THE THEORY OF ULTRALOGICS
(Ultralogics and More)
Standard Edition

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This book is written for a 6 x 9 book formate.
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First formal announcement of many of these results appeared in

(1) Mathematical philosophy, Vol. 2 (No. 6) (1981), #81T-03-529, p. 527.

(2) A useful *-real valued function, Vol. 4 (No. 4) (1983), #83T-26-280, p. 318.

(3) Nonstandard consequence operators I, Vol. 5 (No. 1) (1984), #84T-03-61, p. 129.

(4) Nonstandard consequence operators II, Vol. 5 (No. 2) (1984), #84T-03-93, p. 195.

(5) D-world alphabets I, Vol. 5 (No. 4) (1984), #84T-03-320, p. 269.

(6) D-world alphabets II, Vol. 5 (No. 5) (1984), #84T-03-374, p. 328.

(7) A solution to the grand unification problem, Vol. 7 (No. 2) (1986), #86T-85-41, p. 238.

Some of the first refereed papers relative to MA-model concepts and its mathematical construction.

(1) A special isomorphism between superstructures, Kobe J. Math., 10(2)(1993), 125-129.

(2) Fractals and ultrasmooth microeffects, J. Math. Physics, 30(4), April 1989, 805-808.

(3) Physics is legislated by a cosmogony, Speculations in Science and Technology, 11(1) (1988), 17-24.

(4) Nonstandard consequence operators, Kobe J. Math., 4(1)(1987), 1-14.

(5) Mathematical philosophy and developmental processes, Nature and System, 5(1/2)(1983), 17-36.

CONTENTS

NOTES

Some of the following typographical errors (or confusions) may (do) appear in this edition.

[1] There may be a very few times when it is not apparent as to the necessary structure for the results stated. Of course, one can always stay with the EGS. I may have written $\mathcal{M}$ when I meant $\mathcal{M}_1$ or conversely.

[2] The actual reason for the inverse order discussed on page 5 is that it is usefull when considering adjective reasoning discussed later in this edition.

[3] I give two different superstructure constructions and certain processes used to obtain nonstandard models. It would have been better to concentrated on the second construction which is the one actually used.

[4] Relative to the alphabet $\mathcal{A}$. It is trivially obvious that one can include within this alphabet the written symbols used by an intelligent life form that uses a written language and deductive rules similar to those used by humankind. If one goes to the extreme and requires infinitely many such intelligent life forms, then generalized languages as discussed on page 87 can be utilized.

[5] Beginning in Chapter IV, due to the complexity of the first-order statements, I adopt the process of replacing set-theoretically defined predicates (abbreviations) such as $x \in y,\ x = y,\ 1 \leq x < z$ etc. by $(x \in y),\ (x = y),\ (1 \leq x < z)$. Of course, the $=$ is interpreted as set-theoretic equality.

[7] This book is written as a 6 X 9 formate book.

Chapter 1

INTUITIVE CONCEPTS

1.1 The Alphabet, Words, and Choice Sets.

There exists at the instant of time you read this sentence a finite set of all the symbols you have previously used throughout your life for your various forms of communication and human deduction. You may also include frames from sound motion picture film, TV tape, and the like, if you wish, in the event you require visual or audio stimuli for your deductive processes. Let A_h be the set of such symbols for a given human being at this instant of time and H_t the set of all human beings who exist at this instant of time. Now let $\mathcal{A} = \bigcup\{A_h \bigm| h \in H_t\}$ cal A symbol be the *alphabet* alphabet for humanity at this instant of time. The set $\mathcal{A}$ may be enlarged to include all of the symbols which humanity has ever used, if we wished. However, for our purposes the set $\mathcal{A}$ will suffice. Observe that the set $\mathcal{A}$ is finite and among the numerous references relative to alphabets, I direct your attention to [3] [7] [12] [13] [21].

Assume that there is included in the set $\mathcal{A}$ a distinct symbol which represents a blank symbol, say something like $|||$. Now following the above references, any finite string (with repetition) of elements from $\mathcal{A}$ which is nonempty is called a *word*. word Let $\mathcal{W}$ be the set of all words. In certain applications of these mathematical methods, a word is also called an *intuitive* or *naive readable sentence*. Only a fragment of the set $\mathcal{W}$ is used in this investigation and, in most cases, this fragment will be the set of *meaningful sentences* or a portion of a *formal language*. The metalanguage may be assume to be written in a different color than is $\mathcal{W}$. Also, the concept of the empty word will NOT be employed.

Next, apply the concept that Markov calls the *abstraction of* abstraction of identity *identity* [12] to two words W_1 and W_2. The two words are *equal* definition of equal words if they are composed of the same symbols in the same order written left to right or right to left etc. The finitary character of each word allows for such an identification. Another intuitive string concept required for this discussion is the notion of the *juxtaposition operation* juxtaposition operation or *join* join of two strings such as $W_1,\ W_2$ and this is denoted by W_1W_2 or W_2W_1. Apparently, since about 1914 [22], these two intuitive string concepts relative to word theory have been accepted as a reasonable consequence of the finitary character of such forms.

Following Robinson's procedure in [15] applied to $\mathcal{W}$ rather than to a formal language, assume that there exists an injection i from $\mathcal{W}$ onto $\mathbb{N}$, symbol of natural numbers the set of natural numbers. Intuitively, such an injection exists since $\mathcal{W}$ is countably infinite. Now the term "intuitive" definition of intuitive for this research is utilized to denote the words, various grammatical rules, the informal logical procedures, and the like used in ordinary communication. This is to differentiate our common modes of descriptive communication from the formal theory into which these intuitive objects are mapped. Select a fixed injection for this and all other investigations. Notice that Robinson used any set U of cardinality greater than or equal to that of his set of well-formed formulas. The procedure of intuitively mapping objects such as the set $\mathcal{W}$ onto concrete mathematical objects is well established and has been a major procedure in geometry since Descartes. The Cantor Axiom used by Hilbert and Birkhoff [2] for modern geometry assumes that such as map exists from the set of points in a straight line onto the real numbers. Evidently, this injection $i\colon \mathcal{W} \to \mathbb{N}$ falls into a category of similar content as these well-known geometric assumptions. Indeed, i can be an into Gödel coding.

Prior to continuing this introductory section a brief discussion of the Set Theory being used and related matters appears useful. The general set theory being used is **ZFC** = **ZF** + **AC**. The **ZF** (i.e. Zermelo-Fraenkel axioms) and the Axiom of Choice **AC** may be found listed on pages 2 —19 of [5] among hundreds of other references. Within this general set theory we are working within a model for the axiom system **ZFH** = **ZFA** + **AC** + A **is countably infinite**. The axiom system **ZFA** is the Zermelo-Franenkel axiom system with atoms (i.e. urelements or individuals). The set of atoms is the A in the above formulation. All of the axioms are expressible in a first-order language with the predicates $\in$ and $=$, where $\emptyset$ and A are constants. The almost completely written axioms or modifications to **ZF** + **AC** axioms that yield **ZFA** + **AC** may be found on pages 122 of [5], page 44 of [6] and, due to the use of individuals, the actual system studied throughout [20]. I point out that the language used is a first-order language with the logical axioms for equality.

Assuming the consistency of **ZF**, Gödel showed that there is a model in which the **ZFC** axioms hold. Thus the consistency of **ZF** implies the consistency of **ZFC**. Using **ZFC** our model construction for **ZFH** is a slight modification of that which appears in problem 1 on page 51 of [6]. In the modification, let C be a countably infinite set of infinite subsets of the ω in the **ZFC** model. Moreover, we bijectively map the order relation for ω onto $C - \{a_0\}$. Thus, for a bijection

$f\colon (C - \{a_0\}) \to \omega$ define the well-ordering $<$ on $C - \{a_0\}$ as follows: For each $x,\ y \in (C - \{a_0\}),\ x < y$ iff $f(x) \in f(y)$. By the construction of the model, this well-ordering is also a member of this model. We also have need of an interpretation of $=$ for the set $(C - \{a_0\})$. This is to be the logical equality and as such is interpreted to be the identity relation on $(C - \{a_0\})$ which also exists within this model. The set $(C - \{a_0\})$ with its order and equality relation is to be considered the set of natural numbers $\mathbb{N}$ within this model. On the other hand, we also have within this model its ω that will be used when certain constructions are considered. The interpretation of $=$ for objects not in $(C - \{a_0\})$ is set-theoretic equality and the $\in$ in this model is the same as the $\in$ in our model for **ZFC.** All of this yields a model for the **ZFH** axioms that are, hence, consistent relative to **ZF**. Set $\mathcal{W}$ will be intuitively considered an object in this model. It does not contain the empty set since the empty word is not used. The injection i is to be considered as a intuitive map from $\mathcal{W}$ onto $\mathbb{N}$ within this model for **ZFH.** [See page 16 for detailed refinements.]

All of the new results are obtained by using informal mathematical reasoning relative to **ZFH**, proving results by means of the "observer language" and using acceptable mathematical procedures. We subscribe to the remarks quoted in chapter III of the work by Rosiowa and Sikorski [14] as well as the observations made by Stoll [18, p. 228] relative to Rosser's philosophy of mathematics. That is that the procedures used, even though informal in nature, are capable of formalization, and such things as the "formal" proofs of the formal consistency of **ZFH** relative to **ZF** by means of model theory methods imply that the same informal but acceptable procedures which we use, when convincingly presented, will not produce any contradictions.

Now, intuitively, consider that there is a fixed dictionary D that uses a subset of an individual's personal $Y \subset \mathcal{A}$. Let W_Y be the set of all words generated by Y. Then descriptions of natural processes, of their behaviors and developments, as well as psychological descriptions for human behavior, philosophical descriptions for belief-systems, life styles and any other descriptions that are of interest are elements of the set $\mathcal{P}(W_Y)$, Symbol for power set the power set of W_Y. Such sets are also informally called *describing sets.* The meanings of such sets are understandable by the individual and have great content. Now there exists a set $B_Y \subset \mathcal{P}(W_Y)$ composed of all and only these describing sets. Evidently, the set B_Y is finite. Consider the set $\mathcal{P}(B_Y)$. In some applications of the forthcoming mathematical results, the set $\mathcal{P}(B_Y)$ is called a *free (will) choice set.* For the *universal free (will) choice set*, simply consider the set of all $\mathcal{P}(B_Y)$ as Y

varies over all humanity which exists at a given instant of time. This is still a finite set. Notice that all that has been done to obtain these free choice sets can easily be formulated with respect to the set $\mathbb{N}$. Simply consider $i[W_Y]$, and then the formal describing sets are the corresponding elements of $\mathcal{P}(i[W_Y])$ under the injection i. The intuitive describing sets can be recaptured by considering the inverse image of i on these elements of $\mathcal{P}(i[W_Y])$. The (general) Axiom of Choice is not necessary in order to obtain these free choice sets. However, in the sequel, The Axiom of Choice is used in the construction of the NSP-structure. Consequently, The Axiom of Choice is utilized when we interpret, in some applications, the free choice sets as elements of the NSP-structure.

1.2 Readable Sentences.

For the purposes of this research, not only is the finitary concepts of the abstraction of identity and join accepted but evidently a third fundamental procedure needs to be introduced and investigated. Consider the symbol string 'mathematics'. Now this can be obtained or "read" by numerous applications of the join operation with symbol strings of lesser length. For example, let $W_1 =$ **math**; $W_2 =$ **e**; $W_3 =$ **mat**; $W_4 =$ **ics**. Then **mathematics** $= W_1W_2W_3W_4$. Observe that $W_1,\ W_2,\ W_3,\ W_4$ are the syllables for this word. Clearly, for writing purposes, we could consider **mathematics** $= W_1W_2 \ldots W_{11}$, where $W_j,\ j = 1, \ldots, 11$, are the single letters in the word. The necessity to consider intuitively a symbol string as composed of words of various length joined together from left to right leads, when $i[\mathcal{W}]$ is considered, to the concept of the set of (special) partial sequences.

Let nonempty $H \subset i[\mathcal{W}]$ and $n \in \mathbb{N}$, where $0 \in \mathbb{N}$. For simplicity, let symbol H(exponent)n $H^n = H^{[0,n]}$ be the set of all maps from the segment $[0, n]$ into H. An element of H^n is called a *partial sequence* even though this definition is a slight restriction of the one that usually appears in the literature. Now let $P_H = \bigcup\{H^n \bigm| n \in \mathbb{N}\}$. Symbol P subscript H In general, if $H = i[\mathcal{W}]$, then the H notation will be deleted from such symbols as P_H. Deletion of the H symbols from subscripts The set P_H is a set of partial sequences and is a subset of $P_{i[\mathcal{W}]} = P$. The single symbol P

Let $P_1,\ P_2$ denote the set-theoretic first and second coordinate projection maps. Then the following first-order sentences, where the usual assortment of set-theoretic abbreviations for subset, functions, domains, ranges, etc. are used, hold and represent two basic properties for the object P_H.

$$\forall x((x \in P_H) \to \exists y((y \in \mathbb{N}) \wedge (P_1[x] = [0, y]) \wedge (P_2[x] \subset H)));$$

$$\forall x((x \subset \mathbb{N} \times H) \wedge \exists y((y \in \mathbb{N}) \wedge (P_1[x] = [0,y] \wedge \\ \forall w \forall z \forall z_1(((w \in \mathbb{N}) \wedge (z \in H) \wedge (z_1 \in H) \wedge ((w,z) \in x) \wedge \\ ((w,z_1) \in x)) \to z = z_1) \to x \in P_H). \tag{1.2.1}$$

I point out that ever since Gödel used a natural number coding for certain metamathematical concepts, interpreting naive or intuitive processes involving symbol strings as concepts relative to $\mathbb{N}$ has prevailed and become accepted by mathematicians. As —Kleene— writes: "`Metamathematics has become a branch of number theory.''` [7, p. 205] Consequently, it is clearly justified to use partial sequences to discuss the "ordering" of symbol strings. This ordering will be associated with the finitary ordering obtained by joining words by juxtapositioning them in a specific intuitive order.

Consider $f \in H^n, n \in \mathbb{N}$. Then $f(0), f(1), \ldots, f(n) \in H$. The *order induced by* f is defined to be the simple inverse order generated by f applied to the simple order of $[0,n]$. Formally, for each $f(j),\ f(k) \in f[[0,n]]$, define the order induced by f to be $f(k) \leq_f f(j)$ iff $j \leq k$, Symbol for f(k) less or equal to subscript f f(j) where as usual the order $\leq$ is the simple order on $[0,n]$ induced by the simple order on $\mathbb{N}$. In general, the notation $\leq_f$ will NOT be specifically used to denote this f induced order but, rather, the order will be indicated by writing the symbols $f(n), \ldots, f(0)$ from left to right in an ordered fashion. This corresponds to what the intuitive ordering would be under the inverse of i, i^{-1}, when it is used to recapture the original symbol string. Remember, that everything done with the join operation, partial sequences and the induced order is finitary in character. Apparently, care is required in the selection of the H if all the ways a given word may be partitioned for readability are to be investigated. It is required that $i^{-1}[H]$ contain enough symbols for this purpose. However, an approach is now developed that eliminates this apparent difficulty.

Relate the induced ordering for some $f \in P$ to the join in the following manner. The word W_0 to which $f \in (i[\mathcal{W}])^n,\ n \in \mathbb{N}$, corresponds is

$$W_0 = (i^{-1}((f(n)))(i^{-1}((f(n-1))) \cdots (i^{-1}((f(0))). \tag{1.2.2}$$

Using the abstraction of identity and this ordered join concept, a basic equivalence relation is obvious. For $f,\ g \in P\ f \in H^n,\ g \in H^m$, let

$$f \sim g \text{ iff} \\ (i^{-1}((f(n))) \cdots (i^{-1}((f(0))) = (i^{-1}((g(m))) \cdots (i^{-1}(((0))). \tag{1.2.3}$$

Observe that this process is still finitary in nature and consequently effectively recognizable. Clearly, "$\sim$" is an equivalence relation on P since the identity $=$ is such a relation on $\mathcal{W}$. For each $f \in P$ let $[f]$ denote the corresponding *basic equivalence class.* Now if the *cardinality of an intuitive word* $|W_0| = m + 1 =$ the total number of symbols in W_0 including repetitions and $W_0 = (i^{-1}((f(m))) \cdots (i^{-1}((f(0)))$, then each equivalence class is a nonempty **finite** set and theoretically each element in each equivalence class can be effectively recognized.

Recall that many of the intuitive concepts associated with word theory and algorithms [7] [12] [13] [22] have much less rigorously defined concepts than this equivalence relation even though such word theory concepts have been extensively employed. Before proceeding, however, here are some of the simple facts about these equivalence classes. Consider $[f]$ and let the corresponding word for f be W_0, where $|W_0| = m+1$. Then there exists two unique maps $f_m,\ f_0 \in P$ such that $f_m \sim f_0,\ f_m \in (i[\mathcal{W}])^m,\ f_0 \in (i[\mathcal{W}])^0$ and

$$(1.2.4) \qquad W_0 = (i^{-1}((f_m(m))) \cdots (i^{-1}((f_m(0))) = (i^{-1}(f_0(0))).$$

Furthermore, for each $k \in \mathbb{N}$ such that $0 \leq k \leq m$, there exists $g \in [f]$ such that $g \in (i[\mathcal{W}])^k$. And, for each $k \in \mathbb{N}$ such that $k > m$ there does not exist any $g \in (i[\mathcal{W}])^k$ such that $g \in [f]$. properties of the cardinality of $[f]$ Finally, how is a particular class $[f]$ to be intuitively interpreted? interpretation for a class $[f]$ In order to interpret a class $[f]$ in $\mathcal{W}$, simply select any element in $[f]$, say f_n, and effectively construct the word $(i^{-1}((f_n(n))) \cdots (i^{-1}((f_n(0))) = W_0$. The word W_0 is called the *intuitive or naive interpretation* for the class $[f]$.

Let $\mathcal{E}$ be *set of all equivalence classes generated by $\sim$ on P.* The set $\mathcal{E}$ is called *the set of (formal) readable sentences.* The term "readable sentence" is used in two contexts, the intuitive readable sentence that is a member of $\mathcal{W}$ and the corresponding formal readable sentence in $\mathcal{E}$. The terms "intuitive" or "formal" will not be used where no confusing would occur.

1.3 Human Deduction.

As has become the custom, the concepts of human deduction (i.e. reasoning) will first be discussed intuitively with respect to the set $\mathcal{W}$. Certain metatheorems relative to such processes will be established prior to associating these processes with the more formal set $\mathcal{E}$. Recall Tarski's [21] basic axioms for the undefined (finite) consequence operators C on a set of meaningful sentences A. As usual, let $|A| =$

the cardinality of A, the symbol $\mathcal{P}(A)$ denote the powerset of A, $F(A)$ denote the set of all finite subsets of A and $C\colon \mathcal{P}(A) \to \mathcal{P}(A)$. Tarski first bounds the cardinality of A

$$(1)\ 0 < |A| \leq \aleph_0$$

Now the map C is a (finite) *consequence operator* if $A \neq \emptyset$ and

$$\boxed{\begin{array}{l} (2)\ \text{for each } B \subset A,\ \text{then } B \subset C(B) \subset A, \\[1em] (3)\ \text{for each } B \subset A,\ \text{then } C(C(B)) = C(B), \\[1em] (4)\ \text{for each } B \subset A,\ \text{then } C(B) = \bigcup\{C(F) \bigm| F \in F(B)\} \end{array}}$$

Axioms (2), (3), (4) appear to be considerably more significant than does axiom (1) in these applications and the cardinality of the set of sentences considered is not, usually, so bounded. Actually the consequence operator is generated by a slightly more general concept which is called the deductive process. Let A be a nonempty set. Then any nonempty $k \subset F(A) \times A$ is a *deductive process*. The term *deductive operator* is also used. A deductive process $k \subset F(A) \times A$ is *total* if for each nonempty $B \subset A$ there exists some $F \in F(B)$ and some $a \in A$ such that $(F, a) \in k$. For a total $k \subset F(A) \times A$, let

$$\begin{gathered} \text{(i) } C_k = \{(x, y) \bigm| x \in \mathcal{P}(A) \wedge y \in \mathcal{P}(A) \wedge y \neq \emptyset \wedge \\ \forall w(w \in A \wedge w \in y \leftrightarrow \exists z(z \in F(x) \wedge (z, w) \in k))\} \qquad (1.3.1) \\ \text{And, } (B, \emptyset) \in C_k \leftrightarrow B = \emptyset \leftrightarrow \emptyset \notin P_1[k]. \end{gathered}$$

Notice that the definition of C_k implies that $C_k\colon \mathcal{P}(A) \to \mathcal{P}(A)$.

Assume that A is a nonempty arbitrary set that corresponds to Tarski's set of meaningful sentences with the exception that axiom (1) need not hold for A.

Theorem 1.3.1. *Let total $k \subset F(A) \times A$. Then C_k satisfies Tarski's axiom (4).*

Proof. Obviously, if $B \subset A$, then $C_k(B) \subset A$. Let $C_k(B) = \emptyset$. Then $B = \emptyset$ implies that $C_k(B) = \bigcup\{\emptyset\} = \bigcup\{C_k(F) \mid F \in F(\emptyset)\} = \bigcup\{C_k(\emptyset)\}$. Consequently, assume that $C_k(B) \neq \emptyset$ and $x \in C_k(B)$. Then there exists some $F_0 \in F(B)$ such that $F_0 \subset B \subset A$ and $(F_0, x) \in k$. From the definition of C_k, this implies that $x \in C_k(F_0)$. Thus $C_k(B) \subset \bigcup\{C_k(F) \bigm| F \in F(B)\}$. On the other hand, assume that $x \in \bigcup\{C_k(F) \bigm| F \in F(B)\}$. Then there exists some $F_1 \in F(B)$

such that $x \in C_k(F_1)$. Hence there exists some $F_2 \in F(F_1) \subset F(B)$ and $(F_2, x) \in k$. Consequently, $x \in C_k(B)$ and this completes the proof. ▮

Many of the usual deduction processes, such as propositional deduction, satisfy the following additional properties. Let $k \subset F(A) \times A$ be *ordinary* whenever

(i) if $(A_0, b) \in k$ and $A_0 \subset D \in F(A)$, then $(D, b) \in k$ and

(ii) if $\{(A_0, b_1), \ldots, (A_0, b_n)\} \in k$ and $(\{b_1, \cdots, b_n\}, c) \in k$, then $(A_0, c) \in k$.

Theorem 1.3.2. *Let total and ordinary $k \subset F(A) \times A$. If C_k satisfies axiom (2) of Tarski, then C_k is a consequence operator.*

Proof. Since axiom (4) holds for C_k, now proceed to establish axiom (3). Tarski [21] has shown that axiom (4) implies that if $B \subset D \subset A$, then $C_k(A) \subset C_k(D)$. Thus assume $B \subset A$. Then $B \subset C_k(B)$ implies that $C_k(B) \subset C_k(C_k(B))$. Now if $C_k(C_k(B)) = \emptyset$, then this implies that $C_k(B) = \emptyset$ and $C_k(C_k(B)) = C_k(B)$. Let $x \in C_k(C_k(B))$. Then there exists some finite $F_0 \subset C_k(B)$ such that $(F_0, x) \in k$. If $F_0 = \emptyset$, then $\emptyset \subset B$ implies $x \in C_k(F_0) \subset C_k(B)$. Assume that $F_0 \neq \emptyset$, say $F_0 = \{a_1, \ldots, a_n\}$. Then there exists finitely many $F_i \subset B,\ i = 1, \ldots, n$ such that $(F_i, a_i) \in k$. Let $F' = F_1 \cup \cdots \cup F_n \in F(B) \subset F(A)$. Then ordinary (i) implies that $(F', a_i) \in k,\ i = 1, \ldots, n$. But $(F_0, x) \in k$ and ordinary part (ii) implies that $(F', x) \in k$. Consequently, $x \in C_k(B)$ and this completes the proof. ▮

In passing note that if axiom (2) holds for C_k and (i) of ordinary holds for k, then for each $x \in A,\ (\{x\}, x) \in k$. Obviously, if for each $x \in A,\ (\{x\}, x) \in k$, then axiom (2) holds for C_k. In this case, we say that the deduction process is *singular*. Combining the above results, we have the next theorem.

Theorem 1.3.3. *If for nonempty A the deductive process $k \subset F(A) \times A$ is a total, ordinary and singular, then C_k is a consequence operator for A.*

Is there an obvious deduction process generated by a given consequence operator? Let C be a consequence operator on $\mathcal{P}(A)$. Define $k_c \subset F(A) \times A$ as follows:

(1.3.2) $$(F, a) \in k_c \text{ iff } F \in F(A) \text{ and } a \in C(F).$$

By axiom (2), it follows that k_c is total.

Theorem 1.3.4. *If C is a consequence operator on A, then $C_{k_c} = C$.*

Proof. First assume that $C(B) = \emptyset$. Then axiom (2) implies that $B = \emptyset$. Hence $\emptyset \notin P_1(k_c)$. From this it follows that $C_{k_c}(\emptyset) = \emptyset = C(B)$. Now let $B \subset A$ and suppose that $x \in C(B)$. Then there exists some $F \in F(B)$ such that $x \in C(F)$. Since $F \in F(A)$, then $(F, x) \in k_c$. From the definition of C_{k_c}, it follows that $x \in C_{k_c}(B)$; which yields that $C(B) \subset C_{k_c}$. Now if $C_{k_c}(B) = \emptyset$, then $C_{k_c}(B) \subset C(B)$. Hence suppose that $y \in C_{k_c}(B)$. Then there exists $F \in F(B)$ such that $(F, y) \in k_c$. Since $F \in F(A)$, then $y \in C(F)$ implies that $C_{k_c}(B) \subset C(B)$. Therefore, for each $B \in \mathcal{P}(A)$, $C_{k_c}(B) = C(B)$ implies that $C_{k_c} = C$ and the proof is complete. ▮

From the above results, a consequence operator can be thought of as being determined by a deductive process and, in certain cases, conversely. When deductive processes are described in a metalanguage, then such properties as total, ordinary or singular can often be easily established. Sometimes we say that a deductive process $k \subset F(A) \times A$ satisfies the Tarski axioms (2), (3), or (4) if C_k satisfies (2), (3), or (4), respectively.

Our next task is to find an appropriate correspondence between any $k \subset F(\mathcal{W}) \times \mathcal{W}$ and some $K \subset F(\mathcal{E}) \times \mathcal{E}$ such that the axioms of Tarski and set-theoretic properties are preserved. It is clear that this relation should be defined relative to the quotient map determined by $\sim$. Thus for each $w \in \mathcal{W}$, let $f_w \in P$ be such that $f_w \in (i[\mathcal{W}])^0$ and $f_w(0) = i(w)$. Consider the intuitive bijection $\Theta\colon \mathcal{W} \to \mathcal{E}$ a *basic bijection*, defined by $\Theta(w) = [f_w]$ for each $w \in \mathcal{W}$. In the usual manner, extend Θ to its corresponding set functions and the like. Since it is not assumed that all readers of this book are aware of these definitions, we present them in the context of consequence operator theory. However, for the next few results, the map will not be restricted to the specially defined map Θ but rather we establish them for any arbitrary injection $\beta\colon A \to X$. Recall that

(i) if $B \subset A$, then $\beta[B] = \{\beta(x) \mid x \in B\}$,

(ii) for $\mathcal{B} \subset \mathcal{P}(A)$, $\beta[\mathcal{B}] = \{\beta[x] \mid x \in \mathcal{B}\}$,

(iii) for $k \subset F(A) \times A$, let $\beta k = \{(\beta[x], \beta(y)) \mid (x, y) \in k\} = \{(z, w) \mid z \in F(x) \land w \in X \land \exists x \exists y (x \in F(A) \land y \in A \land z = \beta[x] \land w = \beta(y) \land (x, y) \in k)\}$,

(iv) for $C\colon \mathcal{P}(A) \to \mathcal{P}(A)$, let $\beta C = \{(\beta[x], \beta[y]) \mid (x, y) \in C\}$.

By considering the inverse, β^{-1}, it is evident that if k is total on A, then βk is total on $\beta[A]$; if k is ordinary on A, then βk is ordinary on $\beta[A]$; and if k is singular on A, then βk is singular on $\beta[A]$.

Numerous propositions are immediate consequences of the set-theoretic definitions associated with the injection β. However, even though the following propositions hold true from elementary algebraic results, we prove them explicitly for they refer to intuitive structures and metatheoretic results. Furthermore, they are of considerable importance to the foundations of this subject.

As before, let β be an injection on A into X, let $k \subset F(A) \times A$ and $C: \mathcal{P}(A) \to \mathcal{P}(A)$.

Theorem 1.3.5. *Let $B \subset \beta[A]$. Then*

$$\beta^{-1}[(\beta C)(B)] = C(\beta^{-1}[B]), \text{ and}$$

$$(\beta C)(\beta[\beta^{-1}[B]]) = (\beta C)(B) = \beta[C(\beta^{-1}[B])]. \tag{1.3.3}$$

Proof. Let $B \subset \beta[A]$. Then $\beta^{-1}[\beta[B]] \in \mathcal{P}(A)$. Hence, $(\beta^{-1}[B], C(\beta^{-1}[B])) \in C$. Consequently,

$$(\beta[\beta^{-1}[B]], \beta[C(\beta^{-1}[B])]) \in \beta C \tag{1.3.4}$$

implies since βC is a map that

$$\beta[C(\beta^{-1}[B])] = (\beta C)(\beta[\beta^{-1}[B]]) = (\beta C)(B). \tag{1.3.5}$$

Thus

$$\beta^{-1}[\beta[C(\beta^{-1}[B])]] = \beta^{-1}[(\beta C)(B)] = C(\beta^{-1}[B]) \tag{1.3.6}$$

and this completes the proof. ▮

The next two results are important consequences of Theorem 1.3.5.

Theorem 1.3.6. *If C is a consequence operator on $\mathcal{P}(A)$, (i.e. it satisfies axioms (2), (3), (4) of Tarski), then βC is a consequence operator on $\mathcal{P}(\beta[A])$.*

Proof. Observe that $\beta[\mathcal{P}(A)] = \mathcal{P}(\beta[A])$ and that $\beta C: \mathcal{P}(\beta[A]) \to \mathcal{P}(\beta[A])$.

(i) Let $B \subset \beta[A]$. Then $\beta^{-1}[B] \subset A$ implies that $\beta^{-1}[B] \subset C(\beta^{-1}[B]) \subset A$. Hence $B \subset \beta[C(\beta^{-1}[B])] = (\beta C)(B) \subset \beta[A]$.

(ii) Let $B \subset \beta[A]$. Then $\beta^{-1}[B] \subset A$ and $C(C(\beta^{-1}[B])) = C(\beta^{-1}[B])$. Consequently

$$\beta[C(C(\beta^{-1}[B]))] = (\beta C)(\beta[C(\beta^{-1}[B])]) =$$
$$(\beta C)(\beta C)(\beta[\beta^{-1}[B]]) = (\beta C)(\beta C)(B) = \tag{1.3.7}$$

$$\beta[C(\beta^{-1}[B])] = (\beta C)(\beta[\beta^{-1}[B]]) = (\beta C)(B).$$

(iii) First, we show that $\beta[F(\beta^{-1}[B])] = f[B]$. Let $F \in F(\beta^{-1}[B])$. Then $\beta[F] \subset B$ and $\beta[F] \in F(B)$. Therefore, $\beta[F(\beta^{-1}[B])] \subset F(B)$. Conversely, let $F \in f(B)$. Then $\beta^{-1}[F] \in F(\beta^{-1}[B])$, since $|F| = |\beta^{-1}[F]|$. Thus $\beta[F(\beta^{-1}[B])] = F(B)$.

For each $B \subset \beta[A]$, $C(\beta^{-1}[B]) = \bigcup\{C(F) \mid F \in F(\beta^{-1}[B])\}$. This implies that

$$\begin{aligned} &\beta[C(\beta^{-1}[B])] = \bigcup\{\beta[C(F)] \mid F \in F(\beta^{-1}[B])\} = \\ &\bigcup\{(\beta C)(\beta[F]) \mid F \in F(\beta^{-1}[B])\} = \\ &\bigcup\{(\beta C)(F) \mid F \in F(B)\} = (\beta C)(B). \end{aligned} \tag{1.3.8}$$

Results (i), (ii), (iii) imply that (βC) is a consequence operator and the proof is complete. ▮

Theorem 1.3.7. *If $k \subset F(A) \times A$ is total, then $\beta(C_k) = C_{\beta k}$.*

Proof. Let $(x, y) \in \beta(C_k)$. Then $(\beta^{-1}[x], \beta^{-1}[y]) \in C_k$. Notice that $\beta^{-1}[y] = \emptyset$ iff $y = \emptyset$. Assume that $z \in \beta^{-1}[y]$. Then there exists some $w_z \in F(\beta^{-1}[x])$ such that $(w_z, z) \in k$. Hence, it follows that for each $\beta(z) \in y$ there exists some $\beta[w_z] \in F(x)$ such that $(\beta[w_z], \beta(z)) \in \beta k$. This leads to $(x, y) \in C_{\beta k}$. Now if $y = \emptyset$, then $\beta^{-1}[y] = \emptyset$ implies that $\beta^{-1}[x] = \emptyset$ and $x = \emptyset$. Hence, $(\emptyset, \emptyset) \in \beta(C_k)$ implies that $(\emptyset, \emptyset) \in C_k$. The definition gives $\emptyset \notin P_1(k)$ and $\beta[\emptyset] = \emptyset \notin P_1(\beta k)$. Thus $(\emptyset, \emptyset) \in C_{\beta k}$.

On the other hand, for each $z \in y \neq \emptyset$ there exists some $F \in F(x)$ such that $(F, z) \in \beta k$. Hence $(\beta^{-1}[F], \beta^{-1}(z)) \in k$ and again $z \in y$ iff $\beta^{-1}(z) \in \beta^{-1}[y]$; $\beta^{-1}[F] \in F(\beta^{-1}[x])$ imply that $(\beta^{-1}[x], \beta^{-1}[y]) \in C_k$. Also, if $y = \emptyset$, then $x = \emptyset$ (i.e. $(\emptyset, \emptyset) \in C_{\beta k}$) and $\emptyset \notin P_1(\beta k)$. Hence $\emptyset \notin P_1(k)$ and $(\emptyset, \emptyset) \in \beta(C_k)$ and the proof is complete. ▮

I will not continue with this piecemeal approach but rather use a more general result established within the next chapter, where $\mathcal{E}$ will be defined on a set A_1 (i.e. $i[\mathcal{W}] = A_1$) that is isomorphic to $\mathbb{N}$ and $A_1 \cap \mathbb{N} = \emptyset$.

NOTES

Chapter 2

THE G-STRUCTURE

2.1 A Basic Construction.

A primary construction will be a *superstructure*. For X, a superstructure is constructed as follows: Let *ground set* $X = X_0$. Then by induction, let $X_{n+1} = X_n \cup \mathcal{P}(X_n)$. Now let $\mathcal{X} = \bigcup\{X_n \mid n \in \omega\}$. The set $\mathcal{X}$ is a *superstructure over* X_0. Within our model for **ZFA** a set B is X_0-*transitive* if for each $x \in B$ either $x \in X_0$ or $x \subset B$.

Theorem 2.1.1 *For each $n \in \omega$, the set X_n is X_0-transitive.*

Proof. The proof is by induction. Let $n = 0$. The X_0 is X_0-transitive immediately from the definition. Assume that for n, the set X_n is X_0-transitive. Consider X_{n+1} in the above construction. We need only check any $x \in X_{n+1} - X_0 = (X_n \cup \mathcal{P}(X_n)) - X_0$. Hence, either $x \in X_n - X_0$ or $x \in \mathcal{P}(X_n) - X_0$. In the first case, $x \subset X_n$ by the induction hypothesis. In the second case, $x \subset X_n$ by the definition of the power set operator. Since $X_n \subset X_{n+1}$, it follows that $x \subset X_{n+1}$. Thus by induction the proof is complete. ▮

Theorem 2.1.2 *For each $n \in \omega$, if $y \in x \in X_{n+1} - X_0$, then $y \in X_n$.*

Proof. For $n = 0$, $y \in x \subset X_0 \Rightarrow y \in X_0$. Assume that it holds for $n-1$, $n \geq 1$. Let $y \in x \in X_{n+1} - X_0$. Then $x \in (X_n \cup \mathcal{P}(X_n)) - X_0$. If $x \in X_n - X_0$, then by the induction hypothesis $y \in X_{n-1}$. But $X_{n-1} \subset X_n$ implies that $y \in X_n$. If $x \in \mathcal{P}(X_n)$, then $x \subset X_n$ implies that $y \in X_n$. By induction the proof is complete. ▮

Obviously, since we have only used facts about **ZF** to establish Theorems 2.1.1, 2.1.2, these theorems hold for superstructures within **ZF**. Recall that if A is a set of atoms, then this means that if $x \in A$, then $x \neq \emptyset$ and $y \in x$ is not defined. A nonempty ground set X_0 is n-*atomic* if $x \in X_0$ implies that $x \neq \emptyset$ and if $y \in x \in X_0$, then $y \notin X_n$. Two important observations relative to n-atomic. If X_0 is a set of atoms, then X_0 is n-atomic for each $n \in \omega$. If X_0 is n-atomic, then X_0 is k-atomic for each k such that $0 \leq k \leq n$. For each X_n, $n \geq 0$, let $M_{X_n} = \{(x, y) \mid (x \in X_n) \land (y \in X_n) \land (x \in y)\}$ and $E_{X_n} = \{(x, y) \mid (x \in X_n) \land (y \in X_n) \land (x = y)\}$, where the $=$ is set-theoretic equality on sets and the identity on atoms. In like manner, for ground set Z, defined M_{Z_n} and E_{Z_n} for the respective Z_n. For $n \geq 1$, an isomorphism β_n from $\langle X_n, M_{X_n}, E_{X_n}, \mathbb{N}, \emptyset\rangle$ onto $\langle Z_n, M_{Z_n}, E_{Z_n}, \mathbb{N}, \emptyset\rangle$ is *special* if $\beta_n(X_k) = Z_k$, $0 \leq k \leq n-1$, where $\mathbb{N}$ is a set of atoms. Observe that since $X_k \in X_{k+1} \subset X_n$, it follows that $X_k \in X_n$.

Theorem 2.1.3 *Let A be a set of atoms. Suppose that for nonempty sets $X,\ Z,\ X \cap A = Z \cap A$ and there exists a bijection $\beta: X \to Z = \beta[X]$, where β is a set-theoretic bijection on sets and the identity on any atoms in $X \cap A$. Consider the sets $X_0 = X \cup A,\ Z_0 = Z \cup A$, and A and $\emptyset$ as the constants that denote a set of atoms and the empty set in our* **ZFA** *model.*

(i) If $X_0,\ Z_0$ are 0-atomic, then the structures $\langle X_0, M_{X_0}, E_{X_0}\rangle$ and $\langle Z_0, M_{Z_0}, E_{Z_0}\rangle$ are isomorphic.

(ii) For each $n \geq 1$, if $X_0,\ Z_0$ are n-atomic, then the structures $\langle X_n, M_{X_n}, E_{X_n}, A, \emptyset\rangle$ and $\langle Z_n, M_{Z_n}, E_{Z_n}, A, \emptyset\rangle$ are isomorphic and the isomorphism is special.

Proof. By $\in$-recursion, define the map γ on X as follows:

$$\text{For } x \in X,\ \gamma(x) = \beta(x);$$
$$\text{For } x \in X_0 - X,\ \gamma(x) = x;$$
$$\text{For } x \in \mathcal{X} - X_0,\ \gamma(x) = \gamma[x].$$

Let $\beta_n = \gamma | X_n$, where $n \in \omega$. We need only show that for each $n \in \omega$, if X_0 and Z_0 are n-atomic, then

$$\text{(A) } \beta_n \text{ is an isomorphism from } \langle X_n, M_{X_n}, E_{X_n}\rangle$$
$$\text{onto } \langle Z_n, M_{Z_n}, E_{Z_n}\rangle;$$
$$\text{(B) if } n \geq 1,\ \text{then } \beta_n(A) = A,\ \beta_n(\emptyset) = \emptyset,$$
$$\text{and } \beta_n(X_k) = \beta_n(Z_k)(0 \leq k < n).$$

Clearly β_0 is a bijection from X_0 onto Z_0. Since X_0 and Z_0 are 0-atomic, $M_{X_0} = M_{Z_0} = \emptyset$. Therefore (A) and, obviously, (B) hold for $n = 0$.

Assume that (A) and (B) hold for n, where X_0 and Z_0 are n-atomic. We show that (A) and (B) hold for $n+1$, where now X_0 and Z_0 are $(n+1)$-atomic.

Notice that

[†] $$\text{for any set } x,\ x \in X_0 \text{ implies } x \not\subset X_n,$$

for if $x \in X_0$, then $x \neq \emptyset$ and $x \cap X_n = \emptyset$ by the n-atomicity of X_0. Hence, it cannot be that $x \subset X_0$. Similarly,

[††] $$\text{for any set } z,\ z \in Z_0 \text{ implies } z \not\subset Z_n.$$

Clearly, β_{n+1} is a map from X_{n+1} into Z_{n+1}. Suppose that $x,\ y \in X_{n+1}$ and $\gamma(x) = \gamma(y)$. Then $\gamma(x) = \gamma(y) \notin Z_0$ or $\gamma(x) = \gamma(y) \in Z_0$.

For the first case, $x,\ y \notin X_0$. Hence, by Theorem 2.1.2, $x,\ y \subset X_n$ and $\gamma[x] = \gamma(x) = \gamma(y) = \gamma[y]$. Since β_n is an injection, it follows that $x = y$. In the second case, it follows from [††] that $\gamma(x) \not\subset Z_n$. But as was shown in the course of the first case, $x \notin X_0$ implies that $\gamma(x) = \gamma[x] \subset Z_n$. Hence $x \in X_0$. The same argument shows that $y \in X_0$. Again the injectivity of β_n implies that $x = y$. Consequently, β_{n+1} is an injection from X_{n+1} into Z_{n+1}.

To show that β_{n+1} is a surjection, let $z \in Z_{n+1}$. If $z \in Z_n$, then the surjectivity of β_n yields an $x \in X_n \subset X_{n+1}$ such that $z = \gamma(x) \in Z_{n+1}$. If $z \notin Z_n$, then $z \subset Z_n$. Hence again by the surjectivity of β_n, we have that $z = \gamma[x]$, where $x = \beta_n^{-1}[z] \subset X_n$. By [†], $x \notin X_0$. Hence $z = \gamma[x] = \gamma(x) \in Z_{n+1}$.

If $x,\ y \in X_{n+1}$, and $x \in y$, then $y \notin X_0$ by the $(n+1)$-atomicity of X_0. Hence, $\gamma(x) \in \gamma[y] = \gamma(y)$. Conversely, since $\gamma | X_{n+1} = \beta_{n+1}$ is a bijection onto Z_{n+1}, it suffices to assume that $x,\ y \in X_{n+1}$; $\gamma(x),\ \gamma(y) \in Z_{n+1}$ and $\gamma(x) \in \gamma(y)$. Then from the $(n+1)$-atomicity of Z_0, $\gamma(y) \notin Z_0$ implies that $y \notin X_0$. Hence, $\gamma(x) \in \gamma[y]$, and, thus, $\gamma(x) = \gamma(x')$ for some $x' \in y$. By Theorem 2.1.2, $x' \in X_n$. Since β_{n+1} is an injection, $x = x'$. Thus $x \in y$. It follows immediately from the definition of γ that $\beta_{n+1}(x) = \beta_{n+1}(y)$ if and only if $x = y$. Consequently, (A) is established for $n+1$.

In general, since $A \subset X_0 \subset X_n$, we have that $A \notin X_0$ by [†]. Therefore, $\beta_{n+1}(A) = \gamma[A] = A$. The remainder of (B) is easily verified for $n+1$ and by induction the proof is complete. ▮

A criterion as to when a set X_0 is n-atomic for all $n \in \omega$ is very useful. Obviously, if X_0 is a set of atoms, then X_0 is n-atomic for all $n \in \omega$. For the definition of TC, see page 56.

Theorem 2.1.4 *Suppose that $\emptyset \notin TC(X_0)$. If there exists $n \in \omega$ such that X_0 is not n-atomic, then there exists some $y \in X_0$ such that $TC(y) \cap X_0 \neq \emptyset$.*

Proof. Observe that a straightforward inductive argument shows that for each $n \in \omega$, if $x \in X_n$ and $\emptyset \notin TC(\{x\})$, then $TC(\{x\}) \cap X_0 \neq \emptyset$. Assume the hypotheses of the theorem. Since X_0 is not n-atomic, there exists $x,\ y$ such that $x \in y \in X_0$ and $x \in X_n$. Since $TC(\{x\}) \subset TC(X_0)$ and $\emptyset \notin TC(X_0)$, we have $\emptyset \notin TC(\{x\})$. Hence $TC(\{x\}) \cap X_0 \neq \emptyset$. Since $TC(\{x\}) \subset TC(y)$, it follows that $TC(y) \cap X_0 \neq \emptyset$. ▮

Application of Theorems 2.1.3 and 2.1.4 can eliminate a great deal of tedious work. Intuitively, words in a language behave, in many respects, as it they are themselves atoms. We discuss sets of them, subsets of sets of them, etc. Since the symbol strings carry a positioning, unless we extend the intuitive set-theoretic structure to a much more complex one, it would be difficult to discuss the internal construction of a word in the most simplistic of set-theoretic languages. After all, as a set of elements $\{\text{BOOTS}\} = \{\text{BOTS}\}$ have considerably different meanings. This is why the actual intuitive ordering is indicated by the partial sequences. On the other hand, if words seem to behave like atoms within our basic logic, then certain statements about the number of steps in a formal deduction or the "number" of words used for some purpose needs to be represented by relations with respect to the natural numbers.

Let $\mathbb{N}$ be a set of individuals in our model for **ZFH** that is isomorphic to ω. The set $\mathcal{W}$ is assumed to have symbols that represent aspects of the theory of natural numbers (or rational, real, etc.) In the usual manner, these are assumed to be different than those symbols from $\mathbb{N}$ (or other formal sets) used to analyze the set $\mathcal{W}$. Since the specific type of entity being employed is always obvious, a symbolic distinction will not, generally, be made. Relative to the symbols in countably infinite $\mathcal{W}$, $\mathcal{W} \cap \mathbb{N} = \emptyset$, $\mathcal{W}$ is a set of atoms and $\mathbb{N}$ is a disjoint countably infinite set of atoms. The set $\mathbb{N}$ is the natural numbers within the "intuitive" and the "formal" portion of this model. [See note [1] at end of this section.] Let $X_0 = \mathcal{W} \cup \mathbb{N}$. It is a simply matter to show that separating the original set of atoms in this fashion is consistent relative to **ZF**. Since $\mathcal{W} \cup \mathbb{N}$ are atoms, X_0 is n-atomic for all $n \in \omega$.

We now show that the set $\mathcal{E} \cup \mathbb{N} = Z_0$ satisfies the hypotheses of the contrapositive of Theorem 2.1.4. First consider $\mathcal{E}$. Note that each member of $\mathcal{E}$ is a nonempty set and is a finite set of partial functions. That is a finite set of nonempty sets of ordered pairs. Consider any $y \in \mathcal{E}$. Let $x_0 \in y$. Then x_0 is a nonempty finite set of ordered pairs. Let $x_1 \in x_0$. Then x_1 is a nonempty finite set containing one singleton and one doubleton set. Then if $x_2 \in x_1$, then x_2 is a nonempty finite set of atoms. Hence, if $x_3 \in x_2$, then $x_3 \in i[\mathcal{W}]$. Now none of these sets is the empty set and for each $y \in \mathcal{E}$, $TC(y) \cap \mathcal{E} = \emptyset$. For each $y \in \mathbb{N}$, $TC(y) = \emptyset$. Since $\mathbb{N}$ are atoms, then Z_0 is n-atomic for each $n \in \omega$. Thus, for our superstructure construction, Theorem 2.1.3 now applies for each $n \in \omega$.

The above finite argument is considered an effective procedure as are inductive definitions. What can be claimed to be the effective procedure? Even though some might accept the effective procedure as the inductive definition of members in $\mathcal{E}$, in reality, it is the concept of finite recognizability and the fact that members of $\mathcal{E}$

can be constructed from a concrete physical symbol model. Finite recognizability is the same concept that allows for the acceptance that Gödel numbering generates an effective injection into $\mathbb{N}$. If we assign $g(\text{“(”}) = 3$, $g(\text{“,”}) = 5$, and $g(\text{“)”}) = 7$, then unless it is accepted (i.e. recognized) that the string “(,)” is different from the string “(),” the relation determined by assigning to the strings $2^3 5^5 7^7$ and $2^3 3^7 5^5$ would not be a map. Using a concrete symbol model, then from the construction of $\mathcal{E}$, no object that is either an atom, a nonempty set composed of one or two atoms, an ordered pair composed from these previous sets, or a nonempty set of such ordered pairs, is equal to any nonempty finite set of sets of such ordered pairs. Thus, $Z_0 = \mathcal{E} \cup \mathbb{N}$ is n-atomic for every $n \in \omega$. Due to (1.2.4), there is a bijection $\theta \colon i[\mathcal{W}] \to \mathcal{E}$ that associates each member of $i[\mathcal{W}]$ with a unique member of $\mathcal{E}$. This composition yields that bijection needed for Theorem 2.1.3. Consequently, by Theorem 2.1.3, for each $n \in \omega$ the structures $\langle X_n, \in, =, \mathbb{N}, \emptyset \rangle$ and $\langle Z_n, \in, =, \mathbb{N}, \emptyset \rangle$ are isomorphic.

For each $n \in \omega$, let $(\mathcal{E} \cup \mathbb{N})_n$ be the n'th level in a superstructure based upon ground set $\mathcal{E} \cup \mathbb{N}$. Note that relative to a superstructure based upon $\mathcal{W} \cup \mathbb{N}$ and $(\mathcal{E} \cup \mathbb{N}) = X_n$, for each $n \in \omega$, there is a $m \in \omega$ such that $(\mathcal{E} \cup \mathbb{N})_n \subset X_m \subset X_{m+1}$. Thus, we also have that $(\mathcal{E} \cup \mathbb{N})_n \in X_{m+1}$.

The intuitive properties for the deductive processes with which we are concerned can be described within a first-order language and all hold within some particular $(\mathcal{W} \cup \mathbb{N})_n$. Hence, the same properties hold in the corresponding $(\mathcal{E} \cup \mathbb{N})_n$ through application of the isomorphism which exists between these two structures. It is, in reality, by means of i and θ, that the basic logical properties within our intuitive theory become properties within the formal mathematical theory based upon $\mathbb{N}$. (The term “informal” means a restriction to superstructure entities determined by $\mathcal{W}$. The term “formal” means the entire superstructure.) Assuming finite recognizability, the injection i is created and used to pass informal information into formal information about members of $[f_{i_w}]$ since everything is finitary in character. The set $[f_{i_w}]$ is finite, each $g \in [f_{i_w}]$ is finite. Each $x \in g$ is finite, etc. This intuitive finitary process that is employed when formal statements are made within the formal portion of our **ZFH** about the structure of the ordering of the words.

In all that follows, rather than continually mentioning the existence of isomorphisms and applying them to obtain a corresponding property in some $(\mathcal{E} \cup \mathbb{N})_n$ a special approach is followed. When viewed as a models, every object in the superstructure based upon X_0 has a constant name. These objects are uniquely determined by their set-theoretic construction. Among these constants are the constants $\mathbb{N}$ and $\emptyset$ that are used to represent the natural numbers and the empty set in this section. Since on any specific X_n we have an isomorphism β_n from $\langle X_n, \in, =, \mathbb{N}, \emptyset\rangle$ onto $\langle Z_n \in, =, \mathbb{N}, \emptyset\rangle$, if $x \in X_n$, then $\beta_n(x)$ is the corresponding element in Z_n characterized by the same set-theoretic property. In the same way, every member of Z_n has a constant name within our language.

The following convention is used. The injection $i\colon \mathcal{W} \to \mathbb{N}$ is extended, in the usual manner, to subsets of $\mathcal{W}$. Certain constant symbols used to name objects with specific properties in the intuitive part of the superstructure, except for $\mathbb{N}$, its elements and $\emptyset$ are mapped by extended i into a formal superstructure such as $\mathcal{X}$, where the ground set is $X_0 = \mathcal{W} \cup \mathbb{N}$. The map θ is also extended in the same manner as i. Where applicable, the composition of i followed by θ is denoted by **bold** type face. Also, except for members of such sets as $\mathbb{N}$ and variables, most of the informal notation for functions and the like are also represented in the standard model by **bold** font. For this example, let $\mathrm{L} = \mathcal{W}$ and the consequence operator $\mathrm{C}\colon \mathcal{P}(\mathrm{L}) \to \mathcal{P}(\mathrm{L})$. Then $\mathbf{C}\colon \mathcal{P}(\mathbf{L}) \to \mathcal{P}(\mathbf{L})$ is also a consequence operator. This notational convention is followed throughout the remainder of this book.

From these results, if $\mathrm{A} \subset i[\mathcal{W}]$, then any intuitive deductive process $\subset F(\mathrm{A}) \times \mathrm{A}$ or any consequence operator $C\colon \mathcal{P}()A \to \mathcal{P}()A$ becomes under the isomorphism a deductive process $\mathbf{k} \subset F(\mathbf{A}) \times \mathbf{A}$ or a consequence operator $\mathbf{C}\colon \mathcal{P}(\mathbf{A}) \to \mathcal{P}(\mathbf{A})$. Notice that we do not need to consider the isomorphism on the operators F or $\mathcal{P}$ since $\mathrm{B} \in F(A)$ if and only if a sentence, with appropriate constants, of the following type holds. $\mathrm{B} = \emptyset \vee \forall x(x \in \mathrm{B} \leftrightarrow (x = \mathrm{a}_1 \wedge \mathrm{a}_1 \in \mathrm{A}) \vee (x = \mathrm{a}_2 \wedge \mathrm{a}_2 \in \mathrm{A}) \vee \cdots \vee (x = \mathrm{a}_n \wedge \mathrm{a}_n \in \mathrm{A}))$. Hence $\mathbf{B} \in \mathbf{F}(\mathbf{A})$ if and only if $\mathbf{B} = \emptyset \vee \forall x(x \in \mathbf{B} \leftrightarrow (x = \mathbf{a}_1 \wedge \mathbf{a}_1 \in A) \vee (x = \mathbf{a}_2 \wedge \mathbf{a}_2 \in A) \vee \cdots \vee (x = \mathbf{a}_n \wedge \mathbf{a}_n \in A))$, where the isomorphism does map $\emptyset$ onto $\emptyset$ at level $n = 1$. In like manner, the power set operator. (In most cases since it reveals an order, only $\mathcal{E}$ is employed.) Let $\mathrm{A} \subset \mathcal{W}$ and let K_A denote the set of all deductive processes defined for A. Now let C_A denote the set of all consequence operators defined on $\mathcal{P}(\mathrm{A})$. The set $\mathrm{R}_A = \mathrm{K}_A \cup \mathrm{C}_A$ is a set of all *intuitive human reasoning processes* while $\mathbf{R_A} = \mathbf{K_A} \cup \mathbf{C_A}$ is a set of *formal human reasoning processes.*

2.2 A Remark About 2.1

The basic intuitive procedure in establishing a formal model is not relative to structures with a universe $(\mathcal{E} \cup \mathbb{N})$. What most be done is to express in a structure such as $\langle(\mathcal{W} \cup \mathbb{N})_n, \in, =, \mathbb{N}, \emptyset\rangle$ informal statements about our language $\mathcal{W}$, where $\mathcal{W}$ is termed as an informal ground set of atoms disjoint from $\mathbb{N}$. For named objects within such a superstructure, the same **bold** face convention is used for the corresponding objects within any particular $(\mathcal{E} \cup \mathbb{N})_n$ that involves only the members of $\mathcal{E}$.

One additional remark is in order. In 1978 when the following concepts within the discipline termed nonstandard analysis were developed, they were in the mainstream of complexity. Today, many who work in this area would consider them to be very simplistic in nature. To the neophyte, however, they may seem to be somewhat difficult.

2.3 The Nonstandard Structure

Now that the general and basic concepts for the deductive processes and consequence operators have been developed, its necessary to consider "$\mathcal{W} \cup \mathbb{N}$" as embedded into an additional structure. The same concept that every member of the following type of superstructure corresponds to a constant within our language is to be used. With respect to the previous convention, many of these constants will be denoted in bold.

Recall for a moment how $\mathbb{N}$ is obtained. Let set A be our countably infinite set of atoms, disjoint from $\mathcal{W}$. [Note: Specific members of $\mathcal{W}$ are also considered as atomic since if $w \in \mathcal{W}$, then there is no a in the below constructed superstructure such that $a \in w$.] Consider the set of non-negative integers $\mathbb{N}_0$ as defined by Alexander Abian ("The Theory of Set and Transfinite Arithmetic," W. B. Saunder Co., Philadelphia, 1965). These are identified, by Abian, as the (constructed) natural numbers $\mathbb{N}$ in **ZF** set-theory. Let $f\colon A \to \mathbb{N}_0$ be a bijection which exists from A onto the set $\mathbb{N}_0$ in our model for **ZFA**. Denote $f^{-1}[\mathbb{N}_0]$ by $\mathbb{N}$. The identification f is suppressed in all that follows. Thus, notationally, for the (Abian constructed) integers $\mathbf{Z}$, rational numbers $\mathcal{Q}$, and the real numbers $\mathbb{R}$, we can state that $\mathbb{N} \subset \mathbf{Z} \subset \mathcal{Q} \subset \mathbb{R}$. At the least, the defined natural numbers, integers, rational and real numbers, with their basic structural properties, exist as members of the below constructed superstructure.

We obtain a nonstandard model for a slightly different superstructure with ground set $\mathcal{W} \cup \mathbb{N}$ than considered in section 2.1. This is one of the two basic constructions that appear in the literature. The superstructure levels are slightly different [10. p. 40], [17, p. 110], [19, p. 23]. Let $X_0 = \mathcal{W} \cup \mathbb{N}$ and by induction, let $X_{n+1} = \mathcal{P}(\bigcup\{X_k \mid k = 0, \ldots, n\})$. Finally, let $\mathcal{N} = \bigcup\{X_n \mid n \in \omega\}$. Consider a κ-adequate ultrafilter $\mathcal{U}$, where $\kappa > |\mathcal{N}|$. By Theorem 7.5.2

in [19] or Theorem 1.5.1 in [9] such an ultrafilter exists in our **ZFH** and is determined by the indexing set $J = F(\mathcal{P}(\kappa))$.

Consider the structure $\mathcal{M} = \langle \mathcal{N}, \in, = \rangle$. [Note: Since every member of $\mathcal{N}$ is named by a constant, including the customary ones for specific objects, these constants are suppressed in the notation.] By Theorem 7.5.3 in [19] or Theorem 1.5.2 in [19] the ultrapower construction yields by definition 3.8.1 in [9] a structure $\mathcal{M}_1 = \langle \mathcal{N}^J, \in_\mathcal{U}, =_\mathcal{U} \rangle$ which is a nonstandard model for all sentences, K_0, in a first-order language L with equality and predicates for $\in$ and $=$ which hold in $\langle \mathcal{N}, \in, = \rangle$. Assume that the cardinality of the set of constants of $L \geq |\mathcal{N}|$. Moreover, by means of sequences from J onto $\mathcal{N}$, the structure $\langle \mathcal{N}, \in, = \rangle$ may be considered as ismorophically embedded into $\mathcal{M}_1$ so that $\mathcal{M}_1$ is also an elementary extension of the embedded $\langle \mathcal{N}, \in, = \rangle$. The structure $\mathcal{M}_1$ is also an *enlargement* of $\langle \mathcal{N}, \in, = \rangle$. A proof of The Fundamental Result may be found on page 39 of [19] (Theorem 3.8.3) among other places. Now in [10], Theorem 3.8 establishes this for BOUNDED sentences which hold in $\langle \mathcal{N}, \in, = \rangle$. Notice that the interpretation map from the language onto $\langle \mathcal{N}, \in, = \rangle$ has been suppressed and each member of $\mathcal{N}$ is simply to be considered as named by the constants in L.

The next step is to realize either by analysis of the ultrapower construction directly or by interpreting the appropriate sentences [17, p. 119], that $=_\mathcal{U}$ is an equivalence relation with the substitution property for $\in_\mathcal{U}$. Thus passing to the equivalence class $[x]$ for each $x \in \mathcal{N}^J$, define $[x] \in' [y]$ iff $x \in_\mathcal{U} y$ for each $x,\ y \in \mathcal{N}^J$. w.

Now let $(X_n)'$ be the objects in $\{[x] \mid x \in \mathcal{N}^J\}$ that correspond to X_n in $\mathcal{N}$ under the interpretation map I followed by the quotient map for the equivalence relation as determined by $=_\mathcal{U}$ (i.e. the “prime” mapping.) It follows that $(X_0)'$ behaves like atoms (urelements) and each $(X_n)',\ n > 0$ is well-founded with respect to $\in'$. This comes from interpreting the appropriate bounded sentences such as the results of Lemma 2.1 (iv) [10, p. 40] where $R = X_0$ or property (iii) on page 23 of [19] in order to obtain the $\in'$ well-founded for each $(X_n)',\ n \geq 1$. For example, for each $n \geq 1$, the following sentence

$$\forall x((x \in \mathbb{N}) \to \neg \exists y((y \in X_n) \land (y \in x))) \tag{2.3.1}$$

holds in the structure $\langle \mathcal{N}, \in, = \rangle$. Lastly, each $(X_n)',\ n \geq 1$, is well-founded with respect to $\in'$ since “If $x \in y \in X_n$, then $x \in X_0 \cup X_{n-1}$” $(n \geq 1)$ holds in $\langle \mathcal{N}, \in, = \rangle$. Consequently, the Mostowski Collapsing Lemma [1, p. 247] or [17, p. 120] can be inductively applied to each $(X_n)'$ and obtain a corresponding set ${}^*X_n$. Specify the set ${}^*X_0$ to correspond to $(X_0)'$ and we have a unique collapse. As a result of this, the structure $\langle \bigcup\{ {}^*X_n \mid n \in \omega\}, \in, =\rangle = {}^*\mathcal{M} = \langle {}^*\mathcal{N}, \in, =\rangle$ is a set-theoretic model for all bounded sentences that hold in $\langle \mathcal{N}, \in, =\rangle$. Recall that a bounded sentence in a first-order language L is one for which each quantified variable is restricted to an element of $\mathcal{N}$. The composition of the interpretation map $I^{\mathcal{U}}$, the quotient map $'$ and the collapse yield the $*$ map from the structure $\langle \mathcal{N}, \in, =\rangle$ into ${}^*\mathcal{M}$ and maps any element $a \in \mathcal{N}$ to the element ${}^*a$ preserving all of the usual properties for a normal, enlarging and comprehensive monomorphism. For each $B \in \mathcal{N}$, let ${}^\sigma B = \{ {}^*x \mid x \in B\}$. (This definition does not correspond to that used by some other authors.) The $*$ notation is also not placed on elements of X_0 when they are considered as mapped into ${}^*X_0$ by the $*$ map. Observe that for each $B \in \mathcal{N}$, ${}^\sigma B \subset {}^*B$. Technically, where used, $B \subset {}^*B$ also means ${}^\sigma B \subset {}^*B$.

Now to complete the construction, begin with the set $Y_0 = {}^*X_0$ and construct a superstructure with Y_0 as the ground set as **defined in this first example**. Let $Y_{n+1} = \mathcal{P}(\bigcup\{Y_n \mid n \in \omega\})$; and let $Y = \bigcup\{Y_n \mid n \in \omega\}$.

For the above, some general principles such as the Mostowski Collapsing Lemma have been used in order to obtain ${}^*\mathcal{M}$; however, an explicit construction appears on pages 44 and 45 of [19]. In actuality for the next constructed superstructure, the one used in this book, we intend to use only a small portion of ${}^*\mathcal{M}$. Indeed, we apparently need to use a small hierarchy of the ${}^*X_n$ objects. You could, if you wished, restrict the G-structure to say only the $n < 100$ levels. However, this will not be done for fear of not selecting a correct upper bound for n.

For the results in this book, I advocate for our superstructure a construction as defined in section 2.1, where $X_0 = A_1 \cup \mathbb{N}$, and the nonstandard model as constructed on pages 83 - 88 and Theorem 6.3 in Hurd, A. E. and P. A. Loeb, (1985), "An Introduction to Nonstandard Real Analysis," Academic Press, Orlando. [Note: This construction also appears on pages 42-49 in Loeb and Wolff, (eds) (2000), "Nonstandard Analysis for the Working Mathematician," Kluwer Academic Publishers, London. Also $X_n(2.1) = X_0 \cup X_n(2.3),\ n \geq 1.$‡] This construction simply needs to be restricted to our language with $\in$ and $=$, where $=$ is interpreted as set-theoetic equality on sets and the identity on atoms. For the first superstructure, constructed using the procedure in section 2.1, let $\mathcal{N} = \mathcal{X}$. The second superstructure constructed using this procedure has as its ground set $Y_0 = {}^*X_0$ and, as before, $Y = \bigcup\{Y_n \mid n \in \omega\}$. This leads to the G-structure $\mathcal{Y} = \langle Y, \in, =\rangle$ where since I apply this to logical operations this structure is call the *Grundlegend Structure*. (Note: To find the embedded isomorphic copy of the standard superstructure for the Hurd & Loeb construction, simply restrict the Mostowski collapsing function to the constant sequence $\mathcal{U}$-equivalence classes.)

Now to summarize. The consistency of **ZF** implies the consistency of **ZFH** and one can apparently use a model of **ZFH** to obtain the nonstandard structure ${}^*\mathcal{M} = \langle {}^*\mathcal{N}, \in, = \rangle$. The set ${}^*\mathcal{N}$ is dependent upon the of atoms $\mathbb{N}$, the atoms of **ZFH** with the order induced by ω. Any sentence in a appropriate first-order language in which each quantified variable is restricted to an element of $\mathcal{N}$ (i.e. bounded variable) will, when each constant is replaced by the * of the constant, give a true statement about the structure ${}^*\mathcal{M}$. Moreover, ${}^*\mathcal{M}$, at the least, has bounds for all standardly definable concurrent relations. For notation, we denote for each $n \in X_0$, ${}^*n = n$. In addition, all properties of the * map as listed in [10], [17], [19], among other places, hold true. Next some unusual names for G-structure objects will be adopted in order to reflect our application to languages and logics.

Recall some of the basic terminology associated with $\mathcal{Y}$. For each $A \in \mathcal{N}$, A is called a *standard entity.* The set ${}^*A$ is often called an *(internal) standard entity* or better still an *extended standard entity* in $\mathcal{Y}$. If $b \in {}^*A$, $A \in \mathcal{N}$, then b is called an *internal entity.* Indeed, $b \in \mathcal{Y}$ is internal iff there is some X_n such that $b \in {}^*X_n$. Any entity of $\mathcal{Y}$ which is not internal is called *external.* These terms are generally used throughout nonstandard analysis, but for our present purposes they are modified as follows: Any entity of $\mathcal{Y}$ is a *subtle object,* some appropriate members of $\mathcal{N}$ are *human objects* and any entity in $\mathcal{Y}$ which is not an isomorphically embedded member of $\mathcal{N}$ or the * of a member of $\mathcal{N}$ is a *purely subtle object.* Please refer to the basic references [11], [16], [19] for other terminology and the properties of *. So as to avoid symbolic confusion, from this moment on, the entire or major part of any symbol used to represent objects within a language and within our intuitive model will be denoted by Roman type.

2.4 General Interpretations

Throughout this work on ultralogics, L_0 will denote the usual set of propositional formulas (wff) constructed from the connectives $\neg$, $\vee$, $\wedge$, $\to$, $\leftrightarrow$, say as done by Kleene [7, p. 108] and L_1 is a set of predicate formulas with equality considered as an extension of L_0 as say constructed on page 143 of [7]. We also use the usual assortment of set-theoretic abbreviations when we consider the special predicates $\in$ and $=$. Of course, L_1 is called a *first-order language.* Assume that $\mathrm{L}_1 \subset \mathcal{W}$ and that the set of all predicate symbols is a subset of $\{\mathrm{P}_i \mid i \in \mathbb{N}\}$. It is important to realize that any intuitive set-theoretic deduction process, and the like, that is discussed relative to $\mathcal{W}$ is to be embedded by the map θ to a corresponding process relative to $\mathcal{E}$. This also applies to a member of $\mathcal{W}$ and the i injection. The results of any *-transfer of statements which hold relative to $\mathcal{E}$ or $i[\mathcal{W}]$ are modeled in $^*\mathcal{M}$. Also, any results relative to $\mathcal{E}$ or $i[\mathcal{W}]$ (i.e. with respect to standard objects) can be referred back to corresponding intuitive objects relative to $\mathcal{W}$ by means of either the maps i^{-1} or θ^{-1}. Moreover, in order to simplify notation somewhat any formal first-order statement that explicitly involves $i(w)$ individuals will be written with the i deleted from the notation if no confusing results from such an omission.

For example, the sentence

$$\begin{gathered}\forall x(x \in \mathbb{N} \wedge x \geq 1 \to \exists y(y \in P \wedge \forall w(w \in \mathbb{N} \wedge \\ 0 < w \leq x \to y(w) = \text{very,}||| \wedge y(0) = \text{just}))) \end{gathered} \tag{2.4.1}$$

is a slight simplification of the following sentence

$$\begin{gathered}\forall x(x \in \mathbb{N} \wedge x \geq 1 \to \exists y(y \in P \wedge \forall w(w \in \mathbb{N} \wedge \\ 0 < w \leq x \to y(w) = i(\text{very,}|||) \wedge y(0) = i(\text{just})))) \end{gathered} \tag{2.4.2}$$

Most of the following investigation is concerned with specific elements of P and specific human reasoning processes with respect to $\mathcal{E}$. Using the previous example, the *-transfer yields

$$\begin{gathered}\forall x(x \in {}^*\mathbb{N} \wedge x \geq 1 \to \exists y(y \in {}^*P \wedge \forall w(w \in {}^*\mathbb{N} \wedge \\ 0 < w \leq x \to y(w) = {}^*i(\text{very,}|||) \wedge y(0) = {}^*i(\text{just})))) \end{gathered} \tag{2.4.3}$$

and which holds in $^*\mathcal{M}$. Notice that we do not place * on the order relation $<$ since we assume that it is but an extension of the simple order $<$ on $\mathbb{N}$ satisfying all of the same first-order properties. Also ${}^*i(\text{very,}|||) = i(\text{very,}|||)$, etc. Let $\nu \in {}^*\mathbb{N} - \mathbb{N} = \mathbb{N}_\infty$ (i.e. the infinite numbers). Then there exists in $^*P$ a *-partial sequence, say f, such that for each $w \in {}^*\mathbb{N},\ 0 < w \leq \nu,\ f(w) =$ very, and $f(0) =$ just. Hence even though members of $[f]$ are not readable sentence in our sense, we can read the elements in the range of f as well as reading the intuitive ordering when f is restricted to $[0, n],\ n \in \mathbb{N}$. This gives an intuitive interpretation for such an f when it is so restricted to such standard segments as well as knowledge of the properties of the ordering when not so restricted. Observe also, that if $f \in {}^*P - P$, then there exists some $\nu \in {}^*\mathbb{N}$ such that $f\colon [0, \nu] \to {}^*i[\mathcal{W}] = {}^*(i[\mathcal{W}])$, where $[0, \nu] = \{x \mid (x \in {}^*\mathbb{N}) \wedge (0 \leq x \leq \nu)\}$. Often, in our formal statements, parentheses are suppressed and the strength of connectives notion is used.

2.5 Sets of Behavior Patterns

In certain applications of subtle consequence operators the following construction is useful. This is all relative to what is called *adjective reasoning* and any equivalent form. Let B' denote a list of names or simple phrases that are used to identify specific behavior patterns. These terms are taken from a specific discipline language and are, as usual, to be considered as elements of $\mathcal{W}$. For example, the set B' could be taken from the discipline called *psychology* and each term could identify a specific human behavior pattern, as general as such concepts as "kind" or "generous." You can also include any synonyms that might be equivalent to the members of B'. Now consider B constructed as follows: an element $\mathrm{b} \in \mathrm{B}$ if and only if b is a *qualifiable* form of a member of B'. That is each $\mathrm{b} \in \mathrm{B}$ is a $\mathrm{b}' \in \mathrm{B}'$ where b' is written in a form so that it can be modified by the word *very*. (Or, such words as "great," "greater.") Let $\mathrm{B} = \mathrm{C}_0,\ \mathrm{C}_1 = \{\mathrm{very},|||\mathrm{c} \mid \mathrm{c} \in \mathrm{C}_0\}$. By induction let $\mathrm{C}_{n+1} = \{\mathrm{very},|||\mathrm{c} \mid \mathrm{c} \in \mathrm{C}_n\}$. Then an intuitive set of *modified behavior patterns* is the set $\mathrm{BP} = \bigcup\{\mathrm{C}_n \mid n \in \mathbb{N}\}$. The *formal modified behavior patterns* is the set $\mathbf{BP} = \bigcup\{\mathbf{C_n} \mid n \in \mathbb{N}\}$. Notice that each $\mathbf{C_m}$ is a finite set.

In certain cases, the intuitive set BP is associated with a set of formal propositional statements in $\mathcal{W}$. Let L_0 be our propositional language constructed from a denumerable set of atoms $\{\mathrm{P}_i \mid i \in \mathbb{N}\}$. Since B is finite, then there exists an injection $j\colon \mathrm{B} \to \{\mathrm{P}_i \mid i \in \mathbb{N}\}$ and $\{\mathrm{P}_i \mid i \in \mathbb{N}\} - j[B]$ is denumerable. Let $\mathrm{V} \in \{\mathrm{P}_i \mid i \in \mathbb{N}\} - j[\mathrm{B}]$. Let the symbol string "very,|||" correspond to the partial formula "V $\wedge$". Then proceed to construct BP_0 as follows: $\mathrm{E}_0 = j[\mathrm{B}],\ \mathrm{E}_{n+1} = \{(\mathrm{V} \wedge \mathrm{x}) \mid \mathrm{x} \in \mathrm{E}_n\}$. Then, finally, $\mathrm{BP}_0 = \bigcup\{\mathrm{E}_n \mid n \in \mathbb{N}\}$.

In what follows, the modeling of human reasoning processes is often approached from two different points of view. First, from the viewpoint of such sets is BP, as well as many others, we have the constructed set of *meaningful sentences* in the sense of Tarski. Thus, such strings or symbols become our formal language and a simple observer language (i.e. metalanguage) is used to investigate deductive processes on BP. These are mapped to the formal deductive processes on **BP.** However, many of these deductive processes on a given BP can be associated with other formal processes in L_0, especially with respect to BP_0. Hence, whenever possible it is acknowledged that there are at least two "models" for various BP type statements, among others, that are being in investigated. The basic model (and probably the simplest) is that based on BP. Then a somewhat more complex model is based on L_0. The purest is probably more comfortable with the formal languages L_0 and L_1. I feel, however, that BP is as meaningful a set of sentences formed by constructive methods as is the set L_0 and the various forms in BP are easily recognized.

‡This is an important fact. Let $X_{n+1}(X) = X_n(X) \cup \mathcal{P}(X_n(X)),\ n \geq 0,\ X_0(X) = X$ (Def. 2.1) and $X_{n+1} = \mathcal{P}(X_0 \cup \cdots X_n),\ n \geq 0,\ X_0 = X$ (Def. 2.3), where X is the set of individuals. For Def. 2.3, we also have that $X_p \subset X_n,\ 1 \leq p \leq n$ and $X_{n+1} = \mathcal{P}(X \cup X_n), n \geq 0$. We show by induction that $X_n(X) = X \cup X_n,\ n \geq 0$. First, let $n = 0$ on the right. Then $X_0(X) = X,\ X_0 = X \Rightarrow X_0(X) = X \cup X_0$. Now for the specific inductive form, let $n = 0$. Then $X_1(X) = X \cup \mathcal{P}(X_0(X)) = X \cup \mathcal{P}(X) = X \cup \mathcal{P}(X_0) = X \cup X_1$. Assume result holds for n. Then $X_{n+1}(X) = X_n(X) \cup \mathcal{P}(X_n(X)) = X \cup X_n \cup \mathcal{P}(X \cup X_n) = X \cup X_n \cup X_{n+1} = X \cup X_{n+1}$ and the result follows by induction.

[1] (14 DEC 2012). The set $\mathcal{W}$ (and later $\mathcal{W}'$) was added to the ground set on this date. This has been done to provide an additional formal structure to enhance analysis. Using the members of a language itself as constituents of a ground set for a model is well established [13, p. 70]. However, it is the set $\mathcal{E}$ that is generally more significant for our purposes than members of the language itself since they represent the significant aspects of the formation of "words" whether they be formed by symbols, diagrams, images or coded sensory information. Hence, in this theory, members of $\mathcal{E}$ and ${}^*\mathcal{E}$ still remain the basic form for a "word" or "hyper-word."

After developing the basic aspects of this approach, it was discovered that Robinson [15, section 3] also developed a nonstandard approach to sets of symbols. I have noted this in more recent versions. (Also see Geiser, J. T. (1968). "Nonstandard logic," J. Symbolic Logic 33(2):236-250.) The idea of incorporating $\mathcal{E}$ as a way to include how languages are constructed is not part of the Robinson foundations.

The set $\mathcal{W}$ can contain the language for various mathematics theories such as an appropriate portion $\mathcal{T}(\omega)$ of the theory of natural numbers. Each member of $\mathcal{T}(\omega)$ corresponds to objects in $\mathcal{N}$. As mentioned, one can consider members of $\mathcal{W}$ as written in a different color than any other symbols used for any other purposes. In some cases, the "prime" notion for the symbols expressing statements about members of $\mathcal{N}$ other than members of $\mathcal{W}$ is employed. For example, the expressions $2' <' 3'$ and $\mathcal{T}'(\omega')$ are in extended $\mathcal{W}'$. External to $\mathcal{M}$, one can state that the expression $2' <' 3'$ is a member of $\mathcal{T}'(\omega')$. Or we state that $2' <' 3'$ holds. (One can actually include an additional model for this purpose.) This corresponds directly to a statement $2 < 3$ that "holds" in $\mathcal{M}$ where $2, <, 3$ are names for the corresponding "formal" objects. In general, if used, "primed" statements of this type are expressed directly in terms of the corresponding "not primed" expressions. Robinson keeps the statements used to discuss behavior of the members of his set of symbols distinct from those in $\mathcal{W}$ by simply defining such a set and leaving the rest to ones intuition.

The use of the embedding i now seems of little significance. The embedding was used so that $\mathcal{E}$ could simply be considered as entities from the theory of natural numbers with its long history of empirical consistency. In the beginning of nonstandard analysis where simplified type theory was employed and formal set-theory was not considered, such a consistency notion might be useful. But since formal set-theory is now being considered, any consistency considerations depends upon the assume consistency of the set-theory axioms being employed. Hence, as demonstrated, the removing of the function i from both the foundations and expressions should not effect any of the interpreted results.

If i is so removed, then the set of equivalence classes is <u>usually</u> denoted by $\underline{\mathcal{W}}$ and they are now partial sequences of members of the language $\mathcal{W}$ rather than the codes produced by application of i. From the viewpoint of the nonstandard model, this would mean that rather than an ultraword being considered as a partial hyper-sequence of members of ${}^*\mathbb{N}$ with some "symbols" being represented by members of ${}^*\mathbb{N} - \mathbb{N}$, some *symbols are represented by members of ${}^*\mathcal{W} - \mathcal{W}$.

There is, of course, a bijection $w\colon \mathcal{W} \to \mathcal{E}$, there $w(a) = [g]$ and there is a $f \in [g]$ such that $f \in T^0$ and $f(0) = i(a)$ (or simply $f(0) = a$. This bijection may be useful for further developments of this Theory of Ultralogics.

Chapter 3

DEDUCTIVE PROCESSES

3.1 Introduction.

In all that follows, intuitive objects are denoted by roman font. We approach the investigation of various special deductive processes by defining them for some $\mathrm{A} \subset \mathcal{W}$ as being $\mathrm{k} \subset F(\mathrm{A}) \times \mathrm{A}$ or $\mathrm{C}\colon \mathcal{P}(\mathrm{A}) \to \mathcal{P}(\mathrm{A})$. These sets are all considered mapped to objects relative to $\mathcal{P}(\mathcal{E}) \times \mathcal{P}(\mathcal{E})$ for formal investigation. In at least one case, a map $C\colon \mathcal{P}(A) \to \mathcal{P}(A)$ is defined for each nonempty $A \in {}^*\mathcal{N}$ and it is shown (trivially) that such a map is a consequence operator. Any $C \in {}^*\mathcal{N}$ which satisfies (in ${}^*\mathcal{N}$) axioms (2), (3), (4) or their *-transform is a *subtle consequence operator* or *subtle reasoning process*, where for convenience C is restricted to $A \subset {}^*\mathcal{E}$.

3.2 The Identity Process.

Let $A \subset \mathcal{E}$ be any nonempty set. For each $B \subset A$, define $I(B) = B$. Obviously, this is the *identity operator* from $\mathcal{P}(A)$ onto $\mathcal{P}(A)$.

Theorem 3.2.1. *Let $A \subset \mathcal{E}$ be nonempty. Then the identity operator on $\mathcal{P}(A)$ is a consequence operator.*

Proof. Let $B \subset A$. Then $I(B) = B$ implies that $B \subset I(B) \subset A$. Moreover, $I(I(B)) = I(B) = B$ for each $B \subset A$. Finally, $B = I(B) = \bigcup\{F \mid F \in F(B)\} = \bigcup\{I(F) \mid F \in F(B)\}$ and the result follows. ▮

Let H be a nonempty set of Tarski type deductive processes. That is if $h \in H$, then $h \subset F(\mathcal{E}_1) \times \mathcal{E}_1$ for some $\mathcal{E}_1 \subset \mathcal{E}$. Also let H_0 be a nonempty set of consequence operators on some $\mathcal{P}(\mathcal{E}_2)$ for $\mathcal{E}_2 \subset \mathcal{E}$. Then ${}^*H \cup {}^*H_0 = D_0$ is considered a set of subtle reasoning processes. Notice that if $C \in H_0$, then ${}^*C \in D_0$ may not be a consequence operator under our definition. The first reason for this is that axiom (4) *-transforms to read that for every internal subset of $B \subset {}^*\mathcal{E}_2$, ${}^*C(B) = \bigcup\{{}^*C(F) \mid F \in {}^*F(B)\}$. For the sentence

$$\forall x(x \in \mathcal{P}(\mathcal{E}_2) \to \forall w(w \in \mathcal{E}_2 \to (w \in C(x) \leftrightarrow \exists y(y \in F(x) \land w \in C(y))))). \tag{3.2.1}$$

holds in $\mathcal{M}$; hence in ${}^*\mathcal{M}$. As is well known a *-finite set need not be finite. However, there is at least one map from $\mathcal{P}(A)$ into $\mathcal{P}(A)$ for any $A \subset {}^*\mathcal{E}_2$ which is a true consequence operator as shown by Theorem 3.2.1. Consider any infinite $A \subset \mathcal{E}_2$. Then no map $C\colon \mathcal{P}({}^*A) \to \mathcal{P}({}^*A)$ can be written as an extended standard map (i.e. the star of a standard map) from $\mathcal{P}(A)$ into $\mathcal{P}(A)$. This follows from the next result.

Theorem 3.2.2. *Let infinite $A \subset \mathcal{E}_2$ and $G: \mathcal{P}(A) \to \mathcal{P}(A)$. Then there exists a subset of ${}^*A$ upon which ${}^*G$ is not defined.*

Proof. Let infinite $A \subset \mathcal{E}_2$, $G: \mathcal{P}(A) \to \mathcal{P}(A)$ and $D(G)$ be the domain of $G = P_1[G]$. For an appropriate X_m, the sentence

$$\forall x (x \in X_m \to (x \in D(G) \leftrightarrow x \in \mathcal{P}(A))). \tag{3.2.2}$$

holds in $\mathcal{M}$; hence in ${}^*\mathcal{M}$. The *-transfer reads

$$\forall x (x \in {}^*X_m \to (x \in D({}^*G) \leftrightarrow x \in {}^*(\mathcal{P}(A)))), \tag{3.2.3}$$

when the elementary properties of the *-map are applied. The ${}^*G$ is only defined on the internal subsets of ${}^*A$. Since ${}^*A - A$ is external then this result follows. ▮

Corollary 3.2.2.1 *There exist purely subtle reasoning processes.*

As to the cardinality of $\mathcal{E}$, it follows immediately that since each $x \in \mathcal{E}$ is finite, then $|\mathcal{E}| = \omega$. Note the following that will be used throughout this investigation. Recall that the identification ${}^\sigma X_0 = X_0$ is being used. Then if $f \in P_H$ it follows, since f is a finite sequence of members of X_0 that ${}^*f = f$ under this identification. Also, since for each $f \in P_H$ the set $[f]$ is finite, then ${}^*[f] = [f]$. Thus ${}^\sigma\mathcal{E} = \mathcal{E}$. This reduction of finite sets of finite sets of partial sequences continues to other cases such as ${}^\sigma(F(P_H)) = \{{}^*A \mid A \in F(P_H)\} = \{A \mid A \in F(P_H)\} = F(P_H)$.

With the above results in mind, it follows that each $x \in {}^*\mathcal{E} - \mathcal{E}$ is a nonfinite *-finite subset of ${}^*P$ for the sentence $\forall x (x \in \mathcal{E} \to x \in F(P))$ holds in ${}^*\mathcal{M}$. Consequently, ${}^*\mathcal{E} \subset {}^*(F(P))$ and $\mathcal{E} \subset F(P)$ imply that ${}^*\mathcal{E} - \mathcal{E} \subset {}^*(F(P)) - F(P)$. Let infinite $A \subset \mathcal{E}$. Then it is an important fact that there exists a *-finite $B \in {}^*(F(A))$ such that ${}^\sigma A \subset B \subset {}^*A \subset {}^*\mathcal{E}$. For let $Q = \{(x, y) \mid y \in F(A) \wedge x \in A \wedge x \in y\}$. Assume that $(a_1, b_1), \ldots, (a_n, b_n) \in Q$. Then letting $b = b_1 \cup \cdots \cup b_n$ it follows that $(a_1, b), \ldots, (a_n, b) \in Q$. Therefore, Q is a standard concurrent relation. Thus there exists some $B \in {}^*(F(A))$ such that ${}^*x \in B$ for each $x \in A$. Now internal B is not finite since ${}^\sigma A$ is not finite. Indeed, as is well know $|B| = |[0, \nu]| \geq 2^\omega$, where $\nu \in {}^*\mathbb{N} - \mathbb{N} = \mathbb{N}_\infty$. Thus $|{}^*A| \geq 2^\omega$. From the above remarks, it also follows that for $E \in {}^*\mathcal{E} - \mathcal{E}$, $|E| \geq 2^\omega$.

A few other useful results are easily obtained. For example,

(i) if $\mathcal{P}: \mathcal{P}(A) \to Y$ and $B \in \mathcal{P}(A)$, then ${}^*(\mathcal{P}(B)) = {}^*\mathcal{P}({}^*B)$.

(ii) Consider the finite power set operator F. If $F: \mathcal{P}(A) \to Y$ and $B \in \mathcal{P}(A)$, then ${}^*(F(B)) = {}^*F({}^*B)$.

(iii) If C is a map from $\mathcal{P}(A)$ into $\mathcal{P}(B)$, then for $D \in \mathcal{P}(A)$ it follows that ${}^*(C(D)) = {}^*C({}^*D)$.

The proofs of (i), (ii) and (iii) are easily obtained. Indeed, all three follow from the formal definition of a map. First, assuming that $A, Y \in \mathcal{N}$. All the objects with which we shall be concerned will also be members of $\mathcal{N}$. Indeed, if necessary to obtain bounded sentences, we know that there is some $n \in \omega$ such that everything needed to characterize (i), (ii), and (iii) are members of $X_0 \cup X_n$. For example, consider (i). Then the two sentences $\forall x(x \in \mathcal{P}(A) \to \exists y(y \in Y \wedge (x, y) \in \mathcal{P}))$, and $\forall x \forall y \forall z(x \in \mathcal{P}(A) \wedge y \in Y \wedge z \in Y \wedge (x, y) \in \mathcal{P} \wedge (x, z) \in \mathcal{P} \to y = z)$ imply, by *-transfer, that ${}^*\mathcal{P}$ is a map from ${}^*\mathcal{P}(A)$ into ${}^*Y$. Further, $(B, \mathcal{P}(B)) \in \mathcal{P}$ implies that $({}^*B, {}^*(\mathcal{P}(B))) \in {}^*\mathcal{P}$. Hence, since ${}^*\mathcal{P}$ is a map, mapping notation yields that ${}^*(\mathcal{P}(B))) = {}^*\mathcal{P}({}^*B)$. Now (ii) follows in like manner. Indeed, the following set of sentences shows that ${}^*F$ generates the hyperfinite subsets of any internal subset of ${}^*A$.

$$\begin{aligned} &\forall x \forall w(x \in \mathcal{P}(\mathcal{P}(A)) \wedge w \in \mathcal{P}(A) \to ((A, x) \in F \leftrightarrow \\ &x = \emptyset \vee \exists y \exists z(z \in X_n \wedge y \in \mathbb{N} \wedge \forall v(v \in A \wedge v \in x \to \\ &\exists i(i \in \mathbb{N} \wedge 0 \le i \le y \wedge z(i) = v))))) \end{aligned} \tag{3.2.4}$$

It is well known that, in general, if $A = \bigcup\{B_i \mid i \in \mathbb{N}\}$, then ${}^*A \neq \bigcup\{{}^*B_i \mid i \in \mathbb{N}\}$. However, if $A = \bigcup\{B\}$, then by *-transfer of the definition it follows that ${}^*A = \bigcup\{{}^*B\} = {}^*(\bigcup\{B\})$. This is incorporated in the proof of the next result.

Theorem 3.2.3 *Let $A \in \mathcal{N}$, $A \subset X_n$. If B is a partition of A, then ${}^*B$ is a partition of ${}^*A$.*

Proof. The sentences

$$\begin{aligned} &\forall x(x \in X_n \to (x \in A \leftrightarrow \exists y(y \in B \wedge x \in y))); \\ &\forall x \forall y(x \in B \wedge y \in B \to x = y \vee x \cap y = \emptyset), \end{aligned} \tag{3.2.5}$$

hold in $\mathcal{M}$; hence in $\mathcal{M}$. Thus by *-transfer, ${}^*B$ is a partition of ${}^*A$ and ${}^*A = \bigcup\{{}^*B\} = {}^*(\bigcup\{B\})$. This completes the proof. ▮

3.3 Adjective Reasoning (Also see page 46.)

Following the ideas of Tarski (20) it appears that the set BP is a meaningful set of sentences. Define for BP an intuitive deductive process as follows: Let A ∈ F(BP). Then A $\vdash_{\mathrm{a}}$ b, b ∈ BP if b ∈ A or b is obtained from some x ∈ A be removing a finite number (≠ 0) of "very,|||" strings from x. Due to its form, this process $\vdash_a$ is termed *adjective reasoning.* Denote the relation in $F(\mathrm{BP}) \times \mathrm{BP}$ obtained by

$\vdash_a$ by the symbol "a." Let nonempty $\mathrm{B} \subset \mathrm{BP}$. Then for each $\mathrm{b} \in \mathrm{B}$ it follows that $(\{\mathrm{b}\}, \mathrm{b}) \in a$. Hence "$a$" is singular and C_a satisfies axiom (2) of the Tarski axioms. Let $(\mathrm{A}, \mathrm{b}) \in \mathrm{a}$ and $\mathrm{A} \subset \mathrm{B} \in \mathrm{F(BP)}$. Then $(\mathrm{B}, \mathrm{b}) \in \mathrm{a}$ since b is obtained entirely from an element in A. Assume that $(\mathrm{A}, \mathrm{b_1}), \ldots, (\mathrm{A}, \mathrm{b_n}) \in \mathrm{a}$ and that $(\{\mathrm{b}_1, \cdots, \mathrm{b}_n\}, \mathrm{c}) \in a$. Then c is either some b_i, $i = 1, \cdots, n$; or c is obtained from some b_i by removing finitely many "very,|||" symbol strings. But either this $\mathrm{b}_i \in \mathrm{A}$ or this b_i is obtained from some $\mathrm{x} \in \mathrm{A}$ also by removing finitely many "very,|||" symbol strings. Thus c is either an element of A or is obtained from A by removing finitely many "very,|||" symbol strings from a member of A. Thus $(\mathrm{A}, \mathrm{c}) \in \mathrm{a}$ and C_a is a consequence operator on $\mathcal{P}(\mathrm{BP})$ by Theorem 1.3.3. For the next results, recall that when no confusion might occur the set ${}^\sigma D$ is denoted by D.

A remark concerning notation is necessary. Two special abbreviations are used in certain explicit formal sentences. The first is the symbol $[x]$ for $x \in P$. This denotes the unique object z that satisfies the sentence

$$\exists! z(z \in \mathcal{E} \land x \in z \land x \in P), \tag{3.3.1}$$

where $\exists! z A(z)$ means $\exists z(z \in \mathcal{E} \land \forall y(y \in \mathcal{E} \to (A(y) \leftrightarrow z = y)))$. Now in the first formula in the proof of Theorem 3.3.2 the formula $\exists! z_1 \exists! z_2((z_1 \in \mathcal{E}) \land (z_2 \in \mathcal{E}) \land (y \in z_1) \land (y_1 \in z_2) \land (z_2 \in \mathbf{C_a}(\{z_1\})))$ could be inserted. Also, the notation $\{x\}$ denotes the unique singleton set that satisfies the following for any $A \in \mathcal{N}$,

$$\begin{aligned} &\forall z(z \in A \to \exists! x(x \in \mathcal{P}(A) \land \\ &\quad \forall w(w \in A \land w \in x \to w = z))). \end{aligned} \tag{3.3.2}$$

We could insert for $z_2 \in \mathbf{C_a}(\{z_1\})$ the formula

$$\begin{aligned} &\exists! z_3(z_3 \in \mathcal{P}(\mathcal{E}) \land \forall w_3(w_3 \in \mathcal{E} \land w_3 \in z_3 \to \\ &\quad w_3 = z_1) \land z_2 \in \mathbf{C_a}(z_3)). \end{aligned} \tag{3.3.3}$$

Of course, these formulas are not inserted, but the appropriate abbreviations are used when needed. Recall that only constants which represent elements of $\mathcal{N}$ are "starred" in either the *-transform or any explicit partial formula obtained from the more general statement. All the internal objects which are not standardly internal take on constant names from an extended language. Thus if $A \in {}^*\mathcal{E} - \mathcal{E}$, then $A = [a]$, where $a \in {}^*P$. The same holds for any singleton set. If $a \in {}^*A - A$, then we write $\{a\}$. For each $A \in \mathcal{N}$ and any nonempty finite $\{a_1, \ldots, a_n\} \subset A$ it follows that ${}^*\{a_1, \ldots, a_n\} = \{{}^*a_1, \ldots, {}^*a_n\} \subset \{{}^*x \mid x \in A\} = {}^\sigma A$.

In what follows, other simplifying processes are employed when writing formal sentences. In many cases, these sentences do not appear to be written in the usual *special* bounded form. In most cases, the additional formal expressions can be easily added. In general, this is done by the addition of another $A \wedge$ type expression and an equivalent formula obtained, or $A \to$ when $\leftrightarrow$ appears. Many of the missing expressions are of these types. Here is one example of this process. Let $T = i[\mathcal{W}]$.

Consider the set of all "natural number" intervals (i.e. segments) $H_1 = \{[0,n] \mid n \in \mathbb{N}\}$. From the construction of the superstructure it follows that there exists some $m \in \omega$ such that $\mathbb{N} \times T \subset X_0 \cup X_m$. Hence $\mathcal{P}(\mathbb{N} \times T) \subset \mathcal{P}(X_0 \cup X_m) = X_{m+1}$, where $m \geq 1$. Since no atoms are in $\mathbb{N} \times T$, the set $\mathbb{N} \times T \subset X_m$. For each $x \in H_1$, $T^x \in \mathcal{P}(\mathbb{N} \times T) \subset X_{m+1}$. Obviously, $H_1 \in X_2$. Hence, we also have that $H_1 \in X_{m+1}$. Observe that for each $x \in H_1$, $w \in T^x \subset x \times T \subset \mathbb{N} \times T \subset X_0 \cup X_m$ implies since no atoms are involved that $w \in X_m \subset X_{m+1}$. Thus, all of the objects being considered in an expression of the type $w \in T^x$ are all members of the set X_{m+1}. Notice that in the formal language $w \in T^x$ is a 2-place predicate replaceable by $\exists y \exists z(y \in \mathbb{N} \wedge 0 \leq y \leq x \wedge z \in T \wedge (y,z) \in w) \wedge \forall y_1 \forall z_1 \forall x_1 (y_1 \in \mathbb{N} \wedge z_1 \in T \wedge x_1 \in T \wedge (y_1, z_1) \in w \wedge (y_1, x_1) \in w \to x_1 = z_1))$. Suppose that you have a formula with the expression "$\wedge(w \in T^x)$" as a subformula. Then replace it by " $\wedge(w \in X_m) \wedge (w \in T^x)$." Now the explicit specially constructed formula usually used in the literature for such bounded formula is obtained by expanding the finite sequences of "$\wedge$" into the equivalent forms "($\to$ (…" since recall that for the propositional calculus an expressing such as $\mathrm{P} \wedge \mathrm{Q} \to \mathrm{S}$ is equivalent to $\mathrm{P} \to (\mathrm{Q} \to \mathrm{S})$. With these processes, all of the formula that seem to have quantified variables with missing bounding objects can be modified into an equivalent bounded form. Further, there are equivalent formula such as $\forall x(x \in C \to \exists y(y \in x \ldots))$ where C is standard that express the requirement that the quantified variables vary over members of our superstructure. Also, $\mathcal{N}$ and ${}^*\mathcal{N}$ are closed under the basic set-theoretic operations.

For each $f \in P$, let $[f]$ denote the equivalence class in $\mathcal{E}$ containing f. For each $g \in {}^*P$, let $[g]$ denote the equivalence class containing g and determined by the partition ${}^*\mathcal{E}$ of ${}^*P$.

Theorem 3.3.1. *For each $f \in P$, it follows that ${}^*[f] = [{}^*f] = [{}^\sigma f] = [f]$.*

Proof. Let $f \in P$. Then $f \in [f] \in \mathcal{E}$ implies that ${}^*f \in {}^*P$ and ${}^*f \in {}^*[f] \in {}^*\mathcal{E}$ by properties of the *-map. Now the sentence

$$f \in P \to \exists! z(z \in \mathcal{E} \wedge f \in z) \tag{3.3.4}$$

holds in $\mathcal{M}$; hence in ${}^*\mathcal{M}$. Thus there is a unique set $A \in {}^*\mathcal{E}$ such that ${}^*f \in A \in {}^*\mathcal{E}$. This set is denoted by $[{}^*f]$ since it contains ${}^*f$, and ${}^*\mathcal{E}$ is a partition. The uniqueness implies that ${}^*[f] = [{}^*f]$ and the finite nature of f yields that $[{}^\sigma f] = [f]$ under our conventions. ▮

Theorem 3.3.2. *There exists a purely subtle* $d \in {}^*\mathbf{BP} - \mathbf{BP}$ *such that* ${}^*\mathbf{C_a}(\{d\}) \cap \mathbf{BP} =$ *an infinite set and* ${}^*\mathbf{C_a}(\{d\}) \cap ({}^*\mathbf{BP} - \mathbf{BP}) =$ *an infinite set.*

Proof. Let "just" be a member of BP and consider the sentence

$$\begin{gathered}\forall x(x \in \mathbb{N} \wedge x > 0 \to \exists y(y \in T^x \wedge \forall w(w \in \mathbb{N} \wedge \\ 0 < w \leq x \to y(w) = \text{very,}|||\wedge y(0) = \text{just}) \wedge \\ \forall z(z \in \mathbb{N} \wedge z \leq x \to \exists y_1(y_1 \in T^z \wedge \forall w_1(w_1 \in \mathbb{N} \wedge \\ 0 < w_1 \leq z \to y_1(w_1) = \text{very,}||| \wedge y_1(0) = \text{just} \wedge \\ [y_1] \in \mathbf{C_a}(\{[y]\})))))). \end{gathered} \tag{3.3.5}$$

which holds in $\mathcal{M}$; hence, in ${}^*\mathcal{M}$. So, let $\nu \in {}^*\mathbb{N} - \mathbb{N}$. Then there exists some *-partial sequence $f \in ({}^*T)^\nu - P$ such that $f(w) = i(\text{very,}|||)$ for each $0 < w \leq \nu$ and $f(0) = i(\text{just})$. Also for each $n \in \mathbb{N},\ n > 0$, there exists a partial sequence $f_n \in T^n$ such that for each $n \in \mathbb{N}$, where $0 < w \leq n,\ f_n(w) = i(\text{very,}|||)$ and $f_n(0) = i(\text{just})$. Notice that if $n, m \in \mathbb{N} - \{0\}$ and $n \neq m$, then ${}^*[f_n] \neq {}^*[f_m]$. Now for each $n \in \mathbb{N},\ n > 0,\ {}^*[f_n] \in {}^\sigma\mathbf{BP}$. Application of Theorem 3.3.1 implies that ${}^*[f_n] = [{}^*f_n] = [f_n]$ and the above sentence yields that for each such $n \in \mathbb{N},\ {}^*[f_n] \in {}^*\mathbf{C_a}(\{[f]\})$. Consequently, ${}^*\mathbf{C_a}(\{[f]\}) \cap \mathbf{BP} =$ an infinite set.

Consider the infinite set $R = \{\nu_n \mid \nu_n = \nu - n \wedge n \in \mathbb{N} \wedge n \geq 1\} \subset {}^*\mathbb{N} - \mathbb{N}$. By *-transform of the above, for each $n \in \mathbb{N},\ n \geq 1$, there exists some $g_n \in ({}^*T)^{\nu_n} - P$ such that $g_n(w) = i(\text{very,}|||)$ for each $w \in {}^*\mathbb{N};\ 0 < w \leq v_n < \nu$ and $g_n(0) = i(\text{just})$. Observe that if $n, m \in \mathbb{N}$ and $n \neq m$, then $\nu_n \neq \nu_m$. Moreover, the following sentence

$$\begin{gathered}\forall x \forall y(x \in \mathbb{N} \wedge y \in \mathbb{N} \wedge x > 0 \wedge y > 0 \wedge x \neq y \to \\ \forall w \forall w_1 \forall z \forall z_1(w \in T^x \wedge w_1 \in T^y \wedge z \in \mathbb{N} \wedge z_1 \in \mathbb{N} \wedge \\ 0 < z \leq x \wedge 0 < z_1 \leq y \wedge w(z) = \text{very,}||| \wedge w_1(z_1) = \text{very,}||| \wedge \\ w(0) = \text{just} \wedge w_1(0) = \text{just} \to [w] \neq [w_1])) \end{gathered} \tag{3.3.6}$$

holds in $\mathcal{M}$; hence in ${}^*\mathcal{M}$. By *-transfer, if $n, m \in {}^*\mathbb{N} - \mathbb{N},\ n, m > 0, n \neq m$, then $[g_n] \neq [g_m]$. Since for each such $n \in {}^*\mathbb{N} - \mathbb{N},\ [g_n] \in {}^*\mathbf{C_a}(\{[f]\})$, it follows that ${}^*\mathbf{C_a}(\{[f]\}) \cap ({}^*\mathbf{BP} - \mathbf{BP}) =$ an infinite set. ▮

Corollary 3.3.2.1 *There exists a purely subtle* $d \in {}^*\mathcal{E} - \mathcal{E}$ *such that* ${}^*\mathbf{C_a}(\{d\}) \cap \mathcal{E} =$ *an infinite set and* ${}^*\mathbf{C_a}(\{d\}) \cap ({}^*\mathcal{E} - \mathcal{E}) =$ *an infinite set.*

Of course, in order to apply these results to descriptions that involve members of BP an interpretation procedure is required. We have previously discussed the intuitive interpretation for any $[f] \in {}^*\mathcal{E} - \mathcal{E}$, where the range of $f \in {}^*P$ is a subset of $i[\mathcal{W}]$. Hence, if $A \in {}^\sigma\mathcal{E}$, then $A = {}^*[g] = [g]$, where $[g] \in \mathcal{E}$. Thus in the usual manner, first interpret ${}^*[g]$ to be $[g] \in \mathcal{E}$ and then proceed to the second step and interpret $[g]$ by selecting any $f \in [g]$ and applying our previously discussed inverse procedure. Clearly, this interpretation method is a one-to-one correspondence from a subset of ${}^*\mathcal{E}$ into $\mathcal{W}$.

The concept of adjective deduction, which is obviously isomorphic to a subsystem of ordinary propositional deduction, was originally introduced to give a measure of the strength of various behavioral properties. These intuitive strengths may not be codifiable by a numerical measure. Thus, intuitively, "very,|||very,|||bold" is a stronger concept than "very,|||bold". The exact same process can be applied to physical concepts as well. Even though it may not be possible to measure the combined strengths of all of the intuitive forces that my be altering the appearance of a physical entity such as a thunderhead, the term "very,|||" could be replaced by other terms such as "greater,|||" or "weaker,|||" coupled with terms such as "force" and the like. The same type of $\vdash_a$ analysis would follow.

With respect to the above remarks, later in this book, we consider the reasoning process called simply S, which is an axiomatically presented subsystem of propositional deduction. The process S is closely associated with adjective reasoning, if the set BP is constructed in a different manner and from different objects. One of the minor problems with these constructions is their relation to formal languages and the use of parentheses within such formal languages. Another illustration of the use of these nonstandard methods that does parallel Robinson's original work along this line requires BP to be formally embedded into a propositional language with the insertion and removal of such parentheses.

3.4 Propositional Reasoning

Let S_0 denote the consequence operator determined by the usual propositional deduction $\vdash$ as defined on say pages 108-109 of [7] (i.e. Group A1 deduction). Technically, $\mathrm{BP}_0 \not\subset \mathrm{L}_0$ since BP_0 is constructed without use of parentheses. Let $\mathrm{L}' = \mathrm{L}_0 \cup \mathrm{BP}_0$. Extend $\vdash$ and S_0 in the obvious manner. Let $\mathrm{A} \in \mathrm{BP}_0 - j[\mathrm{B}]$. Then $\mathrm{A} = \mathrm{V} \wedge \ldots \wedge \mathrm{V} \wedge \mathrm{b}$, where $\mathrm{b} \in j[\mathrm{B}]$ and there are $n \geq 1$ connectives $\wedge$. We now consider inserting parentheses in the following manner called the *insertion procedure.* (1) Moving from left to right put a "(" before each V, keeping count of the number of "(" so placed. (2) Place the same number of ")" after the "b" as your count in step (1). Denote this new symbol by $\mathrm{A}_($. Note that $\mathrm{A}_(\in \mathrm{L}_0$.

Example. Suppose that you are given $\mathrm{A} = \mathrm{V} \wedge \mathrm{V} \wedge \mathrm{V} \wedge \mathrm{V} \wedge \mathrm{b}$. Then $\mathrm{A}_(= (\mathrm{V} \wedge (\mathrm{V} \wedge (\mathrm{V} \wedge (\mathrm{V} \wedge \mathrm{b}))))$.

This process of considering a method of inserting parentheses and doing it in an ordered effective manner is no more complex and no less effective than Kleene's concept of "closure with respect to (just) $\mathrm{x}_1, \ldots, \mathrm{x}_q$" on page 105 of [8]. Now to define in the obvious manner $\vdash' \subset F(\mathrm{L}') \times \mathrm{L}'$. First, consider $\mathrm{BP}_{0(} = \{\mathrm{x}_(\mid \mathrm{x} \in \mathrm{BP}_0 - \mathrm{j}[\mathrm{B}]\} \cup \mathrm{j}[\mathrm{B}]$. Then $\mathrm{BP}_{0(} \subset \mathrm{L}_0$. For $\mathrm{F} \in \mathrm{F}(\mathrm{L}')$, consider $\mathrm{F}_(= \{\mathrm{x}_(\mid \mathrm{x} \in \mathrm{F} \cap (\mathrm{BP}_0)\} \cup (\mathrm{F} - (\mathrm{BP}_0 - \mathrm{j}[\mathrm{B}])) \subset \mathrm{L}_0$. Then (i) if $\mathrm{F}_(\vdash \mathrm{B} \in \mathrm{L}_0$, define $\mathrm{F} \vdash' \mathrm{B}$. (ii) If $\mathrm{F}_(\vdash \mathrm{B}_1 \in \mathrm{L}_0$ and D is B_1 with *all of the parentheses removed* and $\mathrm{D} \in \mathrm{BP}_0$, then let $\mathrm{F} \vdash' \mathrm{D}$. (iii) Finally, remove superfluous parenthesis if you wish [7, p. 74]. Only the procedure in this paragraph is to be used to obtain a $\mathrm{B} \in \mathrm{L}'$ such that $\mathrm{F} \vdash \mathrm{B}$.

Obviously, $\vdash'$ is closely related to $\vdash$, since it is well known that for $\mathrm{A},\ \mathrm{B},\ \mathrm{C} \in \mathrm{L}_0, \vdash (\mathrm{A}\wedge(\mathrm{B}\wedge\mathrm{C})) \leftrightarrow ((\mathrm{A}\wedge\mathrm{B})\wedge\mathrm{C})$. By an abuse of notation we often write $\vdash$ for $\vdash'$, S_0 for $\mathrm{S}_0{}'$ and L_0 for L' as well as suppressing parentheses insertion and removal for the elements of BP_0. The next result follows from the fact that for $\mathrm{A},\ \mathrm{B} \in \mathrm{L}_0,\ \mathrm{A} \wedge \mathrm{B} \vdash \mathrm{B}$.

Theorem 3.4.1 *There exists a purely subtle* $d \in {}^*\mathbf{BP}_0 - \mathbf{BP}_0$ *such that* ${}^*\mathbf{S_0}(\{d\}) \cap \mathbf{BP_0}$ = *an infinite set and also* ${}^*\mathbf{S_0}(\{d\}) \cap ({}^*\mathbf{BP_0} - \mathbf{BP_0})$ = *an infinite set.*

Proof. Make the following changes in the formal first-order sentences explicitly given in the proof of Theorem 3.3.2. First, let $\mathrm{b} \in j[\mathrm{B}]$. Now for the every "just" substitute the symbol "b". Then for every "very,|||" string substitute the symbols "$\mathrm{V} \wedge$ ". With these substitutions made, the proof is exactly as for Theorem 3.3.2. ▮

Corollary 3.4.1.1 *There exists a purely subtle* $d \in {}^*\mathbf{L_0} - \mathbf{L_0}$ *such that* ${}^*\mathbf{S_0}(\{d\}) \cap \mathbf{L_0}$ = *an infinite set and also* ${}^*\mathbf{S_0}(\{d\}) \cap ({}^*\mathbf{L_0} - \mathbf{L_0})$ = *an infinite set.*

[*Remark:* The above theorems for the propositional consequence operator also hold for the consequence operator S and other such variations discussed later in this book.]

It is easily shown that for a propositional formula, say A, that $\mathrm{A} \vdash' \mathrm{A} \wedge \cdots \wedge \mathrm{A}$, with any $n \in \omega$ number of connectives $\wedge$.

Theorem 3.4.2 *For any* $q \in \mathbf{L_0}$ *it follows that* ${}^*\mathbf{S_0}(\{q\}) \cap \mathbf{L_0}$ = *an infinite set and* ${}^*\mathbf{S_0}(\{q\}) \cap ({}^*\mathbf{L_0} - \mathbf{L_0})$ = *and infinite set.*

Proof. Let $q = [f],\ f \in T^0,\ f(0) = i(\mathrm{A}),\ \mathrm{A} \in \mathrm{L_0}$. The sentence

$$\forall z(z \in \mathbb{N} \wedge z > 0 \rightarrow \exists y(y \in T^z \wedge \forall w(w \in \mathbb{N} \wedge 0 < w \leq z \rightarrow y(w) = i(\mathrm{A} \wedge\ ) \wedge y(0) = i(\mathrm{A}) \wedge [y] \in \mathbf{S_0}(\{q\})))) \tag{3.4.1}$$

holds in $\mathcal{M}$; hence in ${}^*\mathcal{M}$. Now proceed in the same manner as in the proof ot Theorem 3.3.2, making the obvious changes, starting with the statement, "Also for each $n \in \mathbb{N},\ n > 0, \ldots$." This completes the proof. ▮

3.5 Modus Ponens Reasoning

The reasoning termed *Modus Ponens* (MP) is, of course, the major step in propositional deduction. One can, however, get more basic than $\mathrm{S_0}$ and define MP reasoning to produce a subsystem of $\mathrm{S_0}$ in the following (intuitive) manner. Simply let MP be the same deduction process as determines $\mathrm{S_0}$ but with no axiom schemata. Use the symbol MP to represent the consequence operator obtained from this process. Then, for each $\mathrm{A} \subset \mathrm{L_0}$, it follows that $\mathrm{A} \subset \mathrm{MP(A)} \subset \mathrm{S_0(A)}$. Thus for each internal $B \subset {}^*\mathbf{L_0},\ B \subset {}^*\mathbf{MP}(B) \subset {}^*\mathbf{S_0}(B) \subset {}^*\mathbf{L_0}$.

Besides applying MP to $\mathrm{L_0}$, it is straightforward to apply it to certain meaningfully constructed collections of intuitive readable sentences. For example, consider the set of symbols $\mathrm{B_1} = \{\mathrm{If|||perfect,|||then|||x.} \mid \mathrm{x} \in \mathrm{BP}\}$. Now apply MP to any finite subset of $\mathrm{BP} \cup \mathrm{B_1} \cup \{\mathrm{perfect}\}$. Clearly, we can associate MP deduction formally to $\mathrm{BP_0}$ in a meaningful way. Simply let $\mathrm{c} \in \{\mathrm{P_i} \mid \mathrm{i} \in \omega\} - (\{\mathrm{V}\} \cup \mathrm{j[B]})$ and $\mathrm{B''} = \{\mathrm{c} \rightarrow \mathrm{x} \mid \mathrm{x} \in \mathrm{BP_0}\}$, etc. We leave to the reader the simple consequences of MP deduction in this case.

3.6 Predicate Deduction

In this section, predicate deduction in $\mathrm{L_1}$, say as defined by Kleene on page 82 page [7], is briefly discussed relative to lengths of formal proofs. Robinson mentions [15, p. 25], what is well known from Gödel's work, that using formal predicate deduction there is for each $n \in \mathbb{N}$ a readable sentence in $\mathrm{L_1}$ that is provable as a theorem from the empty set of hypotheses, but requires n or more steps.

Let $\mathrm{S_1}$ denote the operator determined by predicate deduction with respect to $\mathrm{L_1}$. Then $\mathrm{S_1}(\emptyset)$ is the set of all provable formula (i.e. theorems). Of course, all the properties of $\mathrm{S_1}$ are now referred to $\mathcal{E}$. Hence there exists a relation $R_{L_1} \subset \mathbb{N} \times \mathbf{S_1}(\emptyset)$ with the property that $(x, y) \in R_{L_1}$ iff $y \in \mathbf{S_1}(\emptyset),\ y \in \mathbf{L_1},\ x \in \mathbb{N}$ and $x =$ the length of a formal proof that yields $y \in \mathbf{S_1}(\emptyset)$.

Theorem 3.6.1 *For each* $\nu \in {}^*\mathbb{N} - \mathbb{N}$ *there is a subtle* $d \in {}^*\mathbf{L_1},\ d \in \mathbf{S_1}(\emptyset)$ *and for each* $\lambda \in {}^*\mathbb{N},\ \lambda < \nu,\ (\lambda, d) \notin {}^*R_{L_1}$. *Moreover, there exists some* $\nu_0 \in {}^*\mathbb{N} - \mathbb{N}$ *such that* $\nu_0 \geq \nu$ *and* $(\nu_0, d) \in {}^*R_{L_1}$.

Proof. As stated above the sentence

$$\begin{aligned} &\forall x(x \in \mathbb{N} \to \exists y(y \in \mathbf{S_1}(\emptyset) \land \forall w(w \in \mathbb{N} \land 0 \leq w \leq x \to \\ &(w, y) \notin R_{L_1}) \land \exists z(z \in \mathbb{N} \land z \geq x \land (z, y) \in R_{L_1}))) \end{aligned} \tag{3.6.1}$$

holds in $\mathcal{M}$; hence in ${}^*\mathcal{M}$. The result follows by *-transfer.

We now investigate a little more fully what is meant by the "length of a formal proof." There exists a partial sequence f' of elements of L_1 such that the domain $D(f') = [0, n],\ n \in \mathbb{N}$, [rather than $n \in \omega$], the range of $f' = \mathrm{Rn}(f') \subset \mathrm{L}_1$ and $f'(n) = \mathrm{A} \in \mathrm{S}_1(\emptyset)$ and the length of the formal proof that yields $\mathrm{A} \in \mathrm{S}_1(\emptyset)$ is $n+1$. Of course, f' actually gives the elements of L_1 that appear in such a specific formal proof. Now relate this intuitive partial sequence f' to a corresponding partial sequence in P in the following manner. Let $(x, y) \in f_{L_1}$ iff $y = i(\mathrm{w})$ and $(x, \mathrm{w}) \in \mathrm{f}'$. Denote by $P_{L_1} \subset P$ the set of all such length of proof sequences. Then $(x, y) \in {}^*R_{L_1}$ iff $x \in {}^*\mathbb{N},\ y \in {}^*\mathbf{S_1}(\emptyset)$ and there is some $f \in {}^*P_{L_1}$ such that $y = [f(x)]$. By *-transfer the hyperlength of the proof would be $x + 1$. The set P_{L_1} may be used to characterize the concept intuitively associated with the proof length for objects in $\mathrm{S}_1(\emptyset)$. These sequences have other properties as well but these will not be considered in this investigation. With this in mind, then ${}^*P_{L_1}$ represents the subtle concept of proof length for elements in ${}^*\mathbf{S_1}(\emptyset)$. It's the proof length concept we employ in one application of the results from this chapter. Theorem 3.6.1 can now be stated in an alternate form.

Theorem 3.6.2 *For each* $\nu \in {}^*\mathbb{N} - \mathbb{N}$ *there is a subtle* $d \in {}^*\mathbf{S_1}(\emptyset)$ *such that for each* $\lambda \in {}^*\mathbb{N} - \mathbb{N},\ \lambda < \nu$, *there does not exist some* $g \in {}^*P_{L_1}$ *such that* $d = [g(\lambda)]$. *Moreover, there exists some* $\nu_0 \in {}^*\mathbb{N} - \mathbb{N}$ *such that* $\nu_0 \geq \nu$ *and some* $f \in {}^*P_{L_1}$ *such that* $d = [f(\nu_0)]$.

Corollary 3.6.2.1 *There exists* $d \in {}^*\mathbf{S_1}(\emptyset),\ f \in {}^*P_{L_1}$ *and* $\nu \in {}^*\mathbb{N} - \mathbb{N}$ *such that* $d = [f(\nu)]$ *and for each* $g \in {}^*P_{L_1}$ *and each* $\nu' \in {}^*\mathbb{N}$ *such that* $d = [g(\nu')],\ |D(g)| \geq 2^\omega$.

[*Remarks*: It should be apparent to the reader that statements that hold in $\mathcal{M}$ relative to consequence operators or deductive processes are obtained from the corresponding intuitive reasoning processes by application of θ. The proofs that these statements hold in the intuitive case are straightforward or obvious, and are omitted in all cases. The term "ultralogics" is reserved for various special subtle consequence operators to be used in various cosmogony investigations that will be discussed later in this book.]

Chapter 4

SPECIAL DEDUCTIVE PROCESSES

4.1 Introduction.

There are certain words that intuitively denote an upper [resp. lower] bound to such concepts as "stronger" [resp. "weaker"]. With respect to certain philosophic studies, one such concept is the notion of "perfect" when associated with a language like BP. In what follows, this "perfect" associated with BP is used as a prototype for these other cases. Two types of deductive processes associated with this prototype will be introduced, a very trivial one followed by a much more interesting and significant procedure.

4.2 Reasoning From the Perfect Type W

First, an intuitive extension of BP is defined. Let $\mathrm{BPC|} = \mathrm{BP} \cup \{\mathrm{perfect}\}$ and for convenience denote the readable string "perfect" by the single c. Now we define type W reasoning from the perfect by considering an intuitively defined operator, Π_W, from $\mathcal{P}(\mathrm{BPC|})$ into $\mathcal{P}(\mathrm{BPC|})$.

For any finite $\mathrm{F} \subset \mathrm{BPC|}$:

(i) if $\mathrm{c} \in \mathrm{F}$, then $\Pi_W(\mathrm{F}) = \mathrm{BPC|}$;

(ii) if $\mathrm{c} \notin \mathrm{F}$, then $\Pi_W(\mathrm{F}) = \mathrm{F}$;

(iii) and for arbitrary $\mathrm{A} \subset \mathrm{BPC|}$, let

$$\Pi_W(\mathrm{A}) = \bigcup\{\Pi_W(\mathrm{F}) \mid \mathrm{F} \in F(\mathrm{A})\}$$

Theorem 4.2.1 *The map* $\Pi_W \colon \mathcal{P}(\mathrm{BPC|}) \to \mathcal{P}(\mathrm{BPC|})$ *is a consequence operator.*

Proof. Let $\mathrm{A} \subset \mathrm{BPC|}$. Clearly, axiom (4) holds by the definition. Let $\mathrm{a} \in \mathrm{A}$. Then $\{\mathrm{a}\} \in F(\mathrm{A})$. Now if $\mathrm{a} \neq \mathrm{c}$, then $\Pi_W(\{\mathrm{a}\}) = \{\mathrm{a}\}$. If $\mathrm{a} = \mathrm{c}$, then $\Pi_W(\{a\}) = \mathrm{BPC|}$. In these two cases, (iii) of the definition yields that $\mathrm{a} \in \Pi_W(\mathrm{A})$. Thus, even when $\mathrm{A} = \emptyset$, it follows that $\mathrm{A} \subset \Pi_W(\mathrm{A}) \subset \mathrm{BPC|}$ and axiom (2) holds.

Since axiom (4) holds and $\mathrm{A} \subset \Pi_W(\mathrm{A})$, it follows that $\Pi_W(\mathrm{A}) \subset \Pi_W(\Pi_W(\mathrm{A}))$. Now either $\Pi_W(\mathrm{A}) = \mathrm{A}$; in which case $\Pi_W(\Pi_W(\mathrm{A})) = \Pi_W(\mathrm{A}) = \mathrm{A}$ or $\Pi_W(\mathrm{A}) = \mathrm{BPC|}$; in which case $\Pi_W(\Pi_W(\mathrm{A})) = \Pi_W(\mathrm{BPC|}) = \mathrm{BPC|} = \Pi_W(\mathrm{A})$. Thus axiom (3) holds and this completes the proof. ∎

Recall that $T = i[\mathcal{W}]$ and if $\mathrm{w} \in \mathcal{W}$, then $f_w \in P$ denotes the partial sequence which is an element of T^0 and $f_w(0) = i(w)$, $\mathbf{w} =$

$[f_w]$. Also, due to their finitary character, each $x \in \mathcal{E}$ is often identified with ${}^*x \in {}^\sigma\mathcal{E}$.

Theorem 4.2.2 *For each internal* $B \subset \mathbf{BPC}|$ *if* $\mathbf{c} = [{}^*f_c] = [f_c]$, *then* ${}^*\Pi_W(B) = {}^*\mathbf{BPC}| = {}^*\mathbf{BP} \cup \{\mathbf{c}\} = {}^*\mathbf{BP} \cup \{[f_c]\}$, *where* $[f_c] \in {}^\sigma\mathcal{E}$ *and* ${}^\sigma\mathcal{E} = \mathcal{E}$ *under the basic identification of* ${}^\sigma\mathbb{N}$ *with* $\mathbb{N}$.

Proof. Simply consider the sentence

$$\forall x(x \in \mathcal{P}(\mathbf{BPC}|) \land \mathbf{c} \in x \to \mathbf{\Pi_W}(\{[f_c]\}) = \mathbf{BPC}|) \tag{4.2.1}$$

that holds in $\mathcal{M}$; hence in ${}^*\mathcal{M}$. The result follows by *-transfer. ∎

Corollary 4.2.2.1 *The set* ${}^*\mathbf{\Pi_W}(\{[f_c]\}) = {}^*\mathbf{\Pi_W}(\{[{}^*f_c]\}) = {}^*\mathbf{BPC}|$.

Corollary 4.2.2.2 *For each* $b \in i[\mathrm{B}]$ *and each* $\nu \in {}^*\mathbb{N} - \mathbb{N}$ *there exists a subtle* $f^b \in ({}^*T)^\nu - P$ *such that for each* $x \in {}^*\mathbb{N}$, *where* $0 < x \leq \nu$, $f^b(x) = i(\mathrm{very},|||)$ *and* $f^b(0) = b$. *Moreover,* $[f^b] \in {}^*\mathbf{\Pi_W}(\{[f_c]\})$.

For each $b \in i[\mathrm{B}]$ and a fixed $\nu \in {}^*\mathbb{N} - \mathbb{N}$, apply the axiom of choice and let f^b denote one of the subtle objects that exists by Corollary 4.2.2.2 and satisfies the stated properties. Since B is finite, the set $F_\nu = \{[f^b] \mid b \in i[\mathrm{B}]\}$ is internal. The next result is obvious.

Theorem 4.2.3 *For each* $\nu \in {}^*\mathbb{N} - \mathbb{N}$, *internal* $F_\nu \subset {}^*\mathbf{\Pi_W}(\{[f_c])\})$.

Observe that there exist, at least, 2^ω distinct F_ν sets.

4.3 Strong Reasoning From the Prefect

For the second type of reasoning from the perfect, our attention will be restricted to $\mathrm{L}' = \mathrm{L}_0 \cup \mathrm{BP}_0$ and the set BP_0 that bijectively corresponds to BP. Let a specific $\mathrm{c} \in \{\mathrm{P}_i \mid i \in \omega\} - (\{\mathrm{V}\} \cup j[\mathrm{B}])$. Correspond c to the readable sentence "prefect." Let

$$\mathrm{C}_(= \{(\mathrm{c} \to \mathrm{x}_()\mid \mathrm{x} \in \mathrm{BP}_0 - \mathrm{j[B]}\} \bigcup \{(\mathrm{c} \to \mathrm{x}) \mid \mathrm{x} \in \mathrm{j[B]}\} \bigcup \{\mathrm{c}\}.$$
$$\mathrm{C} = \{\mathrm{c} \to \mathrm{x} \mid \mathrm{x} \in \mathrm{BP}_0 - \mathrm{j[B]}\} \bigcup \{\mathrm{c} \to \mathrm{x} \mid \mathrm{x} \in \mathrm{j[B]}\} \bigcup \{\mathrm{c}\}.$$

Why do we go through the following exercise of inserting and removing parentheses so as to conform more closely to the formula of a formal language? The basic reason is related to some of the results later in this book that refer to counting of symbols by means of the partial sequences. Clearly, parenthesis insertion does correspond to the increase strength idea of adjective reasoning, as does the ordering of the very,||| symbols by the partial sequences. However, in certain deductive processes, all of the axioms for the propositional logic are not used. Hence even though it is certainly of no importance, due to equivalence, when all of the usual axioms are used to write a formal $(\mathrm{V} \wedge (\mathrm{V} \to \mathrm{b}))$ as $\mathrm{V} \wedge \mathrm{V} \to \mathrm{b}$, it may not be possible to establish this equivalence for these restrictive deductive processes. The process we now outline simply removes this formal difficulty at the cost of a more involved finitary process.

Let $\mathrm{BPC}_0 = \mathrm{BP}_0 \cup \mathrm{C}$ and $\mathrm{BPC}_{0(} = \mathrm{BP}_{0(} \cup \mathrm{C}_($. The axioms are elements of the set $\mathrm{Ax} = \{(\mathrm{V} \wedge \mathrm{x}) \to \mathrm{x} \mid \mathrm{x} \in \mathrm{BP}_{0(}\}$ with the suppression of the outer most parentheses for simplicity in application of MP. Let $\vdash_\pi$ denote ordinary propositional deduction but *only using the axiom set* Ax *and only formula from the set* $\mathrm{Ax} \cup \mathrm{BPC}_{0(}$ *in the steps of any proof.* For a specific $\mathrm{A}_(\in \mathrm{BP}_{0(}$, let A be the element of BP_0 formed by removing all parentheses from $\mathrm{A}_($. Define $\Pi \colon \mathcal{P}(\mathrm{BPC}_0) \to \mathcal{P}(\mathrm{BPC}_0)$ as follows: Let finite $\mathrm{F} \subset \mathrm{BPC}_0 \subset \mathrm{L}'$ and $\mathrm{F}_(= (\mathrm{BP}_0 \cap \mathrm{F})_(\cup (\mathrm{C} \cap \mathrm{F})_($. Then for any $\mathrm{D} \subset \mathrm{BPC}_0$, $\mathrm{X} \in \Pi(\mathrm{D})$ iff there exists a finite $\mathrm{F} \subset \mathrm{D}$ and $\mathrm{A} \in \mathrm{BPC}_{0(}$ such that $\mathrm{F}_(\vdash_\pi \mathrm{A}_($ and if $\mathrm{A}_(\in \mathrm{C}_($, then $\mathrm{X} = \mathrm{A}$ or if $\mathrm{A}_(\in \mathrm{BP}_{0(}$, then $\mathrm{A} = \mathrm{X}$. It is not difficult to show that Π is a consequence operator since it is the restriction of formal $\vdash_\pi$ to $\mathrm{BPC}_{0(}$ and Π is called *strong reasoning from the perfect.*

By way of a reminder, *-transfer and the fact that $\mathrm{E}_0 = j[\mathrm{B}]$ and $i[\mathrm{E}_0]$ are finite imply that $f \in {}^*\mathbf{BP_0} - {}^*\mathbf{E_0}$ iff there exists some $n \in {}^*\mathbb{N}, n > 0$, and $b \in i[\mathrm{E}_0]$ such that $f(w) = i(\mathrm{V}\wedge)$ and $f(0) = b$ for each $w \in {}^*\mathbb{N}$ such that $0 < w \leq n$. Further, for each $b \in i[\mathrm{E}_0]$ and each $b \in i[\mathrm{E}_0]$ and each $n \in {}^*\mathbb{N}, n > 0$, there exists a $f \in {}^*\mathbf{BP_0} - {}^*\mathbf{E_0}$ such that for each $w \in {}^*\mathbb{N}$, where $0 < w \leq n$, $f(w) = i(\mathrm{V}\wedge)$ and $f(0) = b$. Notice also that $[g] \in {}^*\mathbf{E_0}$ iff $g \in ({}^*T)^0$, $g(0) = b \in i[\mathrm{E}_0]$ and $|[g]| = 1$. The set ${}^*\mathbf{E_0}$ being nonempty and finite implies that ${}^*\mathbf{E_0} = \{{}^*[g_1], \ldots, {}^*[g_n]\} = {}^\sigma\mathbf{E_0} = \mathbf{E_0}$. Finally, due to the identification of each specific ${}^*i(\mathrm{e})$ with $i(\mathrm{e})$, it follows that ${}^*g = g$, where, as usual, this follows from the finitary character of each equivalence class.

We know that for fixed $n \in {}^*\mathbb{N}$ and any $b \in i[\mathrm{E}_0]$ there exists a unique $g^b \in ({}^*T)^n$ such that $g^b(0) = b$ and if $w \in {}^*\mathbb{N}$ and $0 < w \leq n$, then $g^b(w) = i(\mathrm{V}\wedge)$. Let $G_n = \{[g^b] \mid b \in i[\mathrm{E}_0]\}$ for $n \in {}^*\mathbb{N} - \{0\}$. Now G_n has the same general properties as the previously defined set F_n. In particular, each G_n is internal and if $\nu \in \mathbb{N}_\infty$, then G_ν is purely subtle.

Theorem 4.3.1. *If* $n \in {}^*\mathbb{N} - \{0\}$, $m \in {}^*\mathbb{N} - \{0\}$ *are such that* $0 < m \leq n$ *and* $f \in ({}^*T)^m$ *has the property that for each* $w \in {}^*\mathbb{N}$, *where* $0 < w \leq m$; $f(w) = i(\mathrm{V}\wedge)$ *and* $f(0) = b \in i[\mathrm{E}_0]$, *then* $[f] \in {}^*\mathbf{\Pi}(G_n)$. *Moreover, if* $[g] \in i[\mathrm{E}_0]$, *then* $[g] \in {}^*\mathbf{\Pi}(G_n)$.

Proof. First, since G_n is internal, $n \in {}^*\mathbb{N} - \{0\}$, and ${}^*\mathbf{BP_0} \subset$

${}^*\mathbf{BP_0} \cup {}^*\mathbf{C} = {}^*\mathbf{BPC_0}$ and $G_n \subset {}^*\mathbf{BP_0}$, it follows that G_n is in the domain of ${}^*\mathbf{\Pi}$.

Let $\mathrm{A} \in \mathrm{BP}_0,\ \mathrm{A} = \mathrm{V} \wedge \cdots \wedge \mathrm{V} \wedge \mathrm{b}, \mathrm{b} \in \mathrm{E}_0$ and that are $n \geq 1,\ (n \in \omega)$ connectives $\wedge$. We prove by induction that for any $n \in \omega$ such that $0 < w \leq n$, the symbol string $\mathrm{B} = \mathrm{V} \wedge \cdots \wedge \mathrm{V} \wedge \mathrm{b}$ with $w \geq 1$ connectives $\wedge$ or $\mathrm{B} = \mathrm{b}$ has the property that $\mathrm{A}_(\vdash_\pi \mathrm{B}_($, where if $\mathrm{B} = \mathrm{b}$, then $\mathrm{B}_(= \mathrm{b}$.

Case 1. Let $n = 1$. Then $\mathrm{A} = \mathrm{V} \wedge \mathrm{b}$. Consider $\mathrm{A}_(= (\mathrm{A} \wedge \mathrm{b})$. The following is a proof that $\mathrm{A}_(\vdash_\pi \mathrm{b}$. (i) $(\mathrm{V} \wedge \mathrm{b})$, (ii) $(\mathrm{V} \wedge \mathrm{b}) \to \mathrm{b}$, (iii) b. A proof composed of step (i) only yields the trivial result that $\mathrm{A}_(\vdash_\pi \mathrm{A}_)$.

Case $(n{+}1)$. Suppose that the result holds for n, and $\mathrm{A} = \mathrm{V} \wedge \cdots \wedge \mathrm{V} \wedge \mathrm{b}$ has $n{+}1$ connectives $\wedge$. The formula $\mathrm{A}_(= (\mathrm{V} \wedge (\mathrm{A} \cdots (\mathrm{V} \wedge \mathrm{b}) \cdots))$. Let $\mathrm{B}_(= (\mathrm{V} \wedge (\mathrm{A} \cdots (\mathrm{V} \wedge \mathrm{b}) \cdots))$ have n connectives $\wedge$. The following is a proof that $\mathrm{A}_(\vdash_\pi \mathrm{B}_($. (i) $\mathrm{A}_(= (\mathrm{V} \wedge \mathrm{B}_()$, (ii) $(\mathrm{V} \wedge \mathrm{B}_() \to \mathrm{B}_($, (iii) $\mathrm{B}_($. Thus $\mathrm{A}_(\vdash_\pi \mathrm{B}_($. From the induction hypothesis, $\mathrm{B}_(\vdash_\pi \mathrm{E}_($, where $\mathrm{E}_($ has w connectives $\wedge$ such that $0 < w \leq n$, or $\mathrm{E}_(= \mathrm{b}$. Since $\vdash_\pi$ is transitive, it follows that $\mathrm{A}_(\vdash \mathrm{E}_($. The trivial proof using step (i) yields that $\mathrm{A}_(\vdash_\pi \mathrm{A}_($, and the basic result above follows by induction.

We have shown that for each $b \in i[\mathrm{E}_0]$, the following sentences

$$\begin{gathered}\forall y \forall x \forall w \forall z((x \in \mathbb{N}) \wedge (x > 0) \wedge (y \in \mathbb{N}) \wedge (0 < y \leq x) \wedge \\ (w \in T^x) \wedge \forall w_1((w_1 \in \mathbb{N}) \wedge (0 < w_1 \leq x) \to (w(w_1) = i(\mathrm{V}\wedge)) \wedge \\ (w(0) = b)) \wedge (z \in T^y) \wedge \forall z_1((z_1 \in \mathbb{N}) \wedge (0 < z_1 \leq y) \to \\ z(z_1) = i(\mathrm{V}\wedge)) \wedge (z(0) = b) \to ([z] \in \mathbf{\Pi}(\{[x]\}))).\end{gathered} \tag{4.3.1}$$

$$\begin{gathered}\forall x \forall y \forall w \forall z((x \in \mathbb{N}) \wedge (x > 0) \wedge (w \in T^x) \wedge \forall w_1((w_1 \in \mathbb{N}) \wedge \\ (0 < w_1 \leq x) \to w(w_1) = i(\mathrm{V}\wedge)) \wedge (w(0) = b) \wedge \\ (z \in T^0) \wedge (z(0) = b) \to ([z] \in \mathbf{\Pi}(\{[x]\})))\end{gathered} \tag{4.3.2}$$

hold in $\mathcal{M}$, hence in ${}^*\mathcal{M}$. Since the singleton subsets of G_n are *-finite, it follows that $\bigcup\{{}^*\mathbf{\Pi}(\{[g^b]\}) \mid b \in i[\mathrm{E}_0]\} \subset {}^*\mathbf{\Pi}(G_n)$ by *- transfer of Axiom (4). Let $n \in {}^*\mathbb{N} - \{0\},\ f \in ({}^*T)^m,\ 0 < m \leq n,\ f(w) = i(\mathrm{V}\wedge)$ for each $w \in \mathbb{N},\ 0 < w \leq m$ and $f(0) = b$. Then by *-transfer of sentence (4.3.1), we have that

$$[f] \in {}^*\mathbf{\Pi}(\{[g^b]\}) \subset {}^*\mathbf{\Pi}(G_n). \tag{4.3.3}$$

For $g \in T_0$ such that $g(0) = b$, *-transfer of sentence (4.3.2) yields ${}^*[g] = [{}^*g] = [g] \in {}^*\mathbf{\Pi}(\{[g^b]\}) \subset {}^*\mathbf{\Pi}(G_n)$ and this completes the proof. ▮

Corollary 4.3.1.1 *If* $\nu \in \mathbb{N}_\infty$, *then* $\mathbf{BP_0} \cap G_\nu = \emptyset$, $\mathbf{BP_0} \subset {}^*\mathbf{\Pi}(G_\nu)$ *and* $\mathbf{BP_0} \cup G_\nu \subset {}^*\mathbf{\Pi}(G_\nu)$.

For $\mathrm{A} \in \mathrm{L_0}$, let $\#(\mathrm{A})$ denote the length of the formula A.

Theorem 4.3.2 *Let* $\mathrm{B} \subset \mathrm{BP_0},\ \mathrm{A} = \mathrm{V}\land\cdots\land\mathrm{V}\land\mathrm{b},\ \mathrm{b} \in \mathrm{E_0}$, *where there are* $n \geq 1$ *connections* $\land$ *and for each* $\mathrm{Z} \in \mathrm{B}, \#(\mathrm{A}) > \#(\mathrm{Z})$. *Then* $\mathrm{A} \notin \Pi(\mathrm{B})$.

Proof. (Note: In this proof certain of the indicated parentheses may be superfluous.) Assume the hypothesis of the theorem. We show that there does not exist a finite $\mathrm{F} \subset \mathrm{B}$ such that $\mathrm{F}_(\vdash_\pi \mathrm{A}_($. Assume that there exists a finite $\mathrm{F} \subset \mathrm{B}$ such that $\mathrm{F}_(\vdash_\pi \mathrm{A}_($. A relation $R_m \subset \mathrm{BPC}_{0(} \times \mathrm{BPC}_{0(}$ is called an *m-chained sequence of* MP *processes* if there exists an $m \in \omega,\ m \geq 1$ such that $(X, Y) \in R_m$ has the form $(\mathrm{A_i}, (\mathrm{A_i}) \to (\mathrm{A_{i-1}}))$ for $i = 1, \ldots, m$, where $\mathrm{A_0} = \mathrm{A}_($. Also each $\mathrm{A_i} = \mathrm{V} \land (\mathrm{A_{i-1}})$ and $\mathrm{A_i}$ is a step in the proof of $\mathrm{F}_(\vdash_\pi \mathrm{A}_($, where $i = 1, \ldots, m$. We now show by induction that, for each $m \in \omega,\ m \geq 1$, there exists an m-chained sequence of MP processes in the proof that $\mathrm{F}_(\vdash_\pi \mathrm{A}_($.

Case m =1. We know that $\mathrm{A}_(\in \mathrm{BP}_{0(}$ implies that $\mathrm{A}_($ is not an instance of the use of an axiom since no axioms appear in $\mathrm{BP}_{0(}$. Since $\mathrm{A} \notin \mathrm{B}$ then $\mathrm{A}_(\notin \mathrm{F}_($. Thus $\mathrm{A}_($ being the last step in the proof implies that $\mathrm{A}_($ is the conclusion of an MP process with premises $(\mathrm{D}) \to (\mathrm{A}_()$ and D. Assume that $\mathrm{D} = \mathrm{c}$. Then the single step which contains the primitive c could not be an axiom nor an assumption since $\mathrm{c} \notin \mathrm{B}$. Hence c would be the conclusion of a prior MP process. Therefore a prior step would be of the form $(\mathrm{E}) \to \mathrm{c}$. This is impossible; all steps must be formula in $\mathrm{Ax} \cup \mathrm{BPC}_{0(}$. Thus $\mathrm{D} \neq \mathrm{c}$ implies that $(\mathrm{D}) \to (\mathrm{A}_()$ is an instance of an axiom or the conclusion of a prior MP process. However, since no step can be of the form $(\mathrm{E}) \to ((\mathrm{D}) \to (\mathrm{A}_())$, it follows that $(\mathrm{D}) \to (\mathrm{A}_()$ must be an instance of an axiom. Consequently, $\mathrm{D} = \mathrm{V} \land (\mathrm{A}_() = \mathrm{A_1} \in \mathrm{BPC}_{0(}$ and D is a step in the proof. Therefore, $\mathrm{R_1} = (\mathrm{A_1}, (\mathrm{A_1}) \to (\mathrm{A_0})),\ \mathrm{A_0} = \mathrm{A}_($ is a 1- chained sequence of MP processes.

Assume the result holds for m.

Case m + 1. Let $\mathrm{R_m} = \{(\mathrm{A_m}, (\mathrm{A_m}) \to (\mathrm{A_{m-1}})), \ldots, (\mathrm{A_1}, ((\mathrm{A_1}) \to (\mathrm{A_0}))\}$ be an m-chained sequence of MP processes. Now $\mathrm{A_m} = \mathrm{V} \land (\mathrm{A_{m-1}})$ implies by a simple induction proof that $\#(\mathrm{A_m}) > \#(\mathrm{A_0}) = \#(\mathrm{A}_()$. Hence, $\mathrm{A_m} \notin \mathrm{B}_($. Thus $\mathrm{A_m} \notin \mathrm{F}_($. Moreover, $\mathrm{A_m} \notin \mathrm{C},\ \mathrm{A_m} \notin \mathrm{Ax}$ for the primary connective is $\land$. This implies that $\mathrm{A_m} \in \mathrm{BP}_{0(}$ and A_m must be the conclusion of some MP process with premises $(\mathrm{D}) \to (\mathrm{A_m})$ and D. As in case m =1, it follows that $\mathrm{D} \neq \mathrm{c}$ and that $(\mathrm{D}) \to (\mathrm{A_m})$ must be an instance of an axiom.

Consequently, $\mathrm{D} = \mathrm{V} \wedge (\mathrm{A_m})$. Let $\mathrm{D} = \mathrm{A_{m+1}}$. Since D is a step in the proof, $\mathrm{D} \in \mathrm{BPC}_{0(}$. Thus, $\mathrm{R_{m+1}} = \{(\mathrm{A_{m+1}}, (\mathrm{A_{m+1}}) \to (\mathrm{A_m}))\} \cup \mathrm{R_m}$ is an $m+1$-chained sequence of MP processes.

The length of the proof that $\mathrm{F}_(\vdash_\pi \mathrm{A}_($ is some finite number, say $n \in \omega,\ n \geq 1$. The above shows that there exists an n+1-chained sequence of MP processes for this proof. Since $|P_1(\mathrm{R_{n+1}})| = \mathrm{n} + 1$ (note that for each $i,\ j \in \omega$ such that $0 \leq i < j \leq n+1,\ \#(\mathrm{A_i}) < \#(\mathrm{A_j})$) and each element of $P_1(\mathrm{R_{n+1}})$ is a distinct step in the proof, this contradicts the fact that the proof length is n. Consequently, there does not exist a finite $\mathrm{F} \subset \mathrm{B}$ such that $\mathrm{F}_(\vdash_\pi \mathrm{A}_($. Therefore $\mathrm{A} \notin \Pi(\mathrm{B})$ and this completes the proof of this theorem. ▮

Under our embedding, Theorem 4.3.2 is interpreted by θ as embedded into $\mathcal{M}$. When this is done the length of a formula A, $\#(\mathrm{A})$, is the length of the preimage A of the map i associated with the special partial sequence $f_\mathrm{A}(0) = i(\mathrm{A})$. Let $\mathcal{G} = \{G_n \mid (n > 0) \wedge (n \in \omega)\} \cup \{G_0\},\ G_0 = \mathbf{E_0}$, and $|i[\mathrm{E_0}]| = m+1$. Indeed, $\mathrm{E_0} = \mathrm{j}[\mathrm{B}] = \{\mathrm{b_0}, \ldots, \mathrm{b_m}\}$. Let $\mathcal{G} \subset X_n$. The following sentences hold in $\mathcal{M}$; hence in ${}^*\mathcal{M}$.

$$\begin{aligned}&\forall x(x \in X_n \to ((x \in \mathcal{G}) \leftrightarrow \exists y \exists y_0 \cdots \exists y_m((y \in \mathbb{N}) \wedge (y \geq 0) \wedge (y_0 \in T^y)\\ &\quad \wedge \cdots \wedge (y_m \in T^y) \wedge \forall w((w \in \mathbb{N}) \wedge (0 < w \leq y) \to (y_0(w) =\\ &\quad\quad i(\mathrm{V}\wedge)) \wedge \cdots \wedge (y_m(w) = i(\mathrm{V}\wedge))) \wedge (y_0(0) = \mathrm{b_0})\\ &\quad \wedge \cdots \wedge (y_m(0) = \mathrm{b_m} \wedge ([y_0] \in x) \wedge \cdots \wedge ([y_m] \in x)))).\end{aligned} \tag{4.3.4}$$

Sentence (4.3.4) can also be written as

$$\begin{aligned}&\forall x(x \in X_n \to (x \in \mathcal{G} \leftrightarrow \exists y((y \in \mathbb{N}) \wedge (y \geq 0) \wedge A(y))));\\ &\forall y((y \in \mathbb{N}) \wedge (y \geq 0) \to A(y)),\end{aligned} \tag{4.3.5}$$

where $A(y)$ is the obvious expression taken from (4.3.4). The objects that exist for each "y" in the $A(y)$ expression (i.e. the $y_j \in T^y,\ j = 0, \ldots, m$) are unique with respect to the property expressed in $A(y)$. Obviously, for each $n \in \mathbb{N},\ G_n \in {}^*G$. Moreover, there exists a bijection $F\colon \mathbb{N} \to \mathcal{G}$ such that $F(n) = G_n$. Now let $n,\ m \in \mathbb{N},\ n, m$ and $\mathrm{A} = \mathrm{V} \wedge \ldots \wedge \mathrm{b},\ \mathrm{b} \in \mathrm{E_0}$ has m connectives $\wedge$. Then $\#(\mathrm{A}) > \#(\mathrm{Z})$ for each $[f_Z] \in G_n$. It follows from Theorem 4.3.2 that $[f_A] \notin \mathbf{\Pi}(G_n)$.

Theorem 4.3.3 *If $n,\ m \in {}^*\mathbb{N},\ 0 \leq n < m,\ b \in i[E_0]$ and $f \in ({}^*T)^m$ such that $f(w) = i(\mathrm{V}\wedge)$ for each $w \in {}^*\mathbb{N},\ 0 < w \leq m$, and $f(0) = b$, then it follows that $[f] \notin {}^*\mathbf{\Pi}(G_n)$.*

Proof. Let $b \in i[\mathrm{E_0}]$. From the above discussion, the following sentence

$$\forall x \forall y \forall z((x \in \mathbb{N}) \wedge (y \in \mathbb{N}) \wedge (0 \leq y < x) \wedge (z \in T^x) \wedge$$

$$\forall w((w \in \mathbb{N}) \wedge (0 < w \leq x) \to (z(w) = i(\mathrm{V}\wedge))\wedge$$
$$(z(0) = b)) \to ([z] \notin \mathbf{\Pi}(F(y)))) \tag{4.3.6}$$

holds in $\mathcal{M}$; hence in ${}^*\mathcal{M}$. The result follows by *-transfer. ▮

Corollary 4.3.3.1 *For each* $n \in {}^*\mathbb{N}$, ${}^*\mathbf{\Pi}(G_n) \neq {}^*\mathbf{\Pi}({}^*\mathbf{BP_0})$ *and* ${}^*\mathbf{\Pi}(G_n) = \bigcup\{G_x \mid (x \in {}^*\mathbb{N}) \wedge (0 \leq x \leq n)\}$.

Proof. Since $G_n \subset {}^*\mathbf{BP_0}$, ${}^*\mathbf{\Pi}(G_n) \subset {}^*\mathbf{\Pi}({}^*\mathbf{BP_0})$. Theorems 4.3.1 and 4.3.3 along with the above discussion completely characterizes the elements of ${}^*\mathbf{\Pi}(G_n)$. This completes the proof. ▮

For $\nu \in \mathbb{N}_\infty$, let $\mathcal{G}_\nu = \bigcup\{G_x \mid (x \in \mathbb{N}_\infty) \wedge (x < \nu)\}$. Then $\mathcal{G}_\nu$ is a *purely subtle external object.* This follows from the fact that ${}^*\mathbf{\Pi}(G_\nu)$ is internal, $\mathbf{BP}_0$ is external and ${}^*\mathbf{\Pi}(G_\nu) = \mathcal{G}_\nu \cup \mathbf{BP}_0 \cup G_\nu$. Moreover, observe that if $\lambda, \nu \in \mathbb{N}_\infty, \nu > \lambda$, then ${}^*\mathbf{\Pi}(G_\lambda) \cap G_\nu = \emptyset$ and that $\mathcal{G}_\nu$ and $\mathbf{BP}_0$ are not in the domain of ${}^*\mathbf{\Pi}$ unless we extend ${}^*\mathbf{\Pi}$, say by the identity operator.

Theorem 4.3.4 *The set* ${}^*\mathbf{\Pi}({}^*\mathbf{BP}_0) = {}^*\mathbf{BP}_0$.

Proof. (Note once again that some superfluous parentheses may have been added to some formula in this proof.) It is know that ${}^*\mathbf{BP}_0 \subset {}^*\mathbf{\Pi}({}^*\mathbf{BP}_0)$. Let finite $\mathrm{F} \subset \mathrm{BP}_0$, $\mathrm{A} \in \mathrm{C}$ and assume that $\mathrm{F}_(\vdash_\pi \mathrm{A}$. Then $\mathrm{A} = \mathrm{c}$ or $\mathrm{A} = \mathrm{c} \to (\mathrm{x})$, $\mathrm{x} \in \mathrm{BP}_{0(}$.

Case 1. Assume that $\mathrm{A} = \mathrm{c}$. Since $\mathrm{c} \notin \mathrm{F}_(\subset \mathrm{BP}_{0(}$ and $\mathrm{A} = \mathrm{c}$ is not an instance of an axiom, $\mathrm{A} = \mathrm{c}$ must be the conclusion of an MP process. Thus a prior step is of the form $(\mathrm{D}) \to \mathrm{c}$. This is impossible for $(\mathrm{D}) \to \mathrm{c} \notin \mathrm{Ax} \cup \mathrm{BP}_{0(}$.

Case 2. Assume that $\mathrm{A} = \mathrm{c} \to (\mathrm{x})$, $\mathrm{x} \in \mathbf{BP}_{0(}$. Again $\mathrm{c} \to (\mathrm{x})$ is the conclusion of an MP process. This is impossible since no formula of the type $(\mathrm{D}) \to (\mathrm{c} \to (\mathrm{x}))$ is an element of $\mathrm{Ax} \cup \mathrm{BP}_{0(}$. Hence by *-transfer of the appropriate first-order sentence, after the θ embedding, it follows that ${}^*\mathbf{\Pi}({}^*\mathbf{BP}_0) \subset {}^*\mathbf{BP}_0$. ▮

Corollary 4.3.4.1 *The set* ${}^*\mathbf{\Pi}({}^*\mathbf{BP}_0) \underset{\neq}{\subset} {}^*\mathbf{\Pi}({}^*\mathbf{BPC}_0)$.

It is easy to see that $\Pi(\mathrm{C}) = \mathrm{BPC}_0$. For let $\mathrm{A} \in \mathrm{BP}_0$ and consider the proof (1) c, (2) $\mathrm{c} \to (\mathrm{A}_()$, (3) $\mathrm{A}_($. Thus $\{\mathrm{c}, \mathrm{c} \to (\mathrm{A}_()\} \vdash_\pi \mathrm{A}_($ yields that $\mathrm{A} \in \Pi(\mathrm{C})$. Hence $\Pi(\mathrm{C}) = \mathrm{BPC}_0$. Also, ${}^*\mathbf{\Pi}({}^*\mathbf{C}) = {}^*\mathbf{BPC}_0$.

Theorem 4.3.5 *Let internal* $A \subset {}^*\mathbf{BP}_0$ *and internal* $B \subset {}^*\mathbf{BPC}_0$. *Then* ${}^*\mathbf{\Pi}(A \cup B) = {}^*\mathbf{BPC}_0$ *iff* ${}^*\mathbf{C} \subset B$.

Proof. For the sufficiency, let internal $A \subset {}^*\mathbf{BP}_0$, internal ${}^*\mathbf{C} \subset B$. Then $A \cup B$ is internal and ${}^*\mathbf{\Pi}({}^*\mathbf{C}) = {}^*\mathbf{BPC}_0 \subset {}^*\mathbf{\Pi}(B) \subset {}^*\mathbf{\Pi}(A \cup B) \subset {}^*\mathbf{BPC}_0$. Thus ${}^*\mathbf{BPC}_0 = {}^*\mathbf{\Pi}(A \cup B)$.

For the necessity, assume that internal $A \subset {}^*\mathbf{BP}_0$, internal $B \subset {}^*\mathbf{BPC}_0$ and that ${}^*\mathbf{\Pi}(A \cup B) = {}^*\mathbf{BPC_0}$. Let $\mathrm{A}_1 \subset \mathrm{BP}_0,\ \mathrm{B}_1 \subset \mathrm{BPC}_0$ and $\Pi(\mathrm{A}_1 \cup \mathrm{B}_1) = \mathrm{BPC}_0$. It follows from Theorem 4.3.4 that $\mathrm{B}_1 \not\subset \mathrm{BP}_0$. Indeed, given any finite $\mathrm{F} \subset \mathrm{A}_1 \cup (\mathrm{B}_1 \cap \mathrm{BP}_0)$. If $\mathrm{D} \in \Pi(\mathrm{F})$, then $\mathrm{D} \in \mathrm{BP}_0$. Thus only for a finite $\mathrm{F}_1 \subset \mathrm{B}_1 \cap \mathrm{C}$ can there be an $\mathrm{E} \in \mathrm{C}$ such that $\mathrm{E} \in \Pi(\mathrm{F}_1)$. Hence all that needs to be shown is that $\mathrm{C} \subset \Pi(\mathrm{B}_1 \cap \mathrm{C})$ implies that $\mathrm{B}_1 \cap \mathrm{C} = \mathrm{C}$. So, assume that $\mathrm{B}_1 \cap \mathrm{C} \neq \mathrm{C}$. Hence either $\mathrm{c} \notin \mathrm{B}_1 \cap \mathrm{C}$ or there exists some $\mathrm{A}_(\in \mathrm{BP}_{0(}$ such that $\mathrm{c} \to (\mathrm{A}_() \notin (\mathrm{B}_1 \cap \mathrm{C})$.

Case 1. Assume that $\mathrm{c} \notin \mathrm{B}_1 \cap \mathrm{C}$ and F is any finite subset of $\mathrm{B}_1 \cap \mathrm{C}$ such that $\mathrm{F} \vdash_\pi \mathrm{c}$. Of course, c is the last step in a formal proof. c is the conclusion of some MP process since c is not an assumption nor an axiom. Thus some formula of the form $(\mathrm{D}) \to \mathrm{c}$ must be in a prior step in the formal proof. This is impossible since no formula of this form is an element of $\mathrm{BPC}_{0(}$.

Case 2. Assume that there exists some $\mathrm{A}_(\in \mathrm{BP}_{0(}$ such that $\mathrm{c} \to (\mathrm{A}_() \notin \mathrm{B}_1 \cap \mathrm{C}$ and there exists finite $\mathrm{F} \subset \mathrm{B}_1 \cap \mathrm{C}$ such that $\mathrm{F} \vdash_\pi \mathrm{c} \to (\mathrm{A}_()$. Again $\mathrm{c} \to (\mathrm{A}_()$ is not an assumption nor an axiom. Consequently, $\mathrm{c} \to (\mathrm{A}_()$ is the conclusion of an MP process. Thus there exists some formula of the form $(\mathrm{D}) \to (\mathrm{c} \to (\mathrm{A}_())$ in a prior step. Again this is impossible.

These two cases imply that $\mathrm{B}_1 \cap \mathrm{C} = \mathrm{C}$. Therefore, $\mathrm{C} \subset \mathrm{B}_1$ implies the sentence

$$\begin{aligned} &\forall x \forall y((x \in \mathcal{P}(\mathbf{BP_0})) \land (y \in \mathcal{P}(\mathbf{BPC_0})) \land \\ &(\mathbf{\Pi}(x \cup y) = \mathbf{BPC_0}) \to (\mathbf{C} \subset y)) \end{aligned} \tag{4.3.7}$$

holds in $\mathcal{M}$; hence in ${}^*\mathcal{M}$. The result follows from *-transfer. ▮

Note that all of the results in this section hold for BP and BPC, where C is constructed without parentheses.

4.4 Order

We briefly look at two special types of order relations, the "number of symbols" order and the "better than" order. Previously the concept of the length of a formula or word A (i.e. $\#(\mathrm{A})$) was introduced. This type of order has few properties unless it is restricted to certain interesting types of subsets.

Let nonempty $\mathrm{B},\ \mathrm{D} \subset \mathrm{BPC}$ (or BPC_0), then define $\mathrm{B} \leq_\# \mathrm{D}$ if for each $\mathrm{b} \in \mathrm{B}$ and for each $\mathrm{d} \in \mathrm{D}$, it follows that $\#(\mathrm{b}) \leq \#(\mathrm{d})$. This order is obviously a pre-order in the sense that it is reflexive and transitive. However, in general, it should probably not be considered a partial order since antisymmetry does not imply set equality although it does imply that all the symbol strings have equal length in both B and D. Also other pre-orders of this type appear not to be partial orders for the same reason. If $\leq_\#$ is restricted to certain collections of sets, then it does become a useful partial order under set equality.

Consider the collection $\{G_n \mid n \in {}^*\mathbb{N}\}$. Then the pre-order $\leq_\#$ restricted to this set is isomorphic to the simple order of ${}^*\mathbb{N}$. Indeed, $G_n \leq_\# G_m$ iff $n \leq m$, where $n,\ m \in {}^*\mathbb{N}$ and $\leq$ is the usual extension of the simple order induced on $\mathbb{N}$ by ω. Moreover, notice that $G_n = G_m$ iff $n = m$, and $G_n \neq_\# G_m$ iff $G_n \cap G_m = \emptyset$.

For the collection $\{\mathcal{G}_\nu \mid \nu \in \mathbb{N}_\infty\}$, it follows that $\mathcal{G}_\gamma \subset \mathcal{G}_\lambda$ iff $\gamma \leq \lambda$. Thus $\{\mathcal{G}_\nu \mid \nu \in \mathbb{N}_\infty\}$ is ordered by inclusion when the simple order of the subscripts is considered. Notice also that $\bigcap\{\mathcal{G}_\nu \mid \nu \in \mathbb{N}_\infty\} = \emptyset$.

Let fixed $\nu \in \mathbb{N}_\infty$. Then there exist infinitely many $\mathcal{G}_\lambda$ which differ only be a finite set of subtle objects. Simply consider the set $\{\mathcal{G}_{\nu+n} \mid n \in \mathbb{N}\}$. If $n,\ m \in \mathbb{N}$ and $m > n$, then $|\mathcal{G}_{\nu+m} - \mathcal{G}_{\nu+n}| = (m-n)|G_0| \in \mathbb{N}$. Also there exist infinitely many sets "longer than" any $G_{\nu+n}$, where $n \in \mathbb{N}$ or strictly containing any $\mathcal{G}_{\nu+n}$. To see this consider $\nu^2 < \nu^3 < \cdots < \nu^n < \cdots,\ n \in \mathbb{N}$, and observe that $\nu^2 - \nu = \nu(\nu-1) \in \mathbb{N}_\infty$. Thus the length of an interval $[\nu^m, \nu^n],\ m < n$, is an infinite natural number and for any $n \in \mathbb{N}$, this implies that $\nu + n < \nu^2$. Hence for any $n \in \mathbb{N}$ such that $n > 1$, it follows that $\mathcal{G}_{\nu+n} \subset \mathcal{G}_{\nu^n}$ and $G_{\nu+n} \leq_\# G_{\nu^n}$. It is also interesting to note that for each pair $\nu,\ \lambda$ of infinite natural numbers, such that $\nu \leq \lambda,\ {}^*\mathbf{\Pi}(G_\nu) \subset {}^*\mathbf{\Pi}(G_\lambda)$ and conversely.

The "better than" order is only defined for comparable readable sentences. For this research, the domain of definition is restricted to the set $\mathbf{BP_0}$ [resp. $\mathbf{BP}$]. Two elements $[f],\ [g] \in \mathbf{BP_0}$ [resp. $\mathbf{BP}$] are *comparable* if there exists $b \in i[\mathrm{E}_0]$ [resp. $i[\mathrm{B}]$] such that $f^b \in [f]$ and $g^b \in [g]$. Recall that f^b and g^b are unique element of T^n and T^m, respectively, where n and m count the number "V∧" [resp. "very,|||"] symbol strings. The $f^b,\ g^b$ are restricted to $T^n,\ T^m$, where $\nu = n, m > 0$. For example, $0 < x \leq n,\ f^b(x) = i(\text{very,|||})$ and $f^b(0) = b \in i[\mathrm{B}]$. For two comparable objects $[f],\ [g]$ define $[f] \leq_B [g]$ if $n \leq m$. Two nonempty sets $A,\ D \subset \mathbf{BP_0}$ [resp. $\mathbf{BP}$] have the property that $A \leq_B D$ if for each $[f] \in A$ there exists some $[g] \in D$ such that $[f] \leq_B [g]$. This is the *better than* pre-order and usually $[f] \leq_B [g]$ is stated as follows: "$[g]$ is better than $[f]$" or some similar expression. Actually, for the $\mathbf{BP_0}$ [resp. $\mathbf{BP}$], the "better than order" is a partial order and, in some cases, it is equivalent to the $\leq_\#$ order. Of course, $\leq_B$ and $\leq_\#$ are *-transferred to ${}^*\mathcal{M}$.

For each $\mathbf{b} \in \mathbf{B}$, let $C_b = \{x \mid (x \in \mathbf{BP}) \wedge (\mathbf{b} \leq_B x)\}$.

Theorem 4.4.1 *There exists a purely subtle* $c \in {}^*C_b$ *such that* $C_b\, {}^*\!\leq_B \{c\}$.

Proof. The sentence

$$\forall x(x \in \mathbb{N} \to \exists y(y \in T^x \wedge [y] \in C_b \wedge \forall z \forall w(z \in \mathbb{N} \wedge z \leq x \wedge w \in T^z \wedge$$

$$[w] \in C_b \to [w] \leq_B [y])) \tag{4.4.1}$$

holds in $\mathcal{M}$; hence, in ${}^*\mathcal{M}$.

Let $\nu \in {}^*\mathbb{N} - \mathbb{N}$, then there is a $f \in ({}^*T)^\nu$ and a purely subtle $c = [f] \in {}^*C_b$, where $[f]$ satisfies the remainder of the *-transformed (4.4.1) statement. Let $a \in C_b \subset {}^*C_b$. Then there is some $m \in \mathbb{N}$ and some $g \in T^m$ such that $a = [g]$. Thus, since $m \leq \nu$, then $[g]\, {}^*\!\leq_B [f]$. Consequenely, $C_b\, {}^*\!\leq_B \{c\}$.▮

The following is somewhat trivial and is not formalized as a theorem. (Herrmann, R. A. General Logic-Systems and Finite Consequence Operators, Logica Universalis, 1(2006):201-208 (Partial paper at http://arxiv.org/abs/math/05012559).) Consider the usual representation for $c = [f],\ f \in ({}^*T)^\nu,\ \nu \in {}^*\mathbb{N} - \mathbb{N}$. Intuitively, members of ${}^*C_b(\{c\})$, are obtained by removing *-finitely many (including 0) $i(\mathrm{very}, |||)$ from c. Let $n \in \mathbb{N}$. Then $\{x \mid (x \in {}^*\mathbb{N}) \wedge (n \leq x \leq \nu)\}$ is *-finite. By *-transfer of the appropriate sentence, you have the following for each $m \in \mathbb{N}$. If $m = 0$, then $[g] \in {}^*C_a(\{c\})$, where $g(0) = \mathbf{b}$. If $m \geq 1$, then $[g] \in {}^*C_a(\{c\})$, where $g(0) = \mathbf{b}$ and, for each $j \in \mathbb{N}$, such that $1 \leq j \leq m$, $g(j) = i(\mathrm{very}, |||)$. Thus, for ${}^*C_a(\{c\})$, which is simply a restriction of *-propositional deduction, one has that attribute b as well as all of the $\mathrm{very}, ||| \cdots \mathrm{very}, |||\mathrm{b}$ attributes are rationally related to c. When Theorem 4.4.1 is interpreted, then c is stronger than, better than, greater than, b or any of these standard strengthens of the basic b.

(Note: Adjective reasoning can also be determined by a general logic system. (Herrmann, R. A. General Logic-Systems and Finite Consequence Operators, Logica Universalis, 1(2006):201-208 (Partial paper at http://arxiv.org/abs/math/05012559).) Let $\mathrm{x} \in \mathrm{BP}$ have $\mathrm{n} > 0$ $\mathrm{very}, |||$ strings to the left of a $\mathrm{c} \in \mathrm{C_0}$ (page 24). Then a rule of inference $\mathrm{R_x}$ for x is constructed by reduction as follows: remove one $\mathrm{very}, |||$ from x. Write the result as $\mathrm{x_1}$. Then let $(\mathrm{x}, \mathrm{x_1}) \in \mathrm{R_x}$. Continue this finite reduction until $\mathrm{c} \in \mathrm{C_0}$ is obtained. Hence, the last member of $\mathrm{R_x}$ so constructed is (x, c). By definition, the set of all such finite binary relations $\mathrm{R_x}$ obtained for each such x yields a general logic-system. (This is not a unique construction.) From this system, the corresponding consequence operator $\mathrm{C_a}$ is obtained.)

NOTES