$$\pi = 4\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{n\, n!}{1 + n!\,(n+1)!}\right) \tag{204}$$

$$\frac{\pi^2}{9} + (\ln(2 + 2x))^2 = \sum_{n=1}^{\infty}\sum_{m=1}^{n} \frac{(-1)^{n+1}\left(x + i\sqrt{3}\,(1+x)\right)^m \left(x - i\sqrt{3}\,(1+x)\right)^{n-m+1}}{m\,(n-m+1)} \tag{205}$$

$$-1 < x < -1/2$$

$$\frac{\pi^2}{16} + \left(\ln\left(\sqrt{2} + x\sqrt{2}\right)\right)^2 = \sum_{n=1}^{\infty}\sum_{m=1}^{n} \frac{(-1)^{n+1}\,(x + i(1+x))^m\,(x - i(1+x))^{n-m+1}}{m\,(n-m+1)} \tag{206}$$

$$-1 < x < 0$$

$$\frac{\pi^2}{36} + \left(\ln\left(\frac{2}{\sqrt{3}} + \frac{2x}{\sqrt{3}}\right)\right)^2 = \sum_{n=1}^{\infty}\sum_{m=1}^{n} \frac{(-1)^{n+1}\left(x + \frac{i}{\sqrt{3}}\,(1+x)\right)^m \left(x - \frac{i}{\sqrt{3}}\,(1+x)\right)^{n-m+1}}{m\,(n-m+1)} \tag{207}$$

$$-1 < x < 1/2$$

$$\frac{\pi^2}{9} + (\ln k)^2 = \sum_{n=1}^{\infty}\sum_{m=1}^{n} \frac{(-1)^{n+1}}{m\,(n-m+1)}\left(\frac{1}{2k} - 1 + \frac{i\sqrt{3}}{2k}\right)^m \left(\frac{1}{2k} - 1 - \frac{i\sqrt{3}}{2k}\right)^{n-m+1} \tag{208}$$

$$k \in \mathbb{N} - \{1\}$$

$$\frac{\pi^2}{16} + (\ln k)^2 = \sum_{n=1}^{\infty}\sum_{m=1}^{n} \frac{(-1)^{n+1}}{m\,(n-m+1)}\left(\frac{1}{k\sqrt{2}} - 1 + \frac{i}{k\sqrt{2}}\right)^m \left(\frac{1}{k\sqrt{2}} - 1 - \frac{i}{k\sqrt{2}}\right)^{n-m+1} \tag{209}$$

$$k \in \mathbb{N}$$

$$\frac{\pi^2}{36} + (\ln k)^2 = \sum_{n=1}^{\infty}\sum_{m=1}^{n} \frac{(-1)^{n+1}}{m\,(n-m+1)}\left(\frac{\sqrt{3}}{2k} - 1 + \frac{i}{2k}\right)^m \left(\frac{\sqrt{3}}{2k} - 1 - \frac{i}{2k}\right)^{n-m+1} \tag{210}$$

$$k \in \mathbb{N}$$

$$\begin{aligned} \pi = 2\sum_{n=0}^{\infty} 3^{-2n-1}\Bigg(a_n\left(\frac{1}{2n+1} - \frac{5\cdot 3^{-2}}{2n+3} + \frac{9\cdot 3^{-4}}{2n+5} - \frac{4\cdot 3^{-6}}{2n+7} + \frac{3^{-8}}{2n+9}\right) + \\ b_n\cdot 3^{-2}\left(\frac{1}{2n+3} - \frac{4\cdot 3^{-2}}{2n+5} + \frac{9\cdot 3^{-4}}{2n+7} - \frac{5\cdot 3^{-6}}{2n+9} + \frac{3^{-8}}{2n+11}\right)\Bigg) + \\ 2\int_{1/3}^{3} \frac{x^2(1 - 4x^2 + 9x^4 - 5x^6 + x^8)}{1 - 10x^2 + 37x^4 - 42x^6 + 26x^8 - 8x^{10} + x^{12}}\, dx \end{aligned} \tag{211}$$

$$a_{n+6} = 8a_{n+5} - 26a_{n+4} + 42a_{n+3} - 37a_{n+2} + 10a_{n+1} - a_n\ ,\ n \in \mathbb{N}_0 \tag{212}$$

$$a_0 = 1,\ a_1 = 8,\ a_2 = 38,\ a_3 = 138,\ a_4 = 415,\ a_5 = 1042 \tag{213}$$

$$b_{n+6} = 10b_{n+5} - 37b_{n+4} + 42b_{n+3} - 26b_{n+2} + 8b_{n+1} - b_n\ ,\ n \in \mathbb{N}_0 \tag{214}$$

$$b_0 = 1,\ b_1 = 10,\ b_2 = 63,\ b_3 = 302,\ b_4 = 1083,\ b_5 = 2050 \tag{215}$$

$$\pi = 2\sum_{n=0}^{\infty} 3^{-2n-1}\left(\frac{a_n}{2n+1} + \frac{3^{-2}(-5a_n + b_n)}{2n+3} + \frac{3^{-4}(9a_n - 4b_n)}{2n+5} + \frac{3^{-6}(-4a_n + 9b_n)}{2n+7} + \frac{3^{-8}(a_n - 5b_n)}{2n+9} + \frac{3^{-10}b_n}{2n+11}\right) + 2\int_{1/3}^{3} \frac{x^2(1 - 4x^2 + 9x^4 - 5x^6 + x^8)}{1 - 10x^2 + 37x^4 - 42x^6 + 26x^8 - 8x^{10} + x^{12}}\,dx \tag{216}$$

$$\sqrt{\pi} = e^{-1/4}\sum_{n=0}^{\infty}\left(e^{-n(n-1)}a_n + e^{-n(n+1)}a_{n+1}\right) \tag{217}$$

$$a_n = \int_0^1 e^{-x^2-(2n-1)x}\,dx\ ,\ \ n \in \mathbb{N}_0 \tag{218}$$

$$a_0 = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{j=0}^{m} \binom{m}{j} \frac{(-1)^j}{2m - j + 1} \tag{219}$$

$$a_n = \sum_{m=0}^{\infty} \frac{(-1)^m\,\gamma(2m+1,\,2n-1)}{m!\,(2n-1)^{2m+1}}\ ,\ n \in \mathbb{N} \tag{220}$$

$$\gamma(a,\,x) = \int_0^x e^{-t}\,t^{a-1}\,dt\ ,\ a > 0 \tag{221}$$

$$\gamma(2m+1,\,2n-1) = (2m)!\left(1 - e^{-(2n-1)}\sum_{k=0}^{2m}\frac{(2n-1)^k}{k!}\right)\ ,\ \ m \in \mathbb{N}_0 \tag{222}$$

$$\frac{e^{-1} - e^{-2n}}{2n-1} < a_n < \frac{1 - e^{-(2n-1)}}{2n-1}\ ,\ n \in \mathbb{N} \tag{223}$$

$$\pi = 4\tan^{-1}\left(\frac{1}{\sqrt{n}}\right) + 4\tan^{-1}\left(\frac{n-1}{2n+2}\right) + 4\sum_{k=0}^{\infty}\tan^{-1}\left(\frac{(n-1)^{2k+3}}{c_{k+1}((n-1)^2 c_k + c_{k+2})}\right)\ ,\ \ n \in \mathbb{N} - \{1\} \tag{224}$$

$$c_{k+2} = 2(n+1)\,c_{k+1} - (n-1)^2\,c_k\ ,\ \ c_0 = 1,\ c_1 = 2n+2 \tag{225}$$

$$\frac{z}{3+2z}\sqrt{\frac{1+z}{3-z}}\,\frac{\pi}{4} - \frac{z\ln(1+z)}{2(3+2z)} + \frac{z}{3+2z}\sqrt{\frac{1+z}{3-z}}\left(\tan^{-1}\left(\frac{z-1}{\sqrt{(1+z)(3-z)}}\right) - \tan^{-1}\left(\frac{2 - \sqrt{(1+z)(3-z)}}{z-1}\right)\right) = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\binom{3n}{n}^{-1} \tag{226}$$

$$z = \left(\frac{1}{2} + \frac{\sqrt{69}}{18}\right)^{1/3} + \left(\frac{1}{2} - \frac{\sqrt{69}}{18}\right)^{1/3} \tag{227}$$

$$\sum_{n=1}^{\infty}\frac{1}{n}\binom{3n}{n}^{-1} = \frac{1}{3}F\left(1,\ 1,\ \frac{3}{2};\ \frac{4}{3},\ \frac{5}{3};\ \frac{4}{27}\right) \tag{228}$$

F : función hipergeométrica

$$\pi = \frac{8}{3}\sum_{n=0}^{\infty}\frac{n!}{(2n+1)!}\left(\frac{1}{45}\right)^n \sum_{k=0}^{n}(-1)^k\binom{n}{k}5^{n+k}\,6^{n-k}\,(n+1)_k\,(n+k+2)_{n-k} \tag{229}$$

$$\pi = \frac{8}{\left(\sqrt{17} - 1\right)}\sum_{n=0}^{\infty}\frac{n!}{(2n+1)!}\left(\frac{7 - \sqrt{17}}{48}\right)^n \sum_{k=0}^{n}(-1)^k\binom{n}{k}3^{n+k}\,2^{2n-2k}\,(n+1)_k\,(n+k+2)_{n-k} \tag{230}$$

$$\pi = \frac{12}{\left(\sqrt{31}-1\right)} \sum_{n=0}^{\infty} \frac{n!}{(2n+1)!} \left(\frac{9-\sqrt{31}}{100}\right)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} 2^{2n+2k}\, 5^{n-k}\, (n+1)_k\, (n+k+2)_{n-k} \qquad (231)$$

$$\pi = \frac{3\sqrt{3}}{2} \sum_{n=0}^{\infty} \frac{n!}{(2n+1)!} \left(\frac{1}{4}\right)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} 2^{n-k}\, (n+1)_k\, (n+k+2)_{n-k} \qquad (232)$$

$$\pi = \frac{8}{5} \sum_{n=0}^{\infty} \frac{n!}{(2n+1)!} \left(\frac{1}{5}\right)^n \left(A_n + 7^{-2n-1}\, C_n\right) \qquad (233)$$

$$A_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} 2^{n-k}\, (n+1)_k\, (n+k+2)_{n-k} \qquad (234)$$

$$C_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} 5^{n+k+1}\, 14^{n-k}\, (n+1)_k\, (n+k+2)_{n-k} \qquad (235)$$

$$\pi = 2 \sum_{n=0}^{\infty} \frac{n!}{(2n+1)!} \left(\frac{4}{5}\left(\frac{1}{5}\right)^n + \frac{3}{5}\left(\frac{1}{10}\right)^n\right) \sum_{k=0}^{n} (-1)^k \binom{n}{k} 2^{n-k}\, (n+1)_k\, (n+k+2)_{n-k} \qquad (236)$$

$$\pi = \frac{32}{245} F\left(\begin{matrix} \frac{3}{2}, \frac{7}{4}, 2, 2, 2 \\ 1, \frac{9}{4}, \frac{11}{4}, \frac{11}{4} \end{matrix} \,\middle|\, \frac{1}{4}\right) + \frac{136}{735} F\left(\begin{matrix} \frac{3}{2}, \frac{7}{4}, 2, 2 \\ \frac{9}{4}, \frac{11}{4}, \frac{11}{4} \end{matrix} \,\middle|\, \frac{1}{4}\right) + \frac{8}{3} F\left(\begin{matrix} \frac{1}{2}, \frac{3}{4}, 1, 1 \\ \frac{5}{4}, \frac{7}{4}, \frac{7}{4} \end{matrix} \,\middle|\, \frac{1}{4}\right) \qquad (237)$$

F : es la función hipergeométrica

$$\pi \sqrt{\frac{3}{2}} = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{12^n\, (n!)^2\, (2n+1)} \sum_{k=0}^{n} (-1)^k \binom{n}{k} 2^k\, (2k)!\, (2k+2)_{2n-2k} \qquad (238)$$

$$\pi \sqrt{3} = 6 \sum_{n=0}^{\infty} \frac{(-1)^n}{12^n\, (n!)^2\, (2n+1)} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (2k)!\, (2k+2)_{2n-2k} \qquad (239)$$

$$\frac{\pi}{(\Gamma(3/4))^4} = \sum_{n=0}^{\infty} 2^{-5n} \sum_{k=0}^{n} \binom{2k}{k}^2 \binom{2n-2k}{n-k}^2 \qquad (240)$$

$$\frac{\pi}{(\Gamma(3/4))^4} = \sum_{n=0}^{\infty} 2^{-10n} \binom{2n}{n}^4 + \sum_{n=0}^{\infty} 2^{-10n-4} \binom{2n+2}{n+1}^2 \sum_{k=0}^{n} 2^{5n-5k} \binom{2k}{k}^2 \qquad (241)$$

$$\frac{\pi}{(\Gamma(3/4))^4} = \sum_{n=0}^{\infty} 2^{-10n-4} \left(16 \binom{2n}{n}^4 + \binom{2n+2}{n+1}^2 \sum_{k=0}^{n} 2^{5n-5k} \binom{2k}{k}^2\right) \qquad (242)$$

$$\frac{(\Gamma(1/3))^3}{2\pi\sqrt{3}\,\sqrt[3]{2}} = \sqrt{\frac{1-a}{1+a+a^2}} + a\,F\left(\frac{1}{3}, \frac{1}{2}; \frac{4}{3}; a^3\right) + \frac{\sqrt{1-a^3}}{3} F\left(\frac{1}{2}, \frac{2}{3}, 1; \frac{3}{2}, 2; 1-a^3\right) \qquad (243)$$

F : función hipergeométrica , $0 < a < 1$

$$\frac{\theta}{2}\left(\cot\left(\frac{\theta}{2}\right) - i\right) = \prod_{n=1}^{\infty}\left(1 - i\tan\left(\frac{\theta}{2^{n+1}}\right)\right), \quad 0 < \theta < \pi/2 \qquad (244)$$

$$\frac{\pi}{4}(1-i) = \prod_{n=1}^{\infty}\left(1 - i\tan\left(\frac{\pi}{2^{n+2}}\right)\right) = \left(1 - i\,\frac{\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}}\right)\left(1 - i\,\frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{\sqrt{2+\sqrt{2+\sqrt{2}}}}\right)\ldots \qquad (245)$$

$$\frac{\pi}{8}\left(\sqrt{2}+1-i\right)=\prod_{n=1}^{\infty}\left(1-i\tan\left(\frac{\pi}{2^{n+3}}\right)\right)=\left(1-i\,\frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{\sqrt{2+\sqrt{2+\sqrt{2}}}}\right)\left(1-i\,\frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}}{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}\right)\cdots$$

$$\pi\,F\left(\frac{1}{2},\,\frac{1}{2};\,1;\,\frac{2}{3}\right)=\sqrt{3}\,\sum_{n=0}^{\infty}\frac{c_n}{16^n\,(2\,n+1)}+\sqrt{2}\,\sum_{n=0}^{\infty}\frac{c_n}{24^n\,(2\,n+1)} \tag{247}$$

$$c_n=\sum_{k=0}^{n}\binom{2\,k}{k}\binom{2\,n-2\,k}{n-k}2^k\,3^{n-k}\ ,\ \ F\ :\ \text{función hipergeométrica} \tag{248}$$

$$\frac{\operatorname{sh}1}{\sqrt{3}}+\sum_{n=1}^{\infty}(-1)^n\int_0^{1/\sqrt{3}}\operatorname{sh}\left(x^{2\,n}\right)dx=\frac{\pi}{6}+\frac{1}{\sqrt{3}}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k\,(1/3)^{(2\,n+3)\,k}}{(2\,n+3)\,!\,((4\,n+6)\,k+1)} \tag{249}$$

$$\frac{\operatorname{sen}1}{\sqrt{3}}+\sum_{n=1}^{\infty}(-1)^n\int_0^{1/\sqrt{3}}\operatorname{sen}\left(x^{2\,n}\right)dx=\frac{\pi}{6}+\frac{1}{\sqrt{3}}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{n+k+1}\,(1/3)^{(2\,n+3)\,k}}{(2\,n+3)\,!\,((4\,n+6)\,k+1)} \tag{250}$$

$$\pi=\frac{4\,i}{\mathrm{Li}_2(z)}\left(3\,\mathrm{Li}_3(z)-\sum_{n=1}^{\infty}\frac{H_{n,2}}{n}\,z^n-2\sum_{n=1}^{\infty}\frac{H_n}{n^2}\,z^n\right)\ ,\ \ z=1-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}} \tag{251}$$

$$\pi=\frac{6\,i}{\mathrm{Li}_2(z)}\left(3\,\mathrm{Li}_3(z)-\sum_{n=1}^{\infty}\frac{H_{n,2}}{n}\,z^n-2\sum_{n=1}^{\infty}\frac{H_n}{n^2}\,z^n\right)\ ,\ \ z=1-\frac{\sqrt{3}}{2}+\frac{i}{2} \tag{252}$$

$$\pi=\frac{8\,i}{\mathrm{Li}_2(z)}\left(3\,\mathrm{Li}_3(z)-\sum_{n=1}^{\infty}\frac{H_{n,2}}{n}\,z^n-2\sum_{n=1}^{\infty}\frac{H_n}{n^2}\,z^n\right),\ \ z=1-\frac{\sqrt{2+\sqrt{2}}}{2}+i\,\frac{\sqrt{2-\sqrt{2}}}{2} \tag{253}$$

$$\pi=\frac{12\,i}{\mathrm{Li}_2(z)}\left(3\,\mathrm{Li}_3(z)-\sum_{n=1}^{\infty}\frac{H_{n,2}}{n}\,z^n-2\sum_{n=1}^{\infty}\frac{H_n}{n^2}\,z^n\right),\ \ z=1-\frac{\sqrt{6}+\sqrt{2}}{4}+i\,\frac{\sqrt{6}-\sqrt{2}}{4} \tag{254}$$

$$\pi=2\,i\ln2+\frac{4\,i}{\mathrm{Li}_2(z)}\left(3\,\mathrm{Li}_3(z)-\sum_{n=1}^{\infty}\frac{H_{n,2}}{n}\,z^n-2\sum_{n=1}^{\infty}\frac{H_n}{n^2}\,z^n\right)\ ,\ \ z=\frac{1}{2}+\frac{i}{2} \tag{255}$$

$$\pi=\frac{1}{\mathrm{Im}(\mathrm{Li}_2(z))}\left(2\ln2\,\mathrm{Re}(\mathrm{Li}_2(z))+12\,\mathrm{Re}(\mathrm{Li}_3(z))-4\sum_{n=1}^{\infty}\frac{H_{n,2}}{n}\,2^{-n/2}\cos\left(\frac{n\,\pi}{4}\right)-8\sum_{n=1}^{\infty}\frac{H_n}{n^2}\,2^{-n/2}\cos\left(\frac{n\,\pi}{4}\right)\right) \tag{256}$$

$$\pi=\frac{1}{\mathrm{Re}(\mathrm{Li}_2(z))}\left(-2\ln2\,\mathrm{Im}(\mathrm{Li}_2(z))-12\,\mathrm{Im}(\mathrm{Li}_3(z))+4\sum_{n=1}^{\infty}\frac{H_{n,2}}{n}\,2^{-n/2}\sin\left(\frac{n\,\pi}{4}\right)+8\sum_{n=1}^{\infty}\frac{H_n}{n^2}\,2^{-n/2}\sin\left(\frac{n\,\pi}{4}\right)\right) \tag{257}$$

En ecuaciones (256) *y* (257) : $z=\dfrac{e^{i\,\pi/4}}{\sqrt{2}}=\dfrac{1+i}{2}$

$$\pi=\int_0^{\infty}x^{-3}\left(\sin\left(2\sqrt{\frac{2}{3}}\,x\right)\right)^3dx \tag{258}$$

$$\pi=\int_0^{\infty}\frac{1}{(1+4\,x)\,\sqrt[4]{x^3}}\,dx \tag{259}$$

$$\pi = \int_0^\infty \frac{1}{(1 + 64\,x)\,\sqrt[6]{x^5}}\,dx \tag{260}$$

$$\pi = \int_0^\infty \frac{1}{\left(1 + 64\left(1 + \sqrt{3}\right)^{12} x\right)\sqrt[12]{x^{11}}}\,dx \tag{261}$$

$$\pi = 4\int_0^1 \tan^{-1}\left(\sqrt[4]{x^{-1} - 1}\right)dx \tag{262}$$

$$\pi = 4\int_0^1 \tan^{-1}\left(\left(\frac{x}{1 - x}\right)^{1/n}\right)dx\ ,\ \ n \in \mathbb{R} - \{0\} \tag{263}$$

$$\pi = \int_0^\infty \frac{1}{(1 + x)\sqrt{x}}\,dx \tag{264}$$

$$\pi^2 = \int_0^\infty \frac{x}{1 + e^{x/2\sqrt{3}}}\,dx \tag{265}$$

$$\pi^2 = \int_0^1 \frac{\ln\left(1 + \sqrt[12]{x}\right)}{x}\,dx \tag{266}$$

$$\pi = \int_0^\infty\int_0^\infty \sqrt{x\,y}\ e^{-(x/2)-\left(y/\sqrt[3]{2}\right)}\,dx\,dy \tag{267}$$

$$\pi\left(1 - e^{-1}\right) = 4\sum_{n=1}^{\infty}\sum_{k=1}^{2^{n-1}} I(n,\,k) \tag{268}$$

donde

$$I(n,\,k) = \iint_{R(n,k)} e^{-\left(x^2+y^2\right)}\,dx\,dy \tag{269}$$

$$R(n,\,k) = \left[\frac{\sqrt{2^{2n} - 4\,k^2}}{2^n},\ \frac{\sqrt{2^{2n} - (2\,k - 1)^2}}{2^n}\right]\times\left[\frac{2\,k - 2}{2^n},\ \frac{2\,k - 1}{2^n}\right]\ ,\ \ k = 1,\ \ldots,\ 2^{n-1},\ n \in \mathbb{N} \tag{270}$$

$$I(n,\,k) = F\left(\frac{\sqrt{2^{2n} - 4\,k^2}}{2^n},\ \frac{\sqrt{2^{2n} - (2\,k - 1)^2}}{2^n}\right)F\left(\frac{2\,k - 2}{2^n},\ \frac{2\,k - 1}{2^n}\right) \tag{271}$$

$$F(a,\,b) = \int_a^b e^{-x^2}\,dx \tag{272}$$

$$\pi = \int_0^\infty \left(\ln\left(\frac{1}{x}\right)\right)\ln\left(1 + x^{-2}\right)dx \tag{273}$$

$$\pi = \int_0^\infty (\ln x)\,\ln\left(1 + a^2\,x^{-2}\right)dx \tag{274}$$

donde

$$\frac{1}{a} = e^{-1-e^{-1-e^{-1-\cdots}}}\ ,\ \ a = 3.591121476668\ \ldots\ ,\ \ a\ln a = 1 + a \tag{275}$$

$$y_{n+1} = e^{-1-y_n}\ ,\ \ y_1 = 0\ ,\ \ y_n \to a^{-1} \tag{276}$$

$$\pi = \int_1^{e^2}\int_0^\infty \frac{1}{\cosh(x\,y)}\,dx\,dy \tag{277}$$

$$\pi = \int_1^3 \int_0^\infty \frac{y}{\cosh(x\,y)}\,dx\,dy \tag{278}$$

$$\int_0^\infty \int_0^\infty \frac{1}{\cosh(x^2)\cosh(y^2)}\,dx\,dy = \pi \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{2\,n+1}} \right)^2 \tag{279}$$

$$\pi = \int_0^\infty \frac{\ln\left((e-1-x^2)^2\right)}{1+x^2}\,dx \tag{280}$$

$$\pi = \int_0^\infty \frac{\ln(x)\ln(1+x^2)}{x^2}\,dx \tag{281}$$

$$\pi = \int_0^1 \frac{\ln(1+4\,e(e-1)\,x^2)}{\sqrt{1-x^2}}\,dx \tag{282}$$

$$\pi = y_n^2 \int_0^\pi \frac{\cos(n\,x)}{1+y_n^2-2\,y_n\cos(x)}\,dx\ ,\ \ n \in \mathbb{N} \tag{283}$$

donde los números y_n se definen por :

$$y_1 = \frac{1+\sqrt{5}}{2}\ ,\ \ y_2 = \sqrt{2}\ ,\ \ y_n = \sqrt[n]{1+\sqrt[\frac{n}{n-2}]{1+\sqrt[\frac{n}{n-2}]{1+\sqrt[\frac{n}{n-2}]{1+\ldots}}}}\ ,\ \ n = 3,\ 4,\ 5,\ \ldots \tag{284}$$

$$\pi^2 = \int_0^\infty \frac{\ln x}{(1+x)\,x^a}\,dx \tag{285}$$

$$\pi^2 = \int_0^\infty \frac{\ln(1/x)}{(1+x)\,x^{1-a}}\,dx \tag{286}$$

En las fórmulas (285) *y* (286) el número a se define por :

$$\frac{\sqrt{5}-1}{2} = \left(1-4\,a^2\right)\left(1-\frac{4\,a^2}{3^2}\right)\left(1-\frac{4\,a^2}{5^2}\right)\ldots,\ \ 0<a<1/2\ ,\ \ a = 0.287929\ \ldots,\ \ \cos(a\,\pi) = \frac{\sqrt{5}-1}{2} \tag{287}$$

$$\pi = \int_1^\infty \frac{1}{x^2(x-1)^{1-a}}\,dx \tag{288}$$

donde el número a se define por :

$$a(2-a) = \left(1-4\,a^2\right)\left(1-\frac{4\,a^2}{3^2}\right)\left(1-\frac{4\,a^2}{5^2}\right)\ldots\ ,\ \ 0<a<1\ ,\ \ a = 0.26351555\ \ldots \tag{289}$$

$$\pi = \int_0^\infty \frac{x^a}{(1+x)^3}\,dx \tag{290}$$

$$\pi = \int_0^\infty \frac{x^{1-a}}{(1+x)^3}\,dx \tag{291}$$

En las fórmulas (290) *y* (291) el número a se define por :

$$4-(a(a-1))^2 = 4\left(1-4\,a^2\right)\left(1-\frac{4\,a^2}{3^2}\right)\left(1-\frac{4\,a^2}{5^2}\right)\ldots\ ,\ \ 1<a<2\ ,\ \ a = 1.7644144\ \ldots \tag{292}$$

$$\pi = \left(\frac{(2\,n)\,!}{2^{2\,n-1}\;B_n}\right)^{1/2\,n} \sum_{k=1}^{\infty} \frac{a_{n,k}}{k^{2\,n}}\;,\;\; n \in \mathbb{N} \tag{293}$$

donde los números $a_{n,k}$ se definen por :

$$\sum_{k_1\,k_2\,..\,k_{2\,n}=k}\;\prod_{j=1}^{2\,n} a_{n,k_j} = 1\;,\;\; n,\,k \in \mathbb{N} \tag{294}$$

B_n son los números de Bernoulli :

$$B_n = \left\{\frac{1}{6},\,\frac{1}{30},\,\frac{1}{42},\,\frac{1}{30},\,...\right\} \tag{295}$$

caso particular de (293) :

$$\pi = \sqrt{6}\,\sum_{k=1}^{\infty} a_k\,k^{-2} = \sqrt{6}\,\left(1 + \frac{1/2}{2^2} + \frac{1/2}{3^2} + \frac{3/8}{4^2} + ...\right) \tag{296}$$

$$\pi = \left(\frac{(2\,n)\,!}{(2^{2\,n-1} - 1)\,B_n}\right)^{1/2\,n} \sum_{k=1}^{\infty} \frac{a_{n,k}}{k^{2\,n}}\;,\;\; n \in \mathbb{N} \tag{297}$$

donde los números $a_{n,k}$ se definen por :

$$\sum_{k_1\,k_2\,..\,k_{2\,n}=k}\;\prod_{j=1}^{2\,n} a_{n,k_j} = (-1)^{k-1}\;,\;\; n,\,k \in \mathbb{N} \tag{298}$$

caso particular de (297) :

$$\pi = 2\,\sqrt{3}\,\sum_{k=1}^{\infty} a_k\,k^{-2} = 2\,\sqrt{3}\,\left(1 - \frac{1/2}{2^2} + \frac{1/2}{3^2} - \frac{5/8}{4^2} + ...\right) \tag{299}$$

$$\pi = \left(\frac{2\,(2\,n)\,!}{(2^{2\,n} - 1)\,B_n}\right)^{1/2\,n} \sum_{k=1}^{\infty} \frac{a_{n,k}}{(2\,k-1)^{2\,n}}\;,\;\; n \in \mathbb{N} \tag{300}$$

donde los números $a_{n,k}$ se definen por :

$$\sum_{(2\,k_1-1)\,(2\,k_2-1)\,..\,(2\,k_{2\,n}-1)=2\,k-1}\;\prod_{j=1}^{2\,n} a_{n,k_j} = 1\;,\;\; n,\,k \in \mathbb{N} \tag{301}$$

caso particular de (300) :

$$\pi = 2\,\sqrt{2}\,\sum_{k=1}^{\infty} a_k (2\,k-1)^{-2\,n} = 2\,\sqrt{2}\,\left(1 + \frac{1/2}{3^2} + \frac{1/2}{5^2} + \frac{1/2}{7^2} + \frac{3/8}{9^2} + ...\right) \tag{302}$$

$$\pi\,\mathrm{H1}(x) = \frac{x^2}{\sqrt{1-x^2}} \sum_{n=0}^{\infty} \frac{2^{-2\,n+1}}{2\,n+3} \sum_{k=0}^{n} \binom{2\,k}{k} \binom{2\,n-2\,k}{n-k} \left(-\frac{x^2}{1-x^2}\right)^{n-k} \frac{1}{2\,k+1}\;,\;\; 0 < x < 1 \tag{303}$$

$$\mathrm{H1}(x) = 1 - F\left(-\frac{1}{2},\,\frac{1}{2};\,1;\,x^2\right) \tag{304}$$

$$\pi\,\mathrm{H2}(x) = \sum_{n=0}^{\infty} \frac{2^{-2\,n+1}}{2\,n+3} \sum_{k=0}^{n} \binom{2\,k}{k} \binom{2\,n-2\,k}{n-k} \frac{x^{2\,n-2\,k+2}}{2\,k+1}\;,\;\; 0 < x < 1 \tag{305}$$

$$\mathrm{H2}(x) = F\left(-\frac{1}{2}, \frac{1}{2}; 1; x^2\right) - \sqrt{1-x^2} \tag{306}$$

En (304) *y* (306) , *F* es la función hipergeométrica

$$\pi^2 = 12 \ln\left(\frac{1}{x}\right) \ln(1+x) - 12\,\mathrm{Li}_2(-x) + 6 \sum_{n=0}^{\infty} 2^{-n} \sum_{k=0}^{n} \binom{n+k+1}{n-k} \frac{2^{-k}(1-x)^{n+k+2}}{(2k+1)(n+k+2)}, \quad 0 < x < 1 \tag{307}$$

$$\pi^2 = 12 \ln\left(\frac{1}{x}\right) \ln(1+x) - 12\,\mathrm{Li}_2(-x) - 24 \sum_{n=0}^{\infty} \frac{1}{2n+1} \left(\ln\left(\frac{2}{1+x}\right) - \sum_{k=0}^{2n} \binom{2n+1}{k} \frac{(-1)^k}{2n-k+1} \left(\left(\frac{2}{1+x}\right)^{2n-k+1} - 1\right)\right), \quad 0 < x < 1 \tag{308}$$

$$\pi = 4 \sum_{n=0}^{\infty} 2^{-n} \sum_{k=0}^{n} c_k\, c_{n-k} \tag{309}$$

donde los números c_k se definen por :

$$c_0 = 1, \quad c_{n+1} = \frac{1}{n+1} \sum_{k=0}^{n} (-1)^{k+1} s_{k+1}\, c_{n-k} \tag{310}$$

$$s_1 = \gamma, \quad s_n = \zeta(n), \quad n \geq 2 \tag{311}$$

$$\frac{1}{\pi} = \frac{1}{4} \sum_{n=0}^{\infty} 2^{-n} \sum_{k=0}^{n} d_k\, d_{n-k} \tag{312}$$

donde los números d_k se definen por :

$$d_0 = 1, \quad d_{n+1} = \frac{1}{n+1} \sum_{k=0}^{n} (-1)^{k} s_{k+1}\, d_{n-k} \tag{313}$$

$$s_1 = \gamma, \quad s_n = \zeta(n), \quad n \geq 2 \tag{314}$$

$$\pi = 16 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{-3 - \sqrt{17} + \sqrt{6\left(7+\sqrt{17}\right)}}{4}\right)^{2n+1} \left(\cos\left(\frac{(2n+1)\pi}{6}\right)\right)^2 \tag{315}$$

$$\pi = \frac{16}{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{3 - \sqrt{17} + \sqrt{6\left(7-\sqrt{17}\right)}}{4}\right)^{2n+1} \left(\cos\left(\frac{(2n+1)\pi}{6}\right)\right)^2 \tag{316}$$

$$\pi = 4 h_k + 4 \sum_{n=1}^{\infty} (h_{nk+k} - h_{nk}), \quad h_k = \sum_{n=1}^{k} \frac{(-1)^{n-1}}{2n-1}, \quad k \in \mathbb{N} \tag{317}$$

$$\pi = 4n \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)n+1} + 4 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)((2k+1)n+1)}, \quad n \in \mathbb{N} \tag{318}$$

$$\pi = \lim_{n\to\infty} \frac{2^{n+2}\, n!\, a_n}{(2n)!} \tag{319}$$

$$a_{n+2} = 2a_{n+1} + (2n+1)^2 a_n, \quad a_1 = 1, \quad a_2 = 2 \tag{320}$$

$$G + \frac{\pi}{4} x = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \operatorname{sen}\left(\frac{\pi + (4n+2)x}{4}\right) \cos\left(\frac{\pi - (4n+2)x}{4}\right), \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \tag{321}$$

$$G - \frac{\pi}{4} x = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \operatorname{sen}\left(\frac{\pi - (4n+2)x}{4}\right) \cos\left(\frac{\pi + (4n+2)x}{4}\right), \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2} \tag{322}$$

$$G + \frac{\pi}{4} (\pi - x) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \operatorname{sen}\left(\frac{\pi + (4n+2)x}{4}\right) \cos\left(\frac{\pi - (4n+2)x}{4}\right), \quad \frac{\pi}{2} \le x \le \frac{3\pi}{2} \tag{323}$$

$$G - \frac{\pi}{4} (\pi - x) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \operatorname{sen}\left(\frac{\pi - (4n+2)x}{4}\right) \cos\left(\frac{\pi + (4n+2)x}{4}\right), \quad \frac{\pi}{2} \le x \le \frac{3\pi}{2} \tag{324}$$

$$G + \frac{\pi^2}{4} x = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \operatorname{sen}\left(\frac{(4n+2)x+1}{4} \pi\right) \cos\left(\frac{(4n+2)x-1}{4} \pi\right), \quad |x| \le \frac{1}{2} \tag{325}$$

$$G - \frac{\pi^2}{4} x = -2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \operatorname{sen}\left(\frac{(4n+2)x-1}{4} \pi\right) \cos\left(\frac{(4n+2)x+1}{4} \pi\right), \quad |x| \le 1 \tag{326}$$

$$\zeta(3) \pm \frac{x(x-\pi)(x-2\pi)}{12} = 2 \sum_{n=1}^{\infty} \frac{1}{n^3} \operatorname{sen}\left(\frac{\pi \pm 2nx}{4}\right) \cos\left(\frac{\pi \mp 2nx}{4}\right), \quad 0 \le x \le 2\pi \tag{327}$$

$$\zeta(3) \pm \frac{\pi^3 x(x-1)(x-2)}{12} = \pm 2 \sum_{n=1}^{\infty} \frac{1}{n^3} \operatorname{sen}\left(\frac{2nx \pm 1}{4} \pi\right) \cos\left(\frac{2nx \mp 1}{4} \pi\right), \quad 0 \le x \le 2 \tag{328}$$

$$\ln \pi + \gamma - \ln\left(\underbrace{\sqrt{2 - \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}}}_{\longleftarrow \; - \; - \; k \text{ radicales } - \; - \longrightarrow}\right) = (2k+1) \ln 2 - \ln\left(2^{k+1} - 1\right) + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \ln\left(\left(1 + \frac{2^{-k-1}}{n}\right)\left(1 + \frac{1 - 2^{-k-1}}{n}\right)\right)\right) \tag{329}$$

$k \in \mathbb{N}$, γ: constante gamma de Euler

$$\pi = 4i \sum_{n=0}^{\infty} \frac{1}{2n+1} \left(\frac{2}{\left(\sqrt{2} + i\right)^{2n+1}} - \frac{1}{\left(\sqrt{2}\right)^{2n+1}}\right) \tag{330}$$

$$\pi = 4i \sum_{n=0}^{\infty} \frac{1}{2n+1} \left(\frac{2}{(1+2i)^{2n+1}} - \frac{1}{3^{2n+1}}\right) \tag{331}$$

$$\pi = 4i \sum_{n=1}^{\infty} \frac{\left(\sqrt{2}\right)^n}{n} \left(\frac{1-i}{\sqrt{2} + 1 + i}\right)^n \tag{332}$$

$$\pi = 6i \sum_{n=1}^{\infty} \frac{\left(\sqrt{3} - 1\right)^n}{n} \left(\frac{1-i}{\sqrt{3} + i}\right)^n \tag{333}$$

$$\pi = 2i \sum_{n=1}^{\infty} \frac{1}{n} \left(\left(\frac{1}{9}\right)^n + 2\left(\frac{1-3i}{4}\right)^n\right) \tag{334}$$

$$\pi = 2i \sum_{n=1}^{\infty} \frac{1}{n} \left(\left(1 - \frac{1}{2(1-x)^2}\right)^n + 2(x - i(1-x))^n\right), \quad 0 < x < 1/2 \tag{335}$$

$$\pi = 4i \sum_{n=1}^{\infty} \frac{\left(p - q + \sqrt{p^2 + q^2}\right)^n}{n} \left(\frac{1-i}{p + \sqrt{p^2 + q^2} + qi}\right)^n + 4i \sum_{n=1}^{\infty} \frac{\left(\sqrt{2p^2 + 2q^2} - 2p\right)^n}{n} \left(\frac{1-i}{q - p + \sqrt{2p^2 + 2q^2} + (p+q)i}\right)^n \tag{336}$$

$0 < p < q$

$$\pi = 4\sum_{k=0}^{m-1}\tan^{-1}\left(\frac{b_k}{1+a_k}\right) + 4\sum_{n=1}^{\infty}\frac{1}{n}\sum_{j=0}^{[(n-1)/2]}(-1)^j\binom{n}{2j+1}a_m^{n-2j-1}\,b_m^{2j+1} \tag{337}$$

$$a_{k+1} = a_k^2 - b_k^2\ ,\ \ b_{k+1} = 2\,a_k\,b_k\ ,\ \ a_0 = a\,,\ \ b_0 = 1-a\,,\ \ k\in\mathbb{N}_0\,,\ \ m\in\mathbb{N},\ 0<a<1 \tag{338}$$

$$\frac{16}{\pi} = \sum_{n=0}^{\infty}\frac{42\,n\,(2^{12-a}-1)+5\cdot 2^{12-a}-47}{2^{(12-a)(n+1)}}\sum_{k=0}^{n}\binom{2k}{k}^3 2^{-a\,k} \tag{339}$$

$$6 < a < 12$$

$$\frac{512}{\pi} = \sum_{n=0}^{\infty}\frac{1302\,n+113}{2^{5n}}\sum_{k=0}^{n}\binom{2k}{k}^3 2^{-7k} \tag{340}$$

$$\frac{256}{3\pi} = \sum_{n=0}^{\infty}\frac{210\,n+11}{2^{4n}}\sum_{k=0}^{n}\binom{2k}{k}^3 2^{-8k} \tag{341}$$

$$\frac{128}{7\pi} = \sum_{n=0}^{\infty}\frac{42\,n-1}{2^{3n}}\sum_{k=0}^{n}\binom{2k}{k}^3 2^{-9k} \tag{342}$$

$$\frac{64}{9\pi} = \sum_{n=0}^{\infty}\frac{14\,n-3}{2^{2n}}\sum_{k=0}^{n}\binom{2k}{k}^3 2^{-10k} \tag{343}$$

$$\frac{32}{\pi} = \sum_{n=0}^{\infty}\frac{42\,n-37}{2^{n}}\sum_{k=0}^{n}\binom{2k}{k}^3 2^{-11k} \tag{344}$$

$$\frac{128\sqrt{2}}{\pi} = \sum_{n=0}^{\infty}\frac{42\left(8\sqrt{2}-1\right)n+40\sqrt{2}-47}{\left(8\sqrt{2}\right)^n}\sum_{k=0}^{n}\binom{2k}{k}^3\left(\sqrt{2}\right)^{-17k} \tag{345}$$

$$\frac{64\sqrt[3]{4}}{\pi} = \sum_{n=0}^{\infty}\frac{42\left(4\sqrt[3]{4}-1\right)n+20\sqrt[3]{4}-47}{\left(4\sqrt[3]{4}\right)^n}\sum_{k=0}^{n}\binom{2k}{k}^3\left(\sqrt[3]{4}\right)^{-14k} \tag{346}$$

$$\frac{2^{4+a}\,(2^{12-a}-1)}{\pi} = \sum_{n=0}^{\infty}\binom{2n}{n}^3\frac{A(n)\,B(n)}{(n+1)^3\,2^{a\,n}} \tag{347}$$

$$A(n) = 2^{12-a}\left(5\cdot 2^{12-a}+37\right) - \frac{42\cdot 2^{12-a}-5}{2^{(12-a)\,n}} - \frac{42\,(2^{12-a}-1)\,n}{2^{(12-a)\,n}} \tag{348}$$

$$B(n) = (2^a-64)\,n^3 + (3\cdot 2^a-96)\,n^2 + (3\cdot 2^a-48)\,n + 2^a - 8 \tag{349}$$

$$6 \le a \le 12$$

$$\frac{2^8\cdot 31^2}{\pi} = \sum_{n=0}^{\infty}\binom{2n}{n}^3\frac{A(n)\,B(n)}{(n+1)^3\,2^{7n}} \tag{350}$$

$$A(n) = 6304 - \frac{1302\,n+1499}{2^{5n}}\ ,\ \ B(n) = 8\,n^3 + 36\,n^2 + 42\,n + 15 \tag{351}$$

$$\frac{16}{\pi} = 5\sum_{n=0}^{\infty}\binom{2n}{n}^3 2^{-a\,n} - \sum_{n=0}^{\infty}\frac{42\,(2^{12-a}-1)\,n+5\cdot 2^{12-a}-47}{2^{(12-a)(n+1)}}\sum_{k=n+1}^{\infty}\binom{2k}{k}^3 2^{-a\,k} \tag{352}$$

$$6 < a < 12$$

$$\frac{64}{\pi} = 20\sum_{n=0}^{\infty}\binom{2n}{n}^3 2^{-10n} - \sum_{n=0}^{\infty}\frac{126n-27}{2^{2n}}\sum_{k=n+1}^{\infty}\binom{2k}{k}^3 2^{-10k} \quad (353)$$

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