The Function $\gamma(z)$: The Generalized-Euler-Constant Function , Part 1

Abstract. Some results related with " The Generalized-Euler-Constant Function".

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Resumen. Se muestra una colección de fórmulas relacionadas con la función $\gamma(z)=\sum_{n=1}^{\infty} z^{n-1}\left(\frac{1}{n}-ln\left(\frac{n+1}{n}\right)\right)$ para $|z|\leq 1$. La función $\gamma(z)$ se conoce como: " The Generalized – Euler – Constant Function " , fue definida en 2006 por J. Sondow y P. Hadjicostas , ref.(5).

Notación. $\mathbb{N}=\{1,2,3,...\}$, $\mathbb{R}$: números reales , $\mathbb{C}$: números complejos

1 Introducción

Definiciones para la función $\gamma(z)$: " The Generalized – Euler – Constant Function " :

1.1. Para $|z|\leq 1$, se tiene:

$$\gamma(z)=\sum_{n=1}^{\infty} z^{n-1}\left(\frac{1}{n}-ln\left(\frac{n+1}{n}\right)\right) \tag{1.1}$$

1.2. Para $z\in\mathbb{C}-(1,\infty)$, se tiene:

$$\gamma(z)=\int_0^1\int_0^1 \frac{1-x}{(1-xyz)(-ln(xy))}dxdy \tag{1.2}$$

1.3. Para $z\in\mathbb{C}-(1,\infty)$, se tiene:

$$\gamma(z)=\int_0^1 \frac{1-x+ln(x)}{(1-xz)ln(x)}dx \tag{1.3}$$

La función $\gamma(z)$ tiene los siguientes valores particulares:

$$\gamma(1)=\gamma=\lim_{n\to\infty}\left(1+\frac{1}{2}+...+\frac{1}{n}-ln(n)\right) \tag{1.4}$$

$$\gamma(-1)=ln\left(\frac{4}{\pi}\right) \tag{1.5}$$

Observación.

$$\pi=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$$

$$e=\sum_{n=0}^{\infty}\frac{1}{n!}$$

$$G=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}$$

$$Li_2(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^2}\quad ,-1\le x\le 1$$

$$Li_k(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^k}\quad ,-1\le x\le 1,k\in\mathbb{N}-\{1\}$$

2 Fórmulas – Teoremas

2.1. **Teorema 1.** Para $|z|\le 1$, se tiene

$$\gamma(z)+\gamma(-z)=2\sum_{n=1}^{\infty}z^{2n-2}\left(\frac{1}{2n-1}-ln\left(\frac{2n}{2n-1}\right)\right) \tag{2.1}$$

$$\gamma(z)-\gamma(-z)=2\sum_{n=1}^{\infty}z^{2n-1}\left(\frac{1}{2n}-ln\left(\frac{2n+1}{2n}\right)\right) \tag{2.2}$$

2.2. **Teorema 2.** Para $z\in\mathbb{C}-[1,\infty)$, se tiene

$$\gamma(z)+\gamma(-z)=2\int_0^1\int_0^1\frac{1-x}{(1-x^2y^2z^2)(-ln(xy))}dxdy \tag{2.3}$$

$$\gamma(z)-\gamma(-z)=2z\int_0^1\int_0^1\frac{xy(1-x)}{(1-x^2y^2z^2)(-ln(xy))}dxdy \tag{2.4}$$

2.3. **Teorema 3.** Para $|z|\le 1, z\ne 1$, se tiene

$$e^{-z\gamma(z)}=(1-z)\prod_{n=1}^{\infty}\left(\frac{n+1}{n}\right)^{z^n} \tag{2.5}$$

2.4. **Teorema 4.** Es válida la siguiente identidad

$$\begin{aligned}\left(\frac{1+i\sqrt{3}}{2}\right)\gamma\left(\frac{1+i\sqrt{3}}{2}\right)+\left(\frac{1-i\sqrt{3}}{2}\right)\gamma\left(\frac{1-i\sqrt{3}}{2}\right)&=\\ =-\sum_{n=1}^{\infty}\left[\left(\frac{1+i\sqrt{3}}{2}\right)^n+\left(\frac{1-i\sqrt{3}}{2}\right)^n\right]ln\left(\frac{n+1}{n}\right)& \\ =-2\sum_{n=1}^{\infty}cos\left(\frac{n\pi}{3}\right)ln\left(\frac{n+1}{n}\right)&\end{aligned} \tag{2.6}$$

2.5. **Teorema 5.** Para $|z_1|\le 1, |z_2|\le 1, z_1\ne 1, z_2\ne 1, (1-z_1)(1-z_2)=1$, se tiene

$$z_1\gamma(z_1)+z_2\gamma(z_2)=-\sum_{n=1}^{\infty}\left(z_1^n+z_2^n\right)ln\left(\frac{n+1}{n}\right) \tag{2.7}$$

2.6. **Teorema 6.** Para $-1\le x<\frac{1}{2}$, se tiene

$$x\gamma(x)-\frac{x}{1-x}\gamma\left(-\frac{x}{1-x}\right)=-\sum_{n=1}^{\infty}\left[x^n+\left(-\frac{x}{1-x}\right)^n\right]ln\left(\frac{n+1}{n}\right) \tag{2.8}$$

2.7. **Teorema 7.** Para $|x|\le 1$, se tiene

$$\int_0^x\gamma(t)dt=Li_2(x)-\sum_{n=1}^{\infty}\frac{x^n}{n}ln\left(\frac{n+1}{n}\right) \tag{2.9}$$

$$\int_0^1\gamma(t)dt=\frac{\pi^2}{6}-\sum_{n=1}^{\infty}\frac{1}{n}ln\left(\frac{n+1}{n}\right) \tag{2.10}$$

2.8. **Teorema 8.** Es válida la siguiente identidad

$$\int_0^1 \gamma(it)\,dt = G - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} ln\left(\frac{2n}{2n-1}\right) + i\frac{\pi^2}{48} - i\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n} ln\left(\frac{2n+1}{2n}\right) \quad (2.11)$$

2.9. **Teorema 9.** Para $0 < \theta < 2\pi$, se tiene

$$e^{i\theta}\gamma\left(e^{i\theta}\right) = -\frac{1}{2} ln\left(2 - 2cos(\theta)\right) + i\,tan^{-1}\left(\frac{sen(\theta)}{1-cos(\theta)}\right) - \sum_{n=1}^{\infty} e^{ni\theta}\, ln\left(\frac{n+1}{n}\right) \quad (2.12)$$

2.10. **Teorema 10.** Para $m \in \mathbb{N} - \{1\}$ y $\frac{1}{z} = \sqrt[m]{1 + \sqrt[m]{1 + \sqrt[m]{1 + ...}}}$, se tiene

$$-z\gamma(-z) + z^m\gamma\left(z^m\right) = -\sum_{n=1}^{\infty}\left((-z)^n + z^{mn}\right) ln\left(\frac{n+1}{n}\right) \quad (2.13)$$

2.11. **Teorema 11.** Para $0 < x \le 1$, se tiene

$$\int_0^x \gamma\left(-t^2\right)dt = 2\,tan^{-1}(x) - \frac{ln\left(1+x^2\right)}{x} + \sum_{n=2}^{\infty}(-1)^{n-1}\left(\frac{x^{2n-3}}{2n-3} + \frac{x^{2n-1}}{2n-1}\right) ln(n) \quad (2.14)$$

$$\int_0^1 \gamma\left(-t^2\right)dt = \frac{\pi}{2} - ln(2) + \sum_{n=2}^{\infty}(-1)^{n-1}\left(\frac{1}{2n-3} + \frac{1}{2n-1}\right) ln(n) \quad (2.15)$$

2.12. **Teorema 12.** Para $0 < x \le 1$, se tiene

$$\int_0^x \gamma\left(-t^2\right)dt = 2\,tan^{-1}(x) - \frac{ln\left(1+x^2\right)}{x} - \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n-1}}{2n-1} ln\left(\frac{n+1}{n}\right) \quad (2.16)$$

$$\int_0^1 \gamma\left(-t^2\right)dt = \frac{\pi}{2} - ln(2) - \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1} ln\left(\frac{n+1}{n}\right) \quad (2.17)$$

2.13. **Teorema 13.** Para $0 \le x < 1$, se tiene

$$\begin{aligned}(x+(1-x)i)\gamma(x+(1-x)i) = &-\frac{ln(2)}{2} - ln(1-x) + \frac{\pi}{4}i \\ &- \sum_{n=2}^{\infty}(x+(1-x)i)^{n-1} ln\left(\frac{n}{n-1}\right)\end{aligned} \quad (2.18)$$

2.14. **Teorema 14.** Para $a+bi\in\mathbb{C}-(1,\infty)$, se tiene

$$\gamma(a+bi)=\int_0^1\int_0^1\frac{(1-axy)(1-x)}{\left(1-2axy+\left(a^2+b^2\right)x^2y^2\right)\left(-\ln(xy)\right)}dxdy+ \\ +i\int_0^1\int_0^1\frac{bxy(1-x)}{\left(1-2axy+\left(a^2+b^2\right)x^2y^2\right)\left(-\ln(xy)\right)}dxdy \qquad (2.19)$$

2.15. **Teorema 15.** Para $a+bi\in\mathbb{C}-(1,\infty)$ y $\gamma(a+bi)=u(a,b)+iv(a,b)$, se tiene

$$u(a,b)+\frac{a}{b}v(a,b)=\int_0^1\int_0^1\frac{(1-x)}{\left(1-2axy+\left(a^2+b^2\right)x^2y^2\right)\left(-\ln(xy)\right)}dxdy \qquad (2.20)$$

2.16. **Teorema 16.** Para $|x|\leq 1$, se tiene

$$x(1+xi)\gamma(xi)+x(1-xi)\gamma(-xi)=-2x^3\gamma\left(-x^2\right)-2x\ln(2)+ \\ +2\tan^{-1}(x)+x\ln\left(1+x^2\right) \qquad (2.21)$$

$$(1+i)\gamma(i)+(1-i)\gamma(-i)=\frac{\pi}{2}+2\ln(\pi)-6\ln(2) \qquad (2.22)$$

2.17. **Teorema 17.** Para $|x|\leq 1$, se tiene

$$x^2\left(\frac{\gamma(xi)}{1-xi}-\frac{\gamma(-xi)}{1+xi}\right)=-\frac{2xi}{1+x^2}+2i\tan^{-1}(x)+ \\ +\frac{1}{1-xi}\sum_{n=2}^{\infty}\frac{Li_n(xi)-xi}{n}-\frac{1}{1+xi}\sum_{n=2}^{\infty}\frac{Li_n(-xi)+xi}{n} \qquad (2.23)$$

2.18. **Teorema 18.** Para $|z|<1, z=a+bi, \overline{z}=a-bi$, se tiene

$$\overline{z}\gamma(\overline{z})-z\gamma(z)=-2i\tan^{-1}\left(\frac{b}{1-a}\right)+\sum_{n=2}^{\infty}\left((1-z)z^{n-1}-(1-\overline{z})\overline{z}^{n-1}\right)\ln(n) \qquad (2.24)$$

$$\frac{\pi}{2}=i\left[\left(\frac{1-i}{2}\right)\gamma\left(\frac{1-i}{2}\right)-\left(\frac{1+i}{2}\right)\gamma\left(\frac{1+i}{2}\right)\right] \\ -i\sum_{n=2}^{\infty}\left[\left(\frac{1-i}{2}\right)\left(\frac{1+i}{2}\right)^{n-1}-\left(\frac{1+i}{2}\right)\left(\frac{1-i}{2}\right)^{n-1}\right]\ln(n) \qquad (2.25)$$

2.19. **Teorema 19.** Para $|x| \le 1$, se tiene

$$\int_0^x \gamma(t)\,dt = \sum_{n=2}^{\infty} \frac{(-1)^n}{n} Li_{n+1}(x) \tag{2.26}$$

2.20. **Teorema 20.** Sea $S = \{(x,y) : x \in (-1,1), y \in (-1,1), x+y-xy \in (-1,1)\}$, Para $(x,y) \in S, z = x+y-xy$, se tiene

$$z\gamma(z) = x\gamma(x) + y\gamma(y) + \sum_{n=2}^{\infty} \left((1-x)x^{n-1} + (1-y)y^{n-1} - (1-z)z^{n-1}\right) ln(n) \tag{2.27}$$

2.21. **Teorema 21.** Para $|z| \le 1, z \ne 1$, se tiene

$$\gamma(z) = -\frac{ln(1-z)}{z} - \int_0^1\int_0^1 \frac{(1-x)x^{y/(1-y)}}{(1-xz)(1-y)^2} dydx \tag{2.28}$$

$$ln\left(\frac{\pi}{2}\right) = \int_0^1\int_0^1 \frac{(1-x)x^{y/(1-y)}}{(1+x)(1-y)^2} dydx \tag{2.29}$$

2.22. **Teorema 22.** Para $m \in \mathbb{N}$, se tiene

$$\begin{aligned}(-m \pm (m+1)i)\gamma(-m \pm (m+1)i) = -\frac{ln(2)}{2} - ln(m+1) \pm i\frac{\pi}{4} + \\ + \sum_{n=1}^{\infty}\left(\frac{2m+1 \mp i}{2m+2}\right)^n \sum_{k=0}^{n} (-1)^{k+1}\binom{n}{k} ln(k+1)\end{aligned} \tag{2.30}$$

2.23. **Teorema 23.** Para $0 < y < 1$, se tiene

$$\begin{aligned}tan^{-1}(y) = -\sum_{n=1}^{\infty}(-1)^n y^{2n-1} ln(2n-1) + \sum_{n=1}^{\infty}(-1)^n y^{2n} ln(2n) + \\ + y\int_0^1 \frac{1-x+ln(x)}{(1+y^2x^2)ln(x)} dx\end{aligned} \tag{2.31}$$

2.24. **Teorema 24.** Para $|z| < 1$, se tiene

$$\frac{ln(1-z)+z\gamma(z)}{1-z}+\frac{ln(1+z)-z\gamma(z)}{1+z}=-2\sum_{n=1}^{\infty}z^{2n-2}\,ln(2n-1) \tag{2.32}$$

$$\frac{ln(1-z)+z\gamma(z)}{1-z}-\frac{ln(1+z)-z\gamma(z)}{1+z}=-2\sum_{n=1}^{\infty}z^{2n-1}\,ln(2n) \tag{2.33}$$

$$\begin{aligned}\frac{ln\left(\frac{2}{\sqrt{3}}\right)-i\frac{\pi}{6}+\frac{i}{\sqrt{3}}\gamma\left(\frac{i}{\sqrt{3}}\right)}{1-\frac{i}{\sqrt{3}}}+\frac{ln\left(\frac{2}{\sqrt{3}}\right)+i\frac{\pi}{6}-\frac{i}{\sqrt{3}}\gamma\left(-\frac{i}{\sqrt{3}}\right)}{1+\frac{i}{\sqrt{3}}}&=\\ =-2\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^{n-1}ln(2n-1)&\end{aligned} \tag{2.34}$$

$$\begin{aligned}\frac{ln\left(\frac{2}{\sqrt{3}}\right)-i\frac{\pi}{6}+\frac{i}{\sqrt{3}}\gamma\left(\frac{i}{\sqrt{3}}\right)}{1-\frac{i}{\sqrt{3}}}-\frac{ln\left(\frac{2}{\sqrt{3}}\right)+i\frac{\pi}{6}-\frac{i}{\sqrt{3}}\gamma\left(-\frac{i}{\sqrt{3}}\right)}{1+\frac{i}{\sqrt{3}}}&=\\ =-2\sqrt{3}\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^{n}ln(2n)&\end{aligned} \tag{2.35}$$

2.25. **Teorema 25.** Para $k\in\mathbb{N}-\{1\}$, se tiene

$$\gamma=\zeta(k)-\sum_{n=1}^{\infty}ln\left(1+n^{-k}\right)+\sum_{m\neq n^k,m,n\in\mathbb{N}}\left(\frac{1}{m}-ln\left(\frac{m+1}{m}\right)\right) \tag{2.36}$$

$$\gamma=\frac{\pi^2}{6}-\frac{sh(\pi)}{\pi}+\sum_{m\neq n^2,m,n\in\mathbb{N}}\left(\frac{1}{m}-ln\left(\frac{m+1}{m}\right)\right) \tag{2.37}$$

$$\gamma=\zeta(3)-\frac{ch\left(\pi\sqrt{3}/2\right)}{\pi}+\sum_{m\neq n^3,m,n\in\mathbb{N}}\left(\frac{1}{m}-ln\left(\frac{m+1}{m}\right)\right) \tag{2.38}$$

2.26. **Teorema 26.** Para $m\in\mathbb{N},|z|\leq 1$, se tiene

$$\frac{1}{m}\sum_{k=0}^{m-1}\gamma\left(ze^{2k\pi i/m}\right)=\sum_{n=0}^{\infty}z^{mn}\left(\frac{1}{mn+1}-ln\left(\frac{mn+2}{mn+1}\right)\right) \tag{2.39}$$

$$\frac{\pi}{3\sqrt{3}}-\frac{1}{3}ln\left(\frac{3\sqrt{3}}{2}\right)=\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{3n+1}-ln\left(\frac{3n+2}{3n+1}\right)\right) \tag{2.40}$$

2.27. **Teorema 27.** Sea $0<\frac{1}{k}<8$, sea $S(k)$ el conjunto definido como:

$$S(k)=\left\{(x,y,z)\in\mathbb{R}^3 : x,y,z\in(-1,1) : (1-x)(1-y)(1-z)=\frac{1}{k}\right\}$$

se tiene

$$x\gamma(x)+y\gamma(y)+z\gamma(z)=ln(k)-\sum_{n=2}^{\infty}\left(x^{n-1}+y^{n-1}+z^{n-1}\right)ln\left(\frac{n}{n-1}\right) \quad ,(x,y,z)\in S(k) \tag{2.41}$$

3 Referencias

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