A Very Brief Introduction to Reflections in 2D Geometric Algebra, and their Use in Solving "Construction" Problems

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Abstract

This document is intended to be a convenient collection of explanations and techniques given elsewhere ([1]-[3]) in the course of solving tangency problems via Geometric Algebra.

Geometric-Algebra Formulas for Plane (2D) Geometry

The Geometric Product, and Relations Derived from It

For any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$,
$\boldsymbol{a}\cdot\boldsymbol{b}=\boldsymbol{b}\cdot\boldsymbol{a}$
$\boldsymbol{b}\wedge\boldsymbol{a}=-\boldsymbol{a}\wedge\boldsymbol{b}$
$\boldsymbol{ab}=\boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{a}\wedge\boldsymbol{b}$
$\boldsymbol{ba}=\boldsymbol{b}\cdot\boldsymbol{a}+\boldsymbol{b}\wedge\boldsymbol{a}=\boldsymbol{a}\cdot\boldsymbol{b}-\boldsymbol{a}\wedge\boldsymbol{b}$

$\boldsymbol{ab}+\boldsymbol{ba}=2\boldsymbol{a}\cdot\boldsymbol{b}$
$\boldsymbol{ab}-\boldsymbol{ba}=2\boldsymbol{a}\wedge\boldsymbol{b}$
$\boldsymbol{ab}=2\boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{ba}$
$\boldsymbol{ab}=2\boldsymbol{a}\wedge\boldsymbol{b}-\boldsymbol{ba}$

Definitions of Inner and Outer Products (Macdonald A. 2010 p. 101.)

The inner product
The inner product of a j-vector A and a k-vector B is
$A\cdot B=\langle AB\rangle_{k-j}$. Note that if $j>k$, then the inner product doesn't exist.
However, in such a case $B\cdot A=\langle BA\rangle_{j-k}$ does exist.

The outer product
The outer product of a j-vector A and a k-vector B is
$A\wedge B=\langle AB\rangle_{k+j}$.

Relations Involving the Outer Product and the Unit Bivector, i.

For any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$,
$\boldsymbol{ia}=-\boldsymbol{ai}$
$\boldsymbol{a}\wedge\boldsymbol{b}=[(\boldsymbol{ai})\cdot\boldsymbol{b}]\,\boldsymbol{i}=-\,[\boldsymbol{a}\cdot(\boldsymbol{bi})]\,\boldsymbol{i}=-\boldsymbol{b}\wedge\boldsymbol{a}$

Equality of Multivectors

For any two multivectors $\mathcal{M}$ and $\mathcal{N}$,
$\mathcal{M}=\mathcal{N}$ if and only if for all k, $\langle\mathcal{M}\rangle_k=\langle\mathcal{N}\rangle_k$.

Formulas Derived from Projections of Vectors and Equality of Multivectors

Any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ can be written in the form of "Fourier expansions" with respect to a third vector, $\boldsymbol{v}$:

$\boldsymbol{a}=(\boldsymbol{a}\cdot\hat{\boldsymbol{v}})\,\hat{\boldsymbol{v}}+[\boldsymbol{a}\cdot(\hat{\boldsymbol{v}}\boldsymbol{i})]\,\hat{\boldsymbol{v}}\boldsymbol{i}$ and $\boldsymbol{b}=(\boldsymbol{b}\cdot\hat{\boldsymbol{v}})\,\hat{\boldsymbol{v}}+[\boldsymbol{b}\cdot(\hat{\boldsymbol{v}}\boldsymbol{i})]\,\hat{\boldsymbol{v}}\boldsymbol{i}$.

Using these expansions,

$$\boldsymbol{ab}=\{(\boldsymbol{a}\cdot\hat{\boldsymbol{v}})\,\hat{\boldsymbol{v}}+[\boldsymbol{a}\cdot(\hat{\boldsymbol{v}}\boldsymbol{i})]\,\hat{\boldsymbol{v}}\boldsymbol{i}\}\,\{(\boldsymbol{b}\cdot\hat{\boldsymbol{v}})\,\hat{\boldsymbol{v}}+[\boldsymbol{b}\cdot(\hat{\boldsymbol{v}}\boldsymbol{i})]\,\hat{\boldsymbol{v}}\boldsymbol{i}\}$$

Equating the scalar parts of both sides of that equation,

$\boldsymbol{a} \cdot \boldsymbol{b} = [\boldsymbol{a} \cdot \hat{\boldsymbol{v}}]\,[\boldsymbol{b} \cdot \hat{\boldsymbol{v}}] + [\boldsymbol{a} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})]\,[\boldsymbol{b} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})]$, and
$\boldsymbol{a} \wedge \boldsymbol{b} = \{[\boldsymbol{a} \cdot \hat{\boldsymbol{v}}]\,[\boldsymbol{b} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})] - [\boldsymbol{a} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})]\,[\boldsymbol{b} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})]\}\,\boldsymbol{i}$.

Also, $a^2 = [\boldsymbol{a} \cdot \hat{\boldsymbol{v}}]^2 + [\boldsymbol{a} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})]^2$, and $b^2 = [\boldsymbol{b} \cdot \hat{\boldsymbol{v}}]^2 + [\boldsymbol{b} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})]^2$.

Reflections of Vectors, Geometric Products, and Rotation operators
For any vector $\boldsymbol{a}$, the product $\hat{\boldsymbol{v}}\boldsymbol{a}\hat{\boldsymbol{v}}$ is the reflection of $\boldsymbol{a}$ with respect to the direction $\hat{\boldsymbol{v}}$.

For any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, $\hat{\boldsymbol{v}}\boldsymbol{a}\boldsymbol{b}\hat{\boldsymbol{v}} = \boldsymbol{b}\boldsymbol{a}$, and $\boldsymbol{v}\boldsymbol{a}\boldsymbol{b}\boldsymbol{v} = v^2\boldsymbol{b}\boldsymbol{a}$. Therefore, $\hat{\boldsymbol{v}}e^{\theta\boldsymbol{i}}\hat{\boldsymbol{v}} = e^{-\theta\boldsymbol{i}}$, and $\boldsymbol{v}e^{\theta\boldsymbol{i}}\boldsymbol{v} = v^2e^{-\theta\boldsymbol{i}}$.

A useful relationship that is valid only in plane geometry: $\boldsymbol{abc} = \boldsymbol{cba}$.
Here is a brief proof:

$$\begin{aligned}
\boldsymbol{abc} &= \{\boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \wedge \boldsymbol{b}\}\,\boldsymbol{c} \\
&= \{\boldsymbol{a} \cdot \boldsymbol{b} + [(\boldsymbol{a}\boldsymbol{i}) \cdot \boldsymbol{b}]\,\boldsymbol{i}\}\,\boldsymbol{c} \\
&= (\boldsymbol{a} \cdot \boldsymbol{b})\,\boldsymbol{c} + [(\boldsymbol{a}\boldsymbol{i}) \cdot \boldsymbol{b}]\,\boldsymbol{i}\boldsymbol{c} \\
&= \boldsymbol{c}\,(\boldsymbol{a} \cdot \boldsymbol{b}) - \boldsymbol{c}\,[(\boldsymbol{a}\boldsymbol{i}) \cdot \boldsymbol{b}]\,\boldsymbol{i} \\
&= \boldsymbol{c}\,(\boldsymbol{a} \cdot \boldsymbol{b}) + \boldsymbol{c}\,[\boldsymbol{a} \cdot (\boldsymbol{b}\boldsymbol{i})]\,\boldsymbol{i} \\
&= \boldsymbol{c}\,(\boldsymbol{b} \cdot \boldsymbol{a}) + \boldsymbol{c}\,[(\boldsymbol{b}\boldsymbol{i}) \cdot \boldsymbol{a}]\,\boldsymbol{i} \\
&= \boldsymbol{c}\,\{\boldsymbol{b} \cdot \boldsymbol{a} + [(\boldsymbol{b}\boldsymbol{i}) \cdot \boldsymbol{a}]\,\boldsymbol{i}\} \\
&= \boldsymbol{c}\,\{\boldsymbol{b} \cdot \boldsymbol{a} + \boldsymbol{b} \wedge \boldsymbol{a}\} \\
&= \boldsymbol{cba}.
\end{aligned}$$

1 Introduction

This document discusses reflections of vectors and of geometrical products of two vectors, in two-dimensional Geometric Algebra (GA). It then uses reflections to solve a simple tangency problem.

2 Reflections in 2D GA

2.1 Reflections of a single vector

For any two vectors $\hat{\boldsymbol{u}}$ and $\boldsymbol{v}$, the product $\hat{\boldsymbol{u}}\boldsymbol{v}\hat{\boldsymbol{u}}$ is

$$\hat{\boldsymbol{u}}\boldsymbol{v}\hat{\boldsymbol{u}} = \{2\hat{\boldsymbol{u}} \wedge \boldsymbol{v} + \boldsymbol{v}\hat{\boldsymbol{u}}\}\,\hat{\boldsymbol{u}} \tag{2.1}$$
$$= \boldsymbol{v} + 2\left[(\hat{\boldsymbol{u}}\boldsymbol{i}) \cdot \boldsymbol{v}\right]\boldsymbol{i}\hat{\boldsymbol{u}} \tag{2.2}$$
$$= \boldsymbol{v} - 2\left[\boldsymbol{v} \cdot (\hat{\boldsymbol{u}}\boldsymbol{i})\right]\hat{\boldsymbol{u}}\boldsymbol{i}, \tag{2.3}$$

which evaluates to the reflection of the reflection of $\boldsymbol{v}$ with respect to $\hat{\boldsymbol{u}}$ (Fig. 2.1).

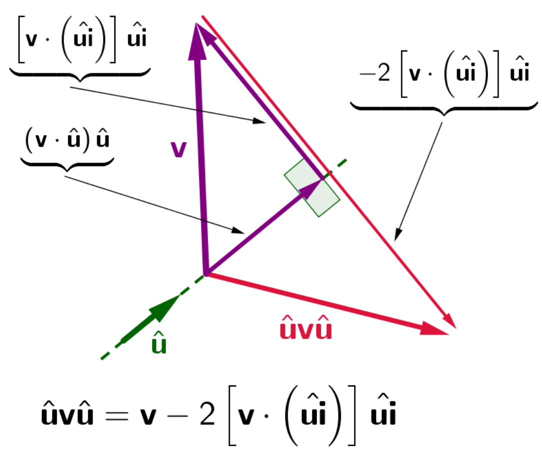

Figure 2.1: Geometric interpretation of $\hat{\boldsymbol{u}}\boldsymbol{v}\hat{\boldsymbol{u}}$, showing why it evaluates to the reflection of $\boldsymbol{v}$ with respect to $\hat{\boldsymbol{u}}$.

We also note that because $\boldsymbol{u} = |\boldsymbol{u}|\,\hat{\boldsymbol{u}}$,

$$\boldsymbol{u}\boldsymbol{v}\boldsymbol{u} = u^2\,(\hat{\boldsymbol{u}}\boldsymbol{v}\hat{\boldsymbol{u}}) = \boldsymbol{u}^2\boldsymbol{v} - 2\left[\boldsymbol{v} \cdot (\boldsymbol{u}\boldsymbol{i})\right]\boldsymbol{u}\boldsymbol{i}. \tag{2.4}$$

2.2 Reflections of a bivector, and of a geometric product of two vectors

The product $\hat{\boldsymbol{u}}\boldsymbol{v}\boldsymbol{w}\hat{\boldsymbol{u}}$ is

$$\begin{aligned}
\hat{\boldsymbol{u}}\boldsymbol{v}\boldsymbol{w}\hat{\boldsymbol{u}} &= \hat{\boldsymbol{u}}\left(\boldsymbol{v}\cdot\boldsymbol{w}+\boldsymbol{v}\wedge\boldsymbol{w}\right)\hat{\boldsymbol{u}} \\
&= \hat{\boldsymbol{u}}\left(\boldsymbol{v}\cdot\boldsymbol{w}\right)\hat{\boldsymbol{u}}+\hat{\boldsymbol{u}}\left(\boldsymbol{v}\wedge\boldsymbol{w}\right)\hat{\boldsymbol{u}} \\
&= \hat{\boldsymbol{u}}^2\left(\boldsymbol{v}\cdot\boldsymbol{w}\right)+\hat{\boldsymbol{u}}\left[\left(\boldsymbol{v}\boldsymbol{i}\right)\cdot\boldsymbol{w}\right]\boldsymbol{i}\hat{\boldsymbol{u}} \\
&= \boldsymbol{v}\cdot\boldsymbol{w}+\hat{\boldsymbol{u}}\left[-\boldsymbol{v}\cdot\left(\boldsymbol{w}\boldsymbol{i}\right)\right]\left(-\hat{\boldsymbol{u}}\boldsymbol{i}\right) \\
&= \boldsymbol{v}\cdot\boldsymbol{w}+\hat{\boldsymbol{u}}^2\left[\left(\boldsymbol{w}\boldsymbol{i}\right)\cdot\boldsymbol{v}\right]\boldsymbol{i} \\
&= \boldsymbol{w}\cdot\boldsymbol{v}+\boldsymbol{w}\wedge\boldsymbol{v} \\
&= \boldsymbol{w}\boldsymbol{v}.
\end{aligned}$$

In other words, the reflection of the geometric product $\boldsymbol{v}\boldsymbol{w}$ is $\boldsymbol{w}\boldsymbol{v}$, and does not depend on the direction of the vector with respect to which it is reflected. We saw that the scalar part of $\boldsymbol{v}\boldsymbol{w}$ was unaffected by the reflection, but the bivector part was reversed.

Further to that point, the reflection of geometric product of $\boldsymbol{v}$ and $\boldsymbol{w}$ is equal to the geometric product of the two vectors' reflections:

$$\begin{aligned}
\hat{\boldsymbol{u}}\boldsymbol{v}\boldsymbol{w}\hat{\boldsymbol{u}} &= \hat{\boldsymbol{u}}\boldsymbol{v}\left(\hat{\boldsymbol{u}}\hat{\boldsymbol{u}}\right)\boldsymbol{w}\hat{\boldsymbol{u}} \\
&= \left(\hat{\boldsymbol{u}}\boldsymbol{v}\hat{\boldsymbol{u}}\right)\left(\hat{\boldsymbol{u}}\boldsymbol{w}\hat{\boldsymbol{u}}\right).
\end{aligned}$$

That observation provides a geometric interpretation (Fig. 2.2) of why reflecting a bivector changes its sign: the direction of the turn from $\boldsymbol{v}$ to $\boldsymbol{w}$ reverses.

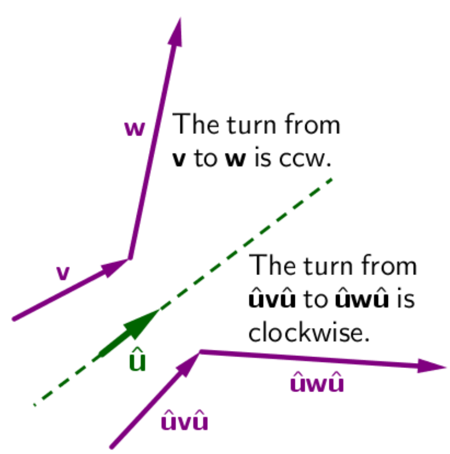

Figure 2.2: Geometric interpretation of $\hat{\boldsymbol{u}}\boldsymbol{v}\boldsymbol{w}\hat{\boldsymbol{u}}$, showing why it evaluates to the reflection of $\boldsymbol{v}$ with respect to $\hat{\boldsymbol{u}}$. Note that $\hat{\boldsymbol{u}}\boldsymbol{v}\boldsymbol{w}\hat{\boldsymbol{u}} = \hat{\boldsymbol{u}}\boldsymbol{v}\left(\hat{\boldsymbol{u}}\hat{\boldsymbol{u}}\right)\boldsymbol{w}\hat{\boldsymbol{u}} = \left(\hat{\boldsymbol{u}}\boldsymbol{v}\hat{\boldsymbol{u}}\right)\left(\hat{\boldsymbol{u}}\boldsymbol{w}\hat{\boldsymbol{u}}\right)$.

3 Use of reflections to solve a simple tangency problem

The problem that we will solve is

> *"Given two coplanar circles, with a point Q on one of them, construct the circles that are tangent to both of the given circles, with point Q as one of the points of tangency"* (Fig. 3.1).

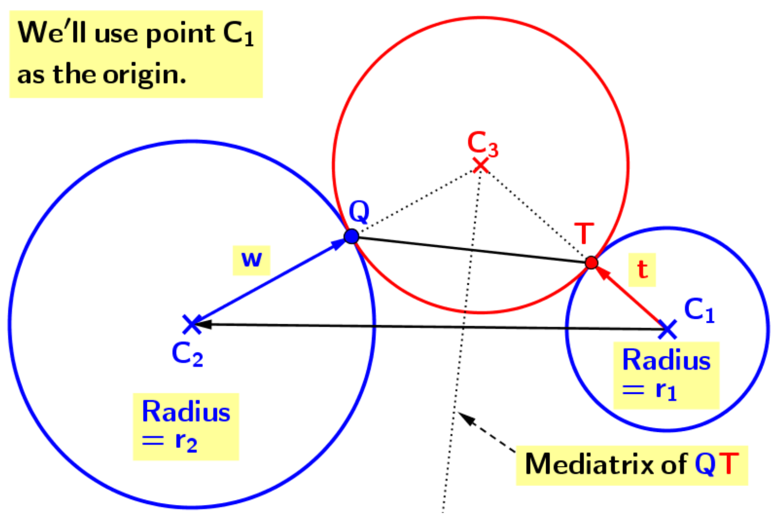

Figure 3.1: Diagram for our problem: *"Given two coplanar circles, with a point Q on one of them, construct the circles that are tangent to both of the given circles, with point Q as one of the points of tangency."*

Several solutions that use rotations are given by [1], but here we will use reflections. The triangle TQC_3 is isosceles, so $\hat{\boldsymbol{t}}$ is the reflection of $\hat{\boldsymbol{w}}$ with respect to the mediatrix of segment $\overline{QT}$. In order to make use of that fact, we need to express the direction of that mediatrix as a vector written in terms of known quantities. We can do so by constructing another isosceles triangle (C_1SC_3) that has the same mediatrix (Fig. 3.2).

The vector from C_1 to S is $\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}$, so the direction of the mediatrix of $\overline{QT}$ is the vector $[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}]\,\boldsymbol{i}$. The unit vector with that direction is $\dfrac{[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}]\,\boldsymbol{i}}{\|\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}\|}$. Therefore, to express $\hat{\boldsymbol{t}}$ as the reflection of $\hat{\boldsymbol{w}}$ with respect

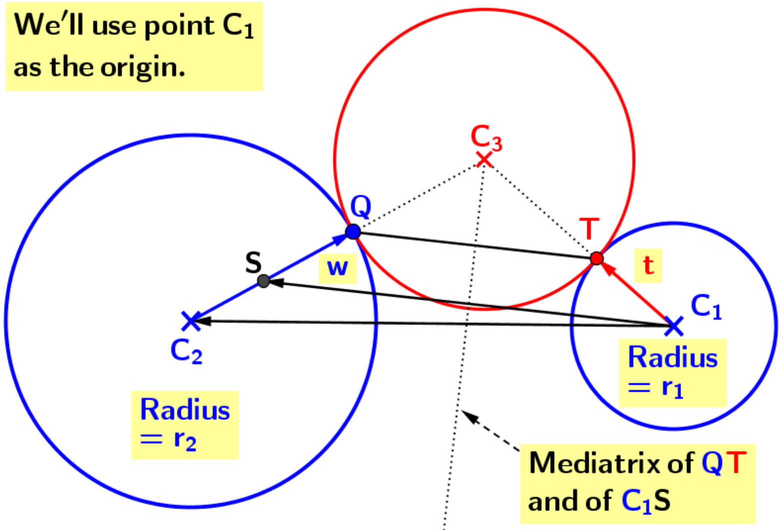

Figure 3.2: Adding segment $\overline{C_1 S}$ to Fig. 3.1 to produce a new isosceles triangle with the same mediatrix as $\overline{QT}$.

to the mediatrix, we write

$$\begin{aligned}\hat{\boldsymbol{t}} &= \left[\frac{[\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}]\,\boldsymbol{i}}{\|\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}\|}\right][\hat{\boldsymbol{w}}]\left[\frac{[\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}]\,\boldsymbol{i}}{\|\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}\|}\right] \\ &= \frac{\{[\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}]\,\boldsymbol{i}\}\,[\hat{\boldsymbol{w}}]\,\{[\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}]\,\boldsymbol{i}\}}{[\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}]^2} \\ &= \frac{[\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}]\,[\hat{\boldsymbol{w}}]\,[\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}]\,\boldsymbol{i}\boldsymbol{i}}{[\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}]^2},\end{aligned}$$

from which

$$\boldsymbol{t}\;\left(= r_1\hat{\boldsymbol{t}}\right) = -r_1\left\{\frac{[\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}]\,[\hat{\boldsymbol{w}}]\,[\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}]}{[\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}]^2}\right\}. \tag{3.1}$$

Interestingly, the geometric interpretation of that result is that $\hat{\boldsymbol{t}}$ and $-\hat{\boldsymbol{w}}$ are reflections of each other with respect to the vector $\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}$. After expanding and rearranging the numerator and denominator of (3.1), then using $\boldsymbol{w} = r_2\hat{\boldsymbol{w}}$, we obtain

$$\boldsymbol{t} = r_1\left\{\frac{\left[{c_2}^2 - (r_2 - r_1)^2\right]\boldsymbol{w} - 2\,[\boldsymbol{c}_2\cdot\boldsymbol{w} + r_2\,(r_2 - r_1)]\,\boldsymbol{c}_2}{r_2{c_2}^2 + 2\,(r_2 - r_1)\,\boldsymbol{c}_2\cdot\boldsymbol{w} + r_2\,(r_2 - r_1)^2}\right\}. \tag{3.2}$$

References

[1] J. Smith, “Rotations of Vectors Via Geometric Algebra: Explanation, and Usage in Solving Classic Geometric ”Construction” Problems” (Version of 11 February 2016). Available at http://vixra.org/abs/1605.0232 .

[2] “Solution of the Special Case ”CLP” of the Problem of Apollonius via Vector Rotations using Geometric Algebra”. Available at http://vixra.org/abs/1605.0314.

[3] “The Problem of Apollonius as an Opportunity for Teaching Students to Use Reflections and Rotations to Solve Geometry Problems via Geometric (Clifford) Algebra”. Available at http://vixra.org/abs/1605.0233.