Mathematical Formulas : Part 4

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Abstract

In this paper we give some formulas related with the constant pi:

$$\pi = 3.141592653589\ldots$$

1. Introducción

Recordamos una fórmula integral para la constante pi:

(1) $$\pi = 8\int_0^\infty \left(1-\sqrt[4]{f(x)}\right)dx$$

donde

(2) $$f(x) = -\frac{1}{3}-\frac{\sqrt[3]{2}}{3x}\,(3-4x^2)R(x)^{-1/3}+\frac{1}{3\sqrt[3]{2}\,x}\,R(x)^{1/3}$$

(3) $$R(x) = 9x+16x^3+3\sqrt{3}\,\sqrt{4-13x^2+32x^4}$$

2. Introducción. En esta nota mostramos una familia de fórmulas para la constante pi, las fórmulas son del tipo:

(4) $$\pi = A\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n}B^n\,u_n v_n$$

donde: $A\in\mathbb{N}, B\in\mathbb{R}^+$, v_n es una sucesión de la forma:

(5) $$v_{n+2} = \alpha\,v_{n+1}+\beta\,v_n$$

y u_n , es la sucesión definida por:

(6) $$u_{n+2} = u_{n+1}+u_n\ \ ,u_1 = 1, u_2 = 3, n\in\mathbb{N}$$

(7) $$u_n = \phi^n + \left(-\frac{1}{\phi}\right)^n \quad , \phi = \frac{1+\sqrt{5}}{2} \ , n \in \mathbb{N}$$

(8) $$\{u_n\} = \{1,3,4,7,11,18,29, \dots\}$$

Fórmulas

(9) $$\begin{cases} \pi = 4\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{\sqrt{3}-1}{2}\right)^n u_n v_n \\ v_{n+2} = -2v_{n+1} - 2v_n \ , v_1 = 1, v_2 = -2 \end{cases}$$

(10) $$\begin{cases} \pi = 4\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{\sqrt{37}-3}{14}\right)^n u_n v_n \\ v_{n+2} = -4v_{n+1} - 5v_n \ , v_1 = 1, v_2 = -4 \end{cases}$$

(11) $$\begin{cases} \pi = 4\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{3\sqrt{2}-2}{14}\right)^n u_n v_n \\ v_{n+2} = -6v_{n+1} - 10v_n \ , v_1 = 1, v_2 = -6 \end{cases}$$

(12) $$\begin{cases} \pi = 4\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{3\sqrt{13}-5}{46}\right)^n u_n v_n \\ v_{n+2} = -8v_{n+1} - 17v_n \ , v_1 = 1, v_2 = -8 \end{cases}$$

(13) $$\begin{cases} \pi = 4\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{\sqrt{43}-3}{34}\right)^n u_n v_n \\ v_{n+2} = -10v_{n+1} - 26v_n \ , v_1 = 1, v_2 = -10 \end{cases}$$

(14) $$\begin{cases} \pi = 4\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{\sqrt{237}-7}{94}\right)^n u_n v_n \\ v_{n+2} = -12v_{n+1} - 37v_n \ , v_1 = 1, v_2 = -12 \end{cases}$$

(15) $$\begin{cases} \pi = 4\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{\sqrt{78}-4}{62}\right)^n u_n v_n \\ v_{n+2} = -14v_{n+1} - 50v_n \ , v_1 = 1, v_2 = -14 \end{cases}$$

(16)
$$\begin{cases} \pi = 4\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{\sqrt{93}-5}{34}\right)^n u_n v_n \\ v_{n+2} = -6v_{n+1} - 13v_n \,, v_1 = 2, v_2 = -12 \end{cases}$$

(17)
$$\begin{cases} \pi = 4\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{\sqrt{173}-7}{62}\right)^n u_n v_n \\ v_{n+2} = -8v_{n+1} - 25v_n \,, v_1 = 3, v_2 = -24 \end{cases}$$

(18)
$$\begin{cases} \pi = 4\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{\sqrt{277}-9}{98}\right)^n u_n v_n \\ v_{n+2} = -10v_{n+1} - 41v_n \,, v_1 = 4, v_2 = -40 \end{cases}$$

(19)
$$\begin{cases} \pi = 4\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{\sqrt{213}-7}{82}\right)^n u_n v_n \\ v_{n+2} = -10v_{n+1} - 29v_n \,, v_1 = 2, v_2 = -20 \end{cases}$$

(20)
$$\begin{cases} \pi = 4\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{\sqrt{62}-4}{46}\right)^n u_n v_n \\ v_{n+2} = -10v_{n+1} - 34v_n \,, v_1 = 3, v_2 = -30 \end{cases}$$

(21)
$$\begin{cases} \pi = 6\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{\sqrt{30\sqrt{3}+12}-3-\sqrt{3}}{12}\right)^n u_n v_n \\ v_{n+2} = -2v_{n+1} - 2v_n \,, v_1 = 1, v_2 = -2 \end{cases}$$

(22)
$$\begin{cases} \pi = 6\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{1}{6}\left(\frac{\sqrt{3}-4}{13}\right)\left(3+2\sqrt{3}-3\sqrt{\frac{20\sqrt{3}+19}{3}}\right)\right)^n u_n v_n \\ v_{n+2} = -4v_{n+1} - 5v_n \,, v_1 = 1, v_2 = -4 \end{cases}$$

(23)
$$\begin{cases} \pi = 6\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{3\left(\sqrt{10\sqrt{3}+\frac{44}{3}}-1-\sqrt{3}\right)}{4\left(4\sqrt{3}+9\right)}\right)^n u_n v_n \\ v_{n+2} = -6v_{n+1} - 10v_n \,, v_1 = 1, v_2 = -6 \end{cases}$$

(24)
$$\begin{cases}\pi = 6\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{3\left(\sqrt{20\sqrt{3}+\frac{41}{3}}-2-\sqrt{3}\right)}{2\left(5\sqrt{3}+36\right)}\right)^{n} u_n v_n \\ v_{n+2} = -6v_{n+1} - 13v_n\ , v_1 = 2, v_2 = -12\end{cases}$$

(25)
$$\begin{cases}\pi = 8\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{\sqrt{8\sqrt{2}-6}-\sqrt{2}}{4}\right)^{n} u_n v_n \\ v_{n+2} = -2v_{n+1} - 2v_n\ , v_1 = 1, v_2 = -2\end{cases}$$

(26)
$$\begin{cases}\pi = 8\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{\sqrt{29-12\sqrt{2}}-2\sqrt{2}+1}{6\sqrt{2}+2}\right)^{n} u_n v_n \\ v_{n+2} = -4v_{n+1} - 5v_n\ , v_1 = 1, v_2 = -4\end{cases}$$

(27)
$$\begin{cases}\pi = 8\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{\sqrt{94-52\sqrt{2}}-3\sqrt{2}+2}{16\sqrt{2}-4}\right)^{n} u_n v_n \\ v_{n+2} = -6v_{n+1} - 10v_n\ , v_1 = 1, v_2 = -6\end{cases}$$

(28)
$$\begin{cases}\pi = 8\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{\sqrt{2\sqrt{2}+31}-3\sqrt{2}+1}{10\sqrt{2}+14}\right)^{n} u_n v_n \\ v_{n+2} = -6v_{n+1} - 13v_n\ , v_1 = 2, v_2 = -12\end{cases}$$

(29)
$$\begin{cases}\pi = 12\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{\sqrt{28-14\sqrt{3}}-3+\sqrt{3}}{4}\right)^{n} u_n v_n \\ v_{n+2} = -2v_{n+1} - 2v_n\ , v_1 = 1, v_2 = -2\end{cases}$$

(30)
$$\begin{cases}\pi = 12\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{\sqrt{153-84\sqrt{3}}-5+2\sqrt{3}}{20-6\sqrt{3}}\right)^{n} u_n v_n \\ v_{n+2} = -4v_{n+1} - 5v_n\ , v_1 = 1, v_2 = -4\end{cases}$$

(31)
$$\begin{cases}\pi = 12\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{\sqrt{348-194\sqrt{3}}-7+3\sqrt{3}}{4\left(11-4\sqrt{3}\right)}\right)^{n} u_n v_n \\ v_{n+2} = -6v_{n+1} - 10v_n\ , v_1 = 1, v_2 = -6\end{cases}$$

(32)
$$\begin{cases}\pi = 12\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{\sqrt{327-176\sqrt{3}}-8+3\sqrt{3}}{44-10\sqrt{3}}\right)^{n} u_n v_n \\ v_{n+2} = -6v_{n+1} - 13v_n\ , v_1 = 2, v_2 = -12\end{cases}$$

3. Introducción . En esta nota mostramos algunas fórmulas del tipo arcotangente para la constante pi:

Las fórmulas son de la forma

(33) $$\pi = k\tan^{-1}(q) \pm k\tan^{-1}(r)$$

donde

(34) $$k \in \mathbb{N}, q \in \mathbb{Q}, r \in \mathbb{R} - \mathbb{Q}\,, \tan^{-1} x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2m+1}\,, x \in (-1,1)$$

Una característica interesante es que:

(35) $$\lim_{n\to\infty} q_n = r \in \mathbb{R} - \mathbb{Q} \;\; \wedge \;\; \lim_{n\to\infty} r_n = 0$$

Ejemplos de estas fórmulas son:

(36) $$\pi = 8\tan^{-1}\left(\frac{2}{5}\right) + 8\tan^{-1}\left(29\sqrt{2} - 41\right)$$

(37) $$\pi = 6\tan^{-1}\left(\frac{1}{2}\right) + 6\tan^{-1}\left(\frac{5\sqrt{3} - 8}{11}\right)$$

Fórmulas

(38) $$\pi = 8\tan^{-1}\left(\frac{a_n}{b_n}\right) + (-1)^n 8\tan^{-1}\left((a_n^2 + b_n^2)\sqrt{2} - (b_n^2 - a_n^2 + 2a_n b_n)\right)$$

donde

(39) $$n \in \mathbb{N}\,, a_{n+1} = b_n, b_{n+1} = a_n + 2b_n, a_1 = 1, b_1 = 2$$

(40) $$\{(a_n, b_n)\} = \{(1,2), (2,5), (5,12), (12,29), (29,70), (70,169), \dots\}$$

Ejemplos:

(41) $$\pi = 8\tan^{-1}\left(\frac{1}{2}\right) - 8\tan^{-1}\left(5\sqrt{2} - 7\right)$$

(42) $$\pi = 8\tan^{-1}\left(\frac{2}{5}\right) + 8\tan^{-1}\left(29\sqrt{2} - 41\right)$$

(43) $$\pi = 8\tan^{-1}\left(\frac{5}{12}\right) - 8\tan^{-1}\left(169\sqrt{2} - 239\right)$$

(44) $$\pi = 8\tan^{-1}\left(\frac{12}{29}\right) + 8\tan^{-1}\left(985\sqrt{2} - 1393\right)$$

La fórmula (38) se puede escribir como:

(45) $$\pi = 8\tan^{-1}\left(\frac{a_n}{b_n}\right) + 8(-1)^n\tan^{-1}\left(b_{2n}\sqrt{2} - a_{2n} - b_{2n}\right)$$

(46) $$\pi = 6\tan^{-1}\left(\frac{a_n}{b_n}\right) + 6\tan^{-1}\left(\frac{(a_n^2 + b_n^2)\sqrt{3} - 4a_n b_n}{3b_n^2 - a_n^2}\right)$$

donde

(47) $$n \in \mathbb{N}, a_{n+1} = 2a_n + b_n, b_{n+1} = 3a_n + 2b_n, a_1 = 1, b_1 = 2$$

(48) $$\{(a_n, b_n)\} = \{(1,2), (4,7), (15,26), (56,97), (209,362), \dots\}$$

Ejemplos:

(49) $$\pi = 6\tan^{-1}\left(\frac{1}{2}\right) + 6\tan^{-1}\left(\frac{5\sqrt{3} - 8}{11}\right)$$

(50) $$\pi = 6\tan^{-1}\left(\frac{4}{7}\right) + 6\tan^{-1}\left(\frac{65\sqrt{3} - 112}{131}\right)$$

(51) $$\pi = 6\tan^{-1}\left(\frac{15}{26}\right) + 6\tan^{-1}\left(\frac{901\sqrt{3} - 1560}{1803}\right)$$

(52) $$\pi = 5\tan^{-1}\left(\frac{a_n}{a_{n+1} - a_n}\right) + 5\tan^{-1}\left(\frac{(a_{n+1} - a_n)\sqrt{5 - 2\sqrt{5}} - a_n}{a_{n+1} - a_n + a_n\sqrt{5 - 2\sqrt{5}}}\right)$$

donde

(53) $$n \in \mathbb{N} \cup \{0\}, a_{n+4} = \frac{1}{5}(20a_{n+3} - 20a_{n+2} + 4a_n)$$

(54) $$\{a_n\} = \left\{1,4,12,32,\frac{404}{5},\frac{992}{5},\dots\right\}$$

Ejemplos:

(55) $$\pi = 5\tan^{-1}\left(\frac{1}{3}\right) + 5\tan^{-1}\left(\frac{3\sqrt{5-2\sqrt{5}}-1}{3+\sqrt{5-2\sqrt{5}}}\right)$$

(56) $$\pi = 5\tan^{-1}\left(\frac{1}{2}\right) + 5\tan^{-1}\left(\frac{2\sqrt{5-2\sqrt{5}}-1}{2+\sqrt{5-2\sqrt{5}}}\right)$$

(57) $$\pi = 5\tan^{-1}\left(\frac{3}{5}\right) + 5\tan^{-1}\left(\frac{5\sqrt{5-2\sqrt{5}}-3}{5+3\sqrt{5-2\sqrt{5}}}\right)$$

(58) $$\pi = 5\tan^{-1}\left(\frac{40}{61}\right) + 5\tan^{-1}\left(\frac{61\sqrt{5-2\sqrt{5}}-40}{61+40\sqrt{5-2\sqrt{5}}}\right)$$

(59) $$\pi = 12\tan^{-1}\left(\frac{a_n}{b_n}\right) + 12\tan^{-1}\left(\frac{2b_n - a_n - b_n\sqrt{3}}{b_n + 2a_n - a_n\sqrt{3}}\right)$$

donde

(60) $$n \in \mathbb{N}\,,\, a_{n+1} = -a_n + b_n\,,\, b_{n+1} = -a_n + 3b_n\,,\, a_1 = 1, b_1 = 3$$

(61) $$\{(a_n, b_n)\} = \{(1,3), (2,8), (6,22), (16,60), (44,164), (120,448), \ldots\}$$

Ejemplos:

(62) $$\pi = 12\tan^{-1}\left(\frac{1}{3}\right) - 12\tan^{-1}\left(\frac{3\sqrt{3}-5}{5-\sqrt{3}}\right)$$

(63) $$\pi = 12\tan^{-1}\left(\frac{1}{4}\right) + 12\tan^{-1}\left(\frac{7-4\sqrt{3}}{6-\sqrt{3}}\right)$$

(64) $$\pi = 12\tan^{-1}\left(\frac{3}{11}\right) - 12\tan^{-1}\left(\frac{11\sqrt{3}-19}{17-3\sqrt{3}}\right)$$

(65) $$\pi = 12\tan^{-1}\left(\frac{b_n - a_n}{b_n + a_n}\right) - 12\tan^{-1}\left(\frac{(a_n^2 + b_n^2)\sqrt{3} - 4a_n b_n}{3b_n^2 - a_n^2}\right)$$

donde

(66) $$n \in \mathbb{N}, a_{n+1} = 2a_n + b_n, b_{n+1} = 3a_n + 2b_n, a_1 = 1, b_1 = 2$$

(67) $$\{(a_n, b_n)\} = \{(1,2), (4,7), (15,26), (56,97), (209,362), \ldots\}$$

Ejemplos:

(68) $$\pi = 12\tan^{-1}\left(\frac{1}{3}\right) - 12\tan^{-1}\left(\frac{5\sqrt{3}-8}{11}\right)$$

(69) $$\pi = 12\tan^{-1}\left(\frac{3}{11}\right) - 12\tan^{-1}\left(\frac{65\sqrt{3}-112}{131}\right)$$

(70) $$\pi = 12\tan^{-1}\left(\frac{11}{41}\right) - 12\tan^{-1}\left(\frac{901\sqrt{3}-1560}{1803}\right)$$

(71) $$\pi = 12\tan^{-1}\left(\frac{a_n}{b_n}\right) + 12\tan^{-1}\left(\frac{2b_n - a_n - b_n\sqrt{3}}{2a_n + b_n - a_n\sqrt{3}}\right)$$

donde

(72) $$n \in \mathbb{N}\,, a_{n+1} = b_n\,, b_{n+1} = -a_n + 4b_n\,, a_1 = 1, b_1 = 4$$

(73) $$\{(a_n, b_n)\} = \{(1,4), (4,15), (15,56), (56,209), (209,780), (780,2911), \dots\}$$

Ejemplos:

(74) $$\pi = 12\tan^{-1}\left(\frac{1}{4}\right) + 12\tan^{-1}\left(\frac{7-4\sqrt{3}}{6-\sqrt{3}}\right)$$

(75) $$\pi = 12\tan^{-1}\left(\frac{4}{15}\right) + 12\tan^{-1}\left(\frac{26-15\sqrt{3}}{23-4\sqrt{3}}\right)$$

(76) $$\pi = 12\tan^{-1}\left(\frac{15}{56}\right) + 12\tan^{-1}\left(\frac{97-56\sqrt{3}}{86-15\sqrt{3}}\right)$$

(77) $$\pi = 5\tan^{-1}\left(\frac{a_n}{b_n}\right) + 5\tan^{-1}\left(\frac{b_n\sqrt{5-2\sqrt{5}} - a_n}{b_n + a_n\sqrt{5-2\sqrt{5}}}\right)$$

donde $\frac{a_n}{b_n}$, son los convergentes de la fracción continua del número $\sqrt{5-2\sqrt{5}}$:

(78) $$\sqrt{5-2\sqrt{5}} = \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}} = [0; 1,2,1,1,1,10,1,1,7, \dots]$$

(79) $$\left\{\frac{a_n}{b_n}\right\} = \left\{1, \frac{2}{3}, \frac{3}{4}, \frac{5}{7}, \frac{8}{11}, \dots\right\}$$

Ejemplos:

(80) $$\pi = 5\tan^{-1}\left(\frac{2}{3}\right) + 5\tan^{-1}\left(\frac{3\sqrt{5-2\sqrt{5}}-2}{3+2\sqrt{5-2\sqrt{5}}}\right)$$

(81) $$\pi = 5\tan^{-1}\left(\frac{3}{4}\right) - 5\tan^{-1}\left(\frac{3-4\sqrt{5-2\sqrt{5}}}{4+3\sqrt{5-2\sqrt{5}}}\right)$$

(82) $$\pi = 5\tan^{-1}\left(\frac{5}{7}\right) + 5\tan^{-1}\left(\frac{7\sqrt{5-2\sqrt{5}}-5}{7+5\sqrt{5-2\sqrt{5}}}\right)$$

(83) $$\pi = 24\tan^{-1}\left(\frac{a_n}{b_n}\right) + 24\tan^{-1}\left(\frac{b_n(\sqrt{3}-\sqrt{2})(\sqrt{2}-1)-a_n}{b_n+a_n(\sqrt{3}-\sqrt{2})(\sqrt{2}-1)}\right)$$

donde $\frac{a_n}{b_n}$, son los convergentes de la fracción continua de $(\sqrt{3}-\sqrt{2})(\sqrt{2}-1)$:

(84) $$(\sqrt{3}-\sqrt{2})(\sqrt{2}-1) = \cfrac{1}{7+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cdots}}}} = [0;7,1,1,2,9,47,1,8,14,\ldots]$$

(85) $$\left\{\frac{a_n}{b_n}\right\} = \left\{\frac{1}{7},\frac{1}{8},\frac{2}{15},\frac{5}{38},\ldots\right\}$$

Ejemplos:

(86) $$\pi = 24\tan^{-1}\left(\frac{1}{7}\right) - 24\tan^{-1}\left(\frac{1-7(\sqrt{3}-\sqrt{2})(\sqrt{2}-1)}{7+(\sqrt{3}-\sqrt{2})(\sqrt{2}-1)}\right)$$

(87) $$\pi = 24\tan^{-1}\left(\frac{1}{8}\right) + 24\tan^{-1}\left(\frac{8(\sqrt{3}-\sqrt{2})(\sqrt{2}-1)-1}{8+(\sqrt{3}-\sqrt{2})(\sqrt{2}-1)}\right)$$

(88) $$\pi = 24\tan^{-1}\left(\frac{2}{15}\right) - 24\tan^{-1}\left(\frac{2-15(\sqrt{3}-\sqrt{2})(\sqrt{2}-1)}{15+2(\sqrt{3}-\sqrt{2})(\sqrt{2}-1)}\right)$$

4. Introducción. En esta nota mostramos fórmulas que involucran a la constante pi:

Fórmulas

(89)
$$\frac{35+7\sqrt[3]{189+63\sqrt[3]{189+63\sqrt[3]{189+\cdots}}}}{\pi} = \frac{5\cdot 64}{7}\prod_{n=1}^{\infty}\left(1-\left(\frac{5}{14}\right)^2\right)\left(1-\left(\frac{1}{14n-7}\right)^2\right)^6$$

(90)
$$\frac{35-7\left\{3+\frac{1}{63}\left(3+\frac{1}{63}\left(3+\frac{1}{63}(3+\cdots)^3\right)^3\right)^3\right\}}{\pi} = \frac{64}{7}\prod_{n=1}^{\infty}\left(1-\left(\frac{1}{14n}\right)^2\right)\left(1-\left(\frac{3}{14n-7}\right)^2\right)^6$$

(91)
$$\frac{-35+7\sqrt[3]{189+63\sqrt[3]{189+\cdots}}-7\left\{3+\frac{1}{63}\left(3+\frac{1}{63}(3+\cdots)^3\right)^3\right\}}{\pi} = \frac{128}{7}\prod_{n=1}^{\infty}\left(1-\left(\frac{1}{7n}\right)^2\right)\left(1-\left(\frac{4}{14n-7}\right)^2\right)\left(1-\left(\frac{5}{14n-7}\right)^2\right)^5$$

(92)
$$\pi\left(\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}\right) = \frac{1}{1+a}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{k}\sum_{s=0}^{m}\binom{n}{k}\binom{k}{m}\binom{m}{s}(-1)^k\left(\frac{a}{1+a}\right)^n a^{-k}f(k,m,s)$$

donde

$$a>1$$

(93)
$$f(k,m,s)=\sum_{\substack{r=1\\ r\neq 5}}^{9}\frac{1}{4k+2m+4s+r}$$

(94)
$$\pi\left(\frac{1}{2\sqrt{2}}-\frac{1}{3\sqrt{3}}\right) = \frac{1}{1+a}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{k}\sum_{s=0}^{m}\binom{n}{k}\binom{k}{m}\binom{m}{s}(-1)^k\left(\frac{a}{1+a}\right)^n a^{-k}f(k,m,s)$$

donde

$$a > 1$$

(95) $$f(k,m,s) = \sum_{r=1}^{4} \frac{(-1)^{r-1}}{4k+2m+4s+r} + \sum_{r=6}^{9} \frac{(-1)^{r-1}}{4k+2m+4s+r}$$

(96) $$\pi\left(\frac{1}{2\sqrt{2}} \pm \frac{1}{3\sqrt{3}}\right) = \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\left(\frac{a}{1+a}\right)^{n-k}\left(\frac{1}{1+a}\right)^{k+1}\left\{\left(\frac{2}{4k+1}+\frac{2}{4k+3}\right) \pm \left(\frac{1}{3k+1}+\frac{1}{3k+2}\right)\right\}$$

$$a > 0$$

Ejemplo: $a = 1/2$

(97) $$\pi\left(\frac{1}{2\sqrt{2}} \pm \frac{1}{3\sqrt{3}}\right) = \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\left(\frac{1}{3}\right)^{n-k}\left(\frac{2}{3}\right)^{k+1}\left\{\left(\frac{2}{4k+1}+\frac{2}{4k+3}\right) \pm \left(\frac{1}{3k+1}+\frac{1}{3k+2}\right)\right\}$$

(98) $$\pi = 4\sum_{n=0}^{\infty}\frac{(-4)^n b^{-4n-1}}{4n+1} + 4\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\left(\frac{a}{4+a}\right)^{n-k}\left(\frac{1}{4+a}\right)^{k+1}\frac{b^{4k+3}}{4k+3}$$

$$a > \frac{b^4-4}{2}\ , b > \sqrt{2}$$

(99) $$\pi\sqrt[4]{8} = 4\sum_{n=0}^{\infty}\frac{(-1)^n 2^{-n}}{4n+1} + 4\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\left(\frac{1}{2}\right)^{n-k}\frac{1}{4k+3}$$

(100) $$\pi = 24\left(1+\sqrt{2}-\sqrt{3}\right) - 24\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}\frac{(-1)^k}{a^k}\frac{\left(\left(1/\sqrt{3}\right)^{2k+2m+1} - \left(\sqrt{2}-1\right)^{2k+2m+1}\right)}{2k+2m+1}$$

$$a > 2/9$$

Ejemplo: $a = \frac{92}{9} - 7\sqrt{2}$

(101) $$\pi = 24\left(1+\sqrt{3}-\sqrt{2}\right) - 24\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}\frac{(-1)^k}{a^k}\frac{\left(\left(\sqrt{2}-1\right)^{2k+2m+1}-\left(2-\sqrt{3}\right)^{2k+2m+1}\right)}{2k+2m+1}$$

$$a > 10 - 7\sqrt{2}$$

Ejemplo: $a = 62 - 7\sqrt{2} - 30\sqrt{3}$

(102) $$\pi = \frac{3\sqrt{3}}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\left(\frac{a}{1+a}\right)^{n-k}\left(\frac{1}{1+a}\right)^{k+1}\left(\frac{1}{3k+1}+\frac{1}{3k+2}\right)$$

$$a > 0$$

Ejemplo: $a = 1/2$

(103) $$\pi = \frac{3\sqrt{3}}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\left(\frac{1}{3}\right)^{n-k}\left(\frac{2}{3}\right)^{k+1}\left(\frac{1}{3k+1}+\frac{1}{3k+2}\right)$$

(104) $$\pi = \frac{3\sqrt{3}}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{a}{1+a}\right)^{n-k}\left(\frac{1}{1+a}\right)^{k+1}\frac{(k!)^2}{(2k+1)!}$$

$$a > -1/2$$

Ejemplos: $a = 0, a = 1/2\,, a = -1/8$

(105) $$\pi = \frac{3\sqrt{3}}{2}\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n+1)!}$$

(106) $$\pi = \frac{3\sqrt{3}}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{3}\right)^{n-k}\left(\frac{2}{3}\right)^{k+1}\frac{(k!)^2}{(2k+1)!}$$

(107) $$\pi = \frac{3\sqrt{3}}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\left(-\frac{1}{7}\right)^{n-k}\left(\frac{8}{7}\right)^{k+1}\frac{(k!)^2}{(2k+1)!}$$

(108) $$\pi = 4\sum_{n=0}^{\infty}\frac{1}{n+1}\left(-\frac{2}{3}\right)^n c_n$$

donde

(109) $$c_{n+1} = -\frac{(6n+4)c_n + (2n+1)c_{n-1}}{4n+4}\,,\ c_0 = \frac{1}{2}\,,\ c_1 = -\frac{1}{2}$$

(110) $$e^{2\pi/3} = \left(\sum_{n=0}^{\infty} \frac{(-1)^n \left(1/\sqrt{3}\right)^n a_n}{n!}\right)^2 + \left(\sum_{n=0}^{\infty} \frac{(-1)^n \left(1/\sqrt{3}\right)^n b_n}{n!}\right)^2$$

(111) $$e^{\pi/2} = \left(\sum_{n=0}^{\infty} \frac{(-1)^n \left(\sqrt{2}-1\right)^n a_n}{n!}\right)^2 + \left(\sum_{n=0}^{\infty} \frac{(-1)^n \left(\sqrt{2}-1\right)^n b_n}{n!}\right)^2$$

(112) $$e^{\pi/3} = \left(\sum_{n=0}^{\infty} \frac{(-1)^n \left(2-\sqrt{3}\right)^n a_n}{n!}\right)^2 + \left(\sum_{n=0}^{\infty} \frac{(-1)^n \left(2-\sqrt{3}\right)^n b_n}{n!}\right)^2$$

En las fórmulas (110),(111),(112), se tiene:

(113) $$a_{n+1} = -2a_n - n\, b_n\, ,\, b_{n+1} = n\, a_n - 2b_n\, ,\, a_0 = 1, b_0 = 0, n \in \mathbb{N} \cup \{0\}$$

(114) $$\pi = \sqrt{3} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{2k+1} \left(\frac{a}{1+a}\right)^{n-k} \left(\frac{3}{1+a}\right)^{k+1} \, , a > 2$$

Ejemplo: $a = 3$

(115) $$\pi = \sqrt{3} \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^{n+1} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{2k+1}$$

(116) $$\pi = 3\sqrt{3} \sum_{n=0}^{\infty} (-1)^n\, c^{-n-1}\, I(a,b,n)$$

donde

(117) $$I(a,b,n) = \sum_{k=0}^{n} \binom{n}{k} (b-a)^{n-k} \left(\frac{(1+a)^{k+n+1} - a^{k+n+1}}{k+n+1}\right)$$

(118) $$I(a,b,n) = \sum_{k=0}^{n} \binom{n}{k} (a-b)^{n-k} \left(\frac{(1+b)^{k+n+1} - b^{k+n+1}}{k+n+1}\right)$$

$$c > \frac{3}{2}\, , a + b = 1\, , c = 1 - a\, b$$

Ejemplos: $c = 3, a = -1, b = 2; c = 2, a = \frac{1+\sqrt{5}}{2}, b = \frac{1-\sqrt{5}}{2}$

(119) $$\pi = \sqrt{3} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k 3^{-k}}{k+n+1}$$

$$(120)\ \pi = 3\sqrt{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2}\right)^{n+1}\sum_{k=0}^{n}\binom{n}{k}\frac{\left(\sqrt{5}\right)^{n-k}}{k+n+1}\left(\left(\frac{3-\sqrt{5}}{2}\right)^{k+n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{k+n+1}\right)$$

$$(121)\qquad \frac{1}{\pi} = \frac{3q}{16p}\sum_{n=0}^{\infty}\left(\frac{1}{3\cdot 2^{12}\cdot p^2}\right)^n u_n(p,q)$$

donde

$$(122) u_n(p,q) = \sum_{k=0}^{n}(-1)^k\binom{2k}{k}^2\binom{4k}{2k}\binom{2n-2k}{n-k}(28k+3)p^{2k}\left(3\cdot 2^{10}(p^2-3q^2)\right)^{n-k}$$

$$(123)\qquad p,q\in\mathbb{N}\ ,|3q^2-p^2|<p^2\ ,u_n(p,q)\in\mathbb{N}$$

Sean $p_m, q_m, m \in \mathbb{N}$, las sucesiones definidas por la recurrencia:

$$(124)\qquad \begin{cases} p_{m+1} = 2p_m + 3q_m \\ q_{m+1} = p_m + 2q_m \\ p_1 = 2\ , q_1 = 1 \end{cases}$$

Los números (p_m, q_m), satisfacen las condiciones de la fórmula (123), $u_n(p_m, q_m)$ se reduce a:

$$(125)\quad u_n(p_m,q_m) = \sum_{k=0}^{n}(-1)^k\binom{2k}{k}^2\binom{4k}{2k}\binom{2n-2k}{n-k}(28k+3)p_m^{2k}(3\cdot 2^{10})^{n-k}$$

Algunos valores de (p_m, q_m) , son:

$$(126)\qquad \{(p_m,q_m)\} = \{(2,1),(7,4),(26,15),(97,56),\ldots\}$$

Ejemplo: $(p,q) = (7,4)$

$$(127)\qquad \frac{1}{\pi} = \frac{3}{28}\sum_{n=0}^{\infty}\left(\frac{1}{196}\right)^n\sum_{k=0}^{n}(-1)^k\binom{2k}{k}^2\binom{4k}{2k}\binom{2n-2k}{n-k}(28k+3)\left(\frac{49}{3072}\right)^k$$

$$(128)\quad \pi = \frac{80}{11}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}\left(\frac{9}{11}\right)^n\left(\frac{20}{9}\right)^k\left(\frac{1}{2}\right)^m\frac{(-1)^{n+k+m}\left(\left(\sqrt{2}\right)^{k+m+1}-1\right)}{k+m+1}$$

$$(129)\quad \pi = \frac{16\left(5+2\sqrt{2}\right)}{17}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}A^nB^k\left(\frac{1}{2}\right)^m\frac{(-1)^{n+k+m}\left(\left(\sqrt{2}\right)^{k+m+1}-1\right)}{k+m+1}$$

$$A=\frac{3+8\sqrt{2}}{17}\quad ,B=\frac{4+8\sqrt{2}}{7}$$

(130) $$\pi=6\sqrt{3}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{n+k+1}\left(2^{n-1}\left(\frac{7}{8}\right)^k-\frac{1}{3}\left(\frac{4}{3}\right)^n\left(\frac{7}{12}\right)^k\right)$$

(131) $$\pi=4(1-a)\sum_{n=1}^{\infty}H_n\sum_{k=0}^{n}\binom{n}{k}(-1)^{\left[\frac{3k+2}{2}\right]}a^{n-k}(1-a)^k$$

$$0<a<1\,,H_n=\sum_{k=1}^{n}\frac{1}{k}$$

(132) $$\pi=\frac{3\sqrt{3}}{2m-1}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{2m-1}{2m}\right)^n\sum_{k=0}^{\left[\frac{n-1}{2}\right]}(-1)^k\binom{n}{2k+1}\left(\frac{3}{(2m-1)^2}\right)^k$$

$$m\in\mathbb{N}-\{1\}$$

Ejemplo: $m=3$

(133) $$\pi=\frac{3\sqrt{3}}{5}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{5}{6}\right)^n\sum_{k=0}^{\left[\frac{n-1}{2}\right]}(-1)^k\binom{n}{2k+1}\left(\frac{3}{25}\right)^k$$

(134) $$\pi=4\sum_{n=1}^{\infty}\left(\frac{\sqrt{5}-1}{2}\right)^n c_n$$

(135) $$\pi=6\sum_{n=1}^{\infty}\left(\frac{\sqrt{9+12\sqrt{3}}-3}{6}\right)^n c_n$$

(136) $$\pi=8\sum_{n=1}^{\infty}\left(\frac{\sqrt{4\sqrt{2}-3}-1}{2}\right)^n c_n$$

(137) $$\pi=12\sum_{n=1}^{\infty}\left(\frac{\sqrt{9-4\sqrt{3}}-1}{2}\right)^n c_n$$

En las fórmulas (134),(135),(136),(137), los números c_n , $n\in\mathbb{N}$, se definen como:

(138) $$c_n = \sum_{k=\left[\frac{n-2}{4}\right]}^{\left[\frac{n-1}{2}\right]} \frac{(-1)^k}{2k+1} \binom{2k+1}{n-2k-1} \text{ , } \binom{n}{k} = 0 \; si \; n < k$$

Algunos valores de c_n , son:

(139) $$\{c_n\} = \left\{1,1,-\frac{1}{3},-1,-\frac{4}{5},\frac{2}{3},\frac{13}{17},1,-\frac{17}{9},-\frac{19}{5},\ldots\right\}$$

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