Chapter 5

CONSEQUENCE OPERATORS

5.1 Basic Definitions

Recall once again the Tarski [21] cardinality independent axioms for a *finite consequence operator* $\mathrm{C}\colon \mathcal{P}(\mathrm{A}) \to \mathcal{P}(\mathrm{A})$ on a nonempty set of meaningful sentences A.

$$\boxed{\begin{array}{l}(2)\ \text{If } \mathrm{B} \subset \mathrm{A},\ \text{then } \mathrm{B} \subset \mathrm{C}(\mathrm{B}) \subset \mathrm{A},\\[1ex] (3)\ \text{If } \mathrm{B} \subset \mathrm{A},\ \text{then } \mathrm{C}(\mathrm{C}(\mathrm{B})) = \mathrm{C}(\mathrm{B}),\\[1ex] (4)\ \text{If } \mathrm{B} \subset \mathrm{A},\ \text{then } \mathrm{C}(\mathrm{B}) = \bigcup\{\mathrm{C}(\mathrm{F}) \mid \mathrm{F} \in \mathrm{F}(\mathrm{B})\}\end{array}}$$

The modern theory of *consequence operators (the term finite dropped)* alters axiom (4) and replaces it by axiom

$$\boxed{(5).\ \text{If } \mathrm{B},\ \mathrm{D} \subset \mathrm{A},\ \text{if } \mathrm{B} \subset \mathrm{D} \text{ then } \mathrm{C}(\mathrm{B}) \subset \mathrm{C}(\mathrm{D}),}$$

It is very important to state that **axioms (2), (3) and (4) imply axiom (5).** Thus a finite consequence operator is a consequence operator, but not conversely. All things that can be established for consequence operators without any further axioms hold for finite consequence operators. For this reasons, some of the following results will be established for consequence operators in general. Of course, consequence operators need not be restricted only to objects that are considered to be language. In the theory here being developed, A can be of two types. Either $\mathrm{A} \subset \mathcal{W}$ or $\mathbf{A} \subset \mathcal{E}$. I shall, however, continue to use Roman notation for all of the objects related to $\mathcal{W}$ so as to differentiate them from the other mathematical entities. In all that follows, the symbol $\mathcal{C}'$ will denote the set of all consequence operators defined on some specified $\mathcal{P}(\mathrm{A})$ and the symbol $\mathcal{C}'_f$ the set of a finite consequence operators. Obviously, $\mathcal{C}'_f \subset \mathcal{C}'$, where, if no mention is made of any other possible, it will also be assumed that each member of $\mathcal{C}'$ is defined on the same $\mathcal{P}(\mathrm{A})$.

On the sets $\mathcal{C}'$ and $\mathcal{C}'_f$, we can define a significant partial order. For $\mathrm{C}_1,\ \mathrm{C}_2 \in \mathcal{C}'$ let $\mathrm{C}_1 \leq \mathrm{C}_2$ if $\mathrm{C}_1(\mathrm{X}) \subset \mathrm{C}_2(\mathrm{X})$ for each $\mathrm{X} \in \mathcal{P}(\mathrm{A})$. This partial order, I term *the stronger than order.* The partial order defined on $\mathcal{C}'_f$ is the restricted stronger than order.

A great deal has been discovered about algebras $\langle \mathcal{C}', \leq \rangle$ and $\langle \mathcal{C}'_f, \leq \rangle$. For example, one can define a compatible meet operation as follows: For each $\mathrm{C}_1,\ \mathrm{C}_2 \in \mathcal{C}'$ [resp. $\mathcal{C}'_f$], let the map $\mathrm{C}_1 \wedge \mathrm{C}_2 \colon \mathcal{P}(\mathrm{A}) \to \mathcal{P}(\mathrm{A})$ be defined by $(\mathrm{C}_1 \wedge \mathrm{C}_2)(\mathrm{X}) = \mathrm{C}_1(\mathrm{X}) \cap \mathrm{C}_2(\mathrm{X})$, where $\mathrm{X} \subset \mathrm{A}$. Each of these algebras has the same upper unit and the same lower unit. The lower unit is but the identity map on $\mathcal{P}(\mathrm{A})$. The upper unit U is the map defined by $\mathrm{U}(\mathrm{X}) = \mathrm{A}$ for each $\mathrm{X} \subset \mathrm{A}$. These algebras are both meet semi-lattices.

Our interest in the above two algebras is not in any deep investigation into there different properties but, rather, will be restricted two chains. In general, $\langle \mathcal{C}', \leq \rangle$ is not closed under composition. [24] However, for chains there is a very simple relation between the stronger than order and composition.

Theorem 5.1.1 *Let $\mathcal{D} \subset \mathcal{C}'$. Then $\mathcal{D}$ is a chain in $\langle \mathcal{C}', \leq \rangle$ iff for each $\mathrm{C}_1, \mathrm{C}_2 \in \mathcal{D}$ either the composition $\mathrm{C}_1\mathrm{C}_2 = \mathrm{C}_1$ or $\mathrm{C}_2\mathrm{C}_1 = \mathrm{C}_2$.*

Proof. For the necessity, assume that hypothesis. Suppose that $\mathrm{C}_1 \leq \mathrm{C}_2$. Then for each $\mathrm{X} \subset \mathrm{A}$, $\mathrm{X} \subset \mathrm{C}_1(\mathrm{X}) \subset \mathrm{C}_2(\mathrm{X})$ implies that $\mathrm{C}_2(\mathrm{X}) \subset \mathrm{C}_2(\mathrm{C}_1(\mathrm{X})) \subset \mathrm{C}_2(\mathrm{C}_2(\mathrm{X})) = \mathrm{C}_2(\mathrm{X})$. Hence $\mathrm{C}_2\mathrm{C}_1 = \mathrm{C}_2$. In like manner, if $\mathrm{C}_2 \leq \mathrm{C}_1$.

For the sufficiency, let $\mathrm{C}_2\mathrm{C}_1 = \mathrm{C}_2$. Then for each $\mathrm{X} \subset \mathrm{A}$, $\mathrm{C}_1(\mathrm{X}) \subset \mathrm{C}_2(\mathrm{C}_1(\mathrm{X})) = (\mathrm{C}_2\mathrm{C}_1)(\mathrm{X}) = \mathrm{C}_2(\mathrm{X})$ implies that $\mathrm{C}_1 \leq \mathrm{C}_2$. In like manner for $\mathrm{C}_1\mathrm{C}_2 = \mathrm{C}_1$ and this completes the proof. ∎

5.2 Basic σ Properties

Since consequence operators are relations between sets, it becomes more essential to incorporate, to a certain degree, the σ operator into much of our discussion. Since we wish to maintain symbolic consistency and avoid trivialities, assume that nonfinite $A \subset \mathcal{W}$. One important result that will be used many times without further elaboration, uses the finitary construction of the equivalence class $[f] \in \mathcal{E}$. From our previous discussion, a readable sentence $[f]$ behaves as follows: ${}^*[f] = [{}^*f] = [f]$ under the identification of the natural numbers. Now if $\emptyset \neq A \subset \mathcal{E}$, then ${}^\sigma A = \{{}^*[f] \mid [f] \in A\} = \{[f] \mid [f] \in A\} = A$. The following result brings together various facts relative to the σ operator all of which follow easily from the definitions and characterizing properties. The proofs will be omitted.

Theorem 5.2.1

(i) *Let* $A \in \mathcal{N}$. *Then* ${}^\sigma(F(A)) = F({}^\sigma A)$. *If also* $A \subset (\mathcal{W} \cup \mathcal{E}$, *then* ${}^\sigma(F(A)) = F(A)$.

(ii) *Let* $\mathrm{C} \in \mathcal{C}'$, $\mathrm{B} \subset \mathrm{X} \subset \mathcal{W}$.

(a) ${}^\sigma(\mathbf{C}(\mathbf{B})) = \mathbf{C}(\mathbf{B})$.

(b) ${}^*\mathbf{C} \bigm| \{{}^*\mathbf{A} \mid \mathbf{A} \in \mathcal{P}(X)\} = \{({}^*\mathbf{A}, {}^*\mathbf{B}) \mid (\mathbf{A}, \mathbf{B}) \in \mathbf{C}\} = {}^\sigma\mathbf{C}$.

(c) *If* $\mathrm{F} \in \mathrm{F}(\mathrm{B})$, *then* ${}^\sigma(\mathbf{C}(\mathbf{F})) \subset ({}^\sigma\mathbf{C})({}^\sigma\mathbf{F}) = ({}^\sigma\mathbf{C})(\mathbf{F})$. *Also* ${}^\sigma(\mathbf{C}(\mathbf{B})) \subset ({}^\sigma\mathbf{C})({}^*\mathbf{B})$ *and, in general,* ${}^\sigma(\mathbf{C}(\mathbf{B})) \neq ({}^\sigma\mathbf{C})({}^*\mathbf{B})$, ${}^\sigma(\mathbf{C}(\mathbf{F})) \neq ({}^\sigma\mathbf{C})(F)$.

(d) *If* $C \in \mathcal{C}'_f$, *then* ${}^\sigma(\mathbf{C}(\mathbf{B})) = \bigcup\{{}^\sigma(\mathbf{C}(\mathbf{F})) \mid \mathbf{F} \in {}^\sigma(F(\mathbf{B}))\} = \bigcup\{{}^\sigma(\mathbf{C}(\mathbf{F})) \mid \mathbf{F} \in F({}^\sigma\mathbf{B})\} = \bigcup\{\mathbf{C}(\mathbf{F}) \mid \mathbf{F} \in F(\mathbf{B})\}$.

(A duplicate theorem holds, where $A \in \mathcal{N}$ and $C \in \mathcal{C}$, where $\mathcal{C}$ set of consequence operators defined on subsets of W if W is included as a subset of the ground set. The difference is that the "bold" notion does not appear.)

Throughout the reminder of this section, in order to escape trivialities, we remove the upper unit U from the collections of consequence operators. Let $\mathcal{C} = \mathcal{C}' - \{\mathrm{U}\}$ and $\mathcal{C}_f = \mathcal{C}'_f - \{\mathrm{U}\}$. One of the consequences of this last requirement shows that if $D \in \mathcal{N}$ satisfies the axioms for a consequence operator and $G \subset \mathcal{C}$, then there does not exist a consequence operator $C \in {}^*\mathbf{G}$ such that ${}^\sigma\mathbf{D} = C$. To see this simply note that from Theorem 5.2.1 ${}^\sigma\mathbf{D}$ is defined on extended standard sets while each member of ${}^*\mathbf{G}$ is defined on the internal subsets of A. Since A is not finite, there exists internal subsets of A that are not equal to any extended standard set. **There is also one useful general fact.** Consider *any* sets $A, B \in \mathcal{N}$ such that $A \subset B$. Then ${}^*A - {}^\sigma A \subset {}^*B - {}^\sigma B$. For suppose that there exists some $X \in ({}^*A - {}^\sigma A)$ such that $X \in {}^\sigma B$. Then $X = {}^*D$ for some $D \in B$. Thus ${}^*D \in {}^*A$ implies that $D \in A$. From this, we have the contradiction that $X = {}^*D \in {}^\sigma A$. Also note that there does not exist $D \in \mathcal{N}$ such that ${}^*D \in ({}^*B - {}^\sigma B)$ (i.e. each member of $({}^*B - {}^\sigma B)$ is an internal pure nonstandard object or a pure subtle object.)

5.3 Major Results

For the algebras $\langle \mathcal{C}, \leq \rangle$ and $\langle \mathcal{C}_f, \leq \rangle$ two types of chains will be studied. Denote by K any nonempty chain contained in either of these algebras and by K_∞ a chain with the following property. For each $\mathrm{C} \in \mathrm{K}_\infty$ there exists $\mathrm{C}' \in \mathrm{K}_\infty$ such that $\mathrm{C} < \mathrm{C}'$.

Theorem 5.3.1 *There exists* $C_0 \in {}^*\mathbf{K}$ *such that for each* $\mathrm{C} \in \mathrm{K}$, ${}^*\mathbf{C} \leq C_0$. *There exists some* $C_\infty \in {}^*\mathbf{K}_\infty$ *such that* C_∞ *is a purely subtle consequence operator and for each* $C \in K_\infty$, ${}^*\mathbf{C} < C_\infty$. *Each member of* ${}^*\mathbf{K}$ *and* ${}^*\mathbf{K}_\infty$ *are subtle consequence operators.*

Proof. Let $R = \{(x,y) \mid (x \in \mathbf{K}) \wedge (y \in \mathbf{K}) \wedge (x \leq y)\}$ and $R_\infty = \{(x,y) \mid (x \in \mathbf{K}_\infty) \wedge (y \in \mathbf{K}_\infty) \wedge (x < y)\}$. In the usual manner, it follows that R is concurrent on $\mathbf{K}$ and R_∞ is concurrent on $\mathbf{K}_\infty$. Consequently, there is some $C_0 \in {}^*\mathbf{K}$ and some $C_\infty \in {}^*\mathbf{K}_\infty$ such that for each $\mathrm{C} \in \mathrm{K}$ and each $\mathrm{C}' \in \mathrm{K}$, ${}^*\mathbf{C} \leq \mathrm{C}_0$, ${}^*\mathbf{C}' < \mathrm{C}_\infty$ since ${}^*\mathcal{M}$ is an enlargement. Further, it follows that $C_\infty \in {}^*\mathbf{K}_\infty - {}^\sigma\mathbf{K}_\infty$ implies that C_∞ is a purely subtle consequence operator. Note that each member of ${}^*\mathbf{K} \cup {}^*\mathbf{K}_\infty$ is defined on the set of all internal subsets of ${}^*\mathbf{A}$. This completes our proof. ∎

Notice that C_∞ is stronger than or "more powerful than" any $\mathrm{C} \in \mathrm{K}_\infty$ in the following sense. If $\mathrm{B} \subset \mathrm{A}$, then for each $\mathrm{C} \in \mathrm{K}_\infty$, it follows that $\mathbf{C}(\mathbf{B}) \subset {}^*(\mathbf{C}(\mathbf{B})) = {}^*\mathbf{C}({}^*\mathbf{B}) \subset C_\infty({}^*\mathbf{B})$. Also for each $\mathrm{C} \in \mathrm{K}_\infty$ there exists some internal $E_C \subset {}^*\mathbf{A}$ and ${}^*\mathbf{C}(E_C) \underset{\neq}{\subset} C_\infty(E_C)$. Recall that for $\mathrm{C} \in \mathcal{C}$, a set $\mathrm{B} \subset \mathrm{A}$ is a *C-deductive system* if $\mathrm{C}(\mathrm{B}) = \mathrm{B}$. Also, when we write the $*$-operator on any map f in the form ${}^*f(x)$ this always means $({}^*f)(x)$ rather than ${}^*(f(x))$.

Theorem 5.3.2 *Let* $\mathrm{C} \in \mathcal{C}_f$ *and* $\mathrm{B} \subset \mathrm{A} \subset \mathcal{W}$. *Then there exists a* $*$*-finite* $F \in {}^*(F(\mathbf{B})) = {}^*F({}^*\mathbf{B})$ *such that* $\mathbf{C}(\mathbf{B}) \subset {}^*\mathbf{C}(F) \subset {}^*\mathbf{C}({}^*\mathbf{B}) = {}^*(\mathbf{C}(\mathbf{B}))$ *and* ${}^*\mathbf{C}(F) \cap \mathbf{A} = \mathbf{C}(\mathbf{B}) = {}^*\mathbf{C}(F) \cap \mathbf{C}(\mathbf{B})$.

Proof. Consider the binary relation $Q = \{(x,y) \mid (x \in \mathbf{C}(\mathbf{B})) \wedge (y \in F(\mathbf{B})) \wedge (x \in \mathbf{C}(y))\}$. By axiom (4), the domain of Q is $\mathbf{C}(\mathbf{B})$. Let $(x_1,y_1),\ldots,(x_n,y_n) \in Q$. By Theorem 1 in [5, p. 64] (i.e. axiom (5)) we have that $\mathbf{C}(y_1) \cup \cdots \cup \mathbf{C}(y_n) \subset \mathbf{C}(y_1 \cup \cdots \cup y_n)$. Since $F = y_1 \cup \cdots \cup y_n \in F(\mathbf{B})$, then $(x_1,F),\ldots,(x_n,F) \in Q$. Thus Q is concurrent on $\mathbf{C}(\mathbf{B})$. Hence there is some $F \in {}^*(F(\mathbf{B}))$ such that ${}^\sigma(\mathbf{C}(\mathbf{B})) = \mathbf{C}(\mathbf{B}) \subset {}^*\mathbf{C}(F) \subset {}^*\mathbf{C}({}^*\mathbf{B}) = {}^*(\mathbf{C}(\mathbf{B}))$. Since ${}^\sigma A = A$, ${}^*\mathbf{C}(F) \cap \mathbf{A} = \mathbf{C}(\mathbf{B}) = {}^*\mathbf{C}(F) \cap \mathbf{C}(\mathbf{B})$. ∎

Corollary 5.3.2.1 *If* $\mathrm{C} \in \mathcal{C}_f$ *and* $\mathrm{B} \subset \mathrm{A} \subset \mathcal{W}$ *is a C-deductive system, then there exists a* $*$*-finite* $F \subset {}^*\mathbf{B}$ *such that* ${}^*\mathbf{C}(F) \cap \mathbf{A} = \mathbf{B}$.

Corollary 5.3.2.2 *Let* $\mathrm{C} \in \mathcal{C}_f$. *Then there exists a* $*$*- finite* $F \subset {}^*\mathbf{A}$ *such that for each* $\mathrm{B} \subset \mathrm{A}$, ${}^*\mathbf{C}(\mathrm{F}) \cap \mathbf{B} = \mathbf{B}$.

Proof. In Theorem 5.3.2, let the "B" be equal to A. Then there exists some $*$-finite $F \subset {}^*\mathbf{A}$ such that ${}^*\mathbf{C}(F) \cap \mathbf{A} = \mathbf{C}(\mathbf{A}) = \mathbf{A}$. Thus ${}^*\mathbf{C}(F) \cap \mathbf{A} \cap \mathbf{B} = {}^*\mathbf{C}(F) \cap \mathbf{B} = \mathbf{A} \cap \mathbf{B} = \mathbf{B}$. ∎

Theorem 5.3.3 *Let* $\mathrm{B} \subset \mathrm{A} \subset \mathcal{W}$.

(i) *There exists a* *-*finite* $F_B \in {}^*(F(\mathbf{B}))$ *and a subtle consequence operator* $C_B \in {}^*\mathbf{K}$ *such that for all* $C \in K$, ${}^\sigma(\mathbf{C}(\mathbf{B})) = \mathbf{C}(\mathbf{B}) \subset C_B(F_B)$.

(ii) *There exists a purely subtle consequence operator* $C_B^\infty \in {}^*\mathbf{K}_\infty$ *such that for all* $\mathrm{C} \in \mathrm{K}_\infty$, ${}^\sigma(\mathbf{C}(\mathbf{B})) = \mathbf{C}(\mathbf{B}) \subset \mathrm{C}_{\mathrm{B}}^\infty(\mathrm{F}_{\mathrm{B}})$.

Proof. (i) Consider the binary relation $Q = \{((x,z),(y,w)) \mid (x \in \mathbf{K}) \wedge (y \in \mathbf{K}) \wedge (w \in F(\mathbf{B})) \wedge (z \in x(w)) \wedge (x(w) \subset y(w))\}$. Let nonempty $\{((x_1,z_1),(y_1,w_1)),\ldots,((x_n,z_n),(y_n,w_n))\} \subset Q$. Notice that $F = w_1 \cup \cdots \cup w_n \in F(\mathbf{B})$ and the set $R = \{x_1,\ldots,x_n\}$ has a largest member D with respect to the $\mathcal{E}$ embedded $\leq$ ordering for the consequence operators. It follows that $z_i \in x_i(w_i) \subset x_i(F) \subset D(F)$ for each $i = 1,\ldots,n$. Hence $\{((x_1,z_1),(D,F)),\ldots,((x_n,z_n),(D,F))\} \subset Q$ implies that Q is concurrent on its domain. Consequently, there exists some $(C_B,F_B) \in {}^*\mathbf{K} \times {}^*(F(\mathbf{B}))$ such that for each $(x,z) \in$ domain of Q, $({}^*(x,z),(C_B,F_B)) \in {}^*Q$. Therefore, each $(u,v) \in {}^\sigma$(domain of Q), $((u,v),(C_B,F_B)) \in {}^*Q$. Let arbitrary $C \in K$ and $\mathrm{b} \in \mathrm{C}(\mathrm{B})$. Then there exists some $F' \in F(\mathrm{B})$ such that $\mathrm{b} \in \mathrm{C}(\mathrm{F}')$. Thus $({}^*\mathbf{C}, {}^*\mathbf{b}) \in {}^\sigma$(domain of Q). Consequently, for each $C \in K$ and $\mathrm{b} \in \mathrm{C}(\mathrm{B})$, $\mathbf{b} = {}^*\mathbf{b} \in {}^*\mathbf{C}(\mathrm{F_B}) \subset \mathrm{C_B}(\mathrm{F_B})$. This all implies that for each $C \in K$, ${}^\sigma(\mathbf{C}(\mathbf{B})) = \mathbf{C}(\mathbf{B}) \subset C_B(F_B)$.

(ii) Change the relation Q to Q' be adding the additional requirement to Q that $x \neq y$. Replace the D in (i) with any D' that is greater than and **not equal** to the largest member of R. Such a D' exists in $\mathbf{K}_\infty$ from the definition of K_∞. Continue the proof in the same manner as in (i) to obtain C_B^∞ and F_B. The fact that C_B^∞ is a purely subtle consequence operator follows as in the proof of Theorem 5.3. ∎

Corollary 5.3.3.1 *There exists a* [*resp. purely*] *subtle consequence operator* $C_A \in {}^*\mathbf{K}$ [*resp.* ${}^*\mathbf{K}_\infty$] *and a* *-*finite* $F_A \in {}^*(F(\mathbf{A}))$ *such that for all* $\mathrm{C} \in \mathrm{K}$ [*resp.* K_∞] *and each* $\mathrm{B} \subset \mathrm{A}$, $\mathbf{B} \subset \mathbf{C}(\mathbf{B}) \subset \mathrm{C_A}(\mathrm{F_A})$.

Proof. Simply let the "B" in Theorem 5.3.3 be equal to A. Then there exists a [resp. purely] subtle $C_A \in {}^*\mathbf{K}$ [resp. ${}^*\mathbf{K}_\infty$] such that for all $\mathrm{C} \in \mathrm{K}$ [resp. K_∞], $\mathbf{C}(\mathbf{A}) \subset C_A(F_A)$. If $\mathrm{B} \subset \mathrm{A}$ and $C \in K$ [resp. K_∞], then $\mathrm{B} \subset \mathrm{C}(\mathrm{B}) \subset \mathrm{C}(\mathrm{A})$. Thus for each $\mathrm{B} \subset \mathrm{A}$ and $\mathrm{C} \in \mathrm{K}$ [resp. K_∞], $\mathbf{B} \subset \mathbf{C}(\mathbf{B}) \subset C_A(F_A)$ and this completes the proof. ∎

Relative to the above results, it is well known that for $\mathbf{B} \subset \mathbf{A}$ that there exists a *-finite $F_1 \subset {}^*\mathbf{B}$ such that $\mathbf{B} \subset F_1 \subset {}^*\mathbf{B}$. Thus for any $\mathrm{C} \in \mathcal{C}$ it follows that $\mathbf{B} \subset \mathbf{C}(\mathbf{B}) \subset {}^*\mathbf{C}(F_1) \subset {}^*\mathbf{C}({}^*\mathbf{B})$. One significance of the above results is that the C_B^∞ is purely subtle and, thus, not the same as any extended standard consequence operator.

5.4 Applications

In what follows, let denumerable L be a language constructed from a denumerable set of primitive symbols $\{\mathrm{P_i} \mid \mathrm{i} \in \omega\}$. As to the construction of L it is, at least, constructed from the binary operation $\to$. Deduction over L is defined in the usual sense. Only finitely many steps are allowed, and if any axiom schema are used, then they do not yield statements of the form $\mathrm{P_i}$ or $\mathrm{P_i} \to \mathrm{P_j},\ \mathrm{i} \neq \mathrm{j}$. Further, deduction from premises is also allowed. There are many examples of such languages. Propositional languages with denumerably many atoms. Indeed, in a predicate language with, at least, one predicate the list of all predicates can be considered the set of primitives from which L is constructed. Of course, simple natural languages are isomorphic to L in the usual sense. There will be one modification, however. The modification is in a rule of inference. Define the $\mathrm{MP_n},\ \mathrm{n} \in \omega$ rule of inference on L as follows:

> If two previous steps of a demonstration (or proof) are of the form $\mathrm{A},\ \mathrm{A} \to \mathrm{B}$ where for each $\mathrm{P_i}$ in the primitive expansions of $\mathrm{A},\ \mathrm{A} \to \mathrm{B},\ \mathrm{i} \leq \mathrm{n}$, then the formula B may be written down as the next step. No other type of MP rule is used.

Given a set of hypotheses $\mathcal{H} \subset \mathrm{L}$ and $\mathrm{X} \in \mathrm{L}$, denote by the symbol $\mathcal{H} \vdash_{\mathrm{n}} \mathrm{X}$ this deductive process. It is immediate that $\vdash_{\mathrm{n}}$ determines a finitary consequence operator $\mathrm{C_n}$ on $\mathcal{P}(\mathrm{L})$. Suppose that $\vdash$ deduction on $\mathcal{P}(\mathrm{L})$ has all of the above properties with the exception that the MP rule of inference is the ordinary modus ponens in unrestricted form. Let $\mathrm{S_0}$ denote the consequence operator determined by $\vdash$.

It is a simple matter to show that for any $\mathrm{B} \subset \mathrm{L},\ \mathrm{S_0(B)} = \bigcup\{\mathrm{C_n(B)} \mid \mathrm{n} \in \omega\}$. Let $\mathrm{B} \subset \mathrm{L}$. Suppose that $\mathrm{X} \in \mathrm{S_0(B)}$. Then $\mathrm{B} \vdash \mathrm{X}$. Now in the formal proof of this fact when all of the formula are written in primitive form there is a maximum $\mathrm{P_i}$ subscript, say $\mathrm{m} \in \omega$. It follows immediately that the same steps yield a formal proof that $\mathrm{B} \vdash_{\mathrm{m}} \mathrm{X}$ using the $\mathrm{MP_m}$ in place of any MP step that appears in the $\vdash$ formal proof. Thus $\mathrm{X} \in \bigcup\{\mathrm{C_n(B)} \mid \mathrm{n} \in \omega\}$. Clearly, if $\mathrm{B} \vdash_{\mathrm{n}} \mathrm{X}$, then $\mathrm{B} \vdash \mathrm{X}$. This implies $\mathrm{S_0(B)} = \bigcup\{\mathrm{C_n(B)} \mid \mathrm{n} \in \omega\}$. Another interesting result is that if B is an $\mathrm{S_0}$-deductive system, then $\mathrm{S_0(B)} = \bigcup\{\mathrm{C_n(B)} \mid \mathrm{n} \in \omega\} = \mathrm{B}$ implies that for each $\mathrm{n} \in \omega,\ \mathrm{C_n(B)} = \mathrm{B}$. Thus B is also a $\mathrm{C_n}$-deductive system.

Let $\mathrm{n},\ \mathrm{m} \in \omega,\ \mathrm{n} \leq \mathrm{m},\ \mathrm{B} \subset \mathrm{L}$. Suppose that $\mathrm{X} \in \mathrm{C_n(B)}$. If there are any $\mathrm{MP_n}$ steps in the formal proof, then these steps can also be obtained by application of $\mathrm{MP_m}$. On the other hand, if no steps were obtained by the $\mathrm{MP_n}$ rule, then the exact same steps yield a formal proof that $\mathrm{B} \vdash_{\mathrm{m}} \mathrm{X}$. From this we have that for each $\mathrm{B} \subset \mathrm{L}$, $\mathrm{C_n(B)} \subset \mathrm{C_m(B)}$. Hence, $\mathrm{C_n} \leq \mathrm{C_m}$. Therefore, $\{\mathrm{C_n} \mid \mathrm{n} \in \omega\}$ is a chain of consequence operators.

Now to show that this chain is of type K_∞. Let $\mathrm{n} < \mathrm{m}$ and let $\mathrm{B} = \{\mathrm{P_n}, \mathrm{P_n} \to \mathrm{P_m}\}$. First, no member of B can be obtained as an instance of an axiom. Further, it cannot be the case that $\mathrm{B} \vdash_{\mathrm{n}} \mathrm{P_m}$ for $\mathrm{MP_n}$ does not apply to $\mathrm{P_n} \to \mathrm{P_m}$ or, indeed, any formula containing $\mathrm{P_m}$. Therefore, $\mathrm{P_m} \notin \mathrm{C_n(B)}$. Obviously, $\mathrm{P_m} \in \mathrm{C_m(B)}$. Hence, $\mathrm{C_n(B)} \neq \mathrm{C_m(B)}$ implies that $\mathrm{C_n} < \mathrm{C_m}$. Thus this chain is of type K_∞. Further, note that $\mathrm{P_m} \in \mathrm{S_0(B)}$. Thus for each $\mathrm{C} \in \{\mathrm{C_n} \mid \mathrm{n} \in \omega\}$ there exists some $\mathrm{B} \subset \mathrm{L}$ such that $\mathrm{C(B)} \neq \mathrm{S_0(B)}$ and, clearly $\mathrm{C} \leq \mathrm{S_0}$. Hence, in general, $\mathrm{C} < \mathrm{S_0}$ for all $\mathrm{C} \in \{\mathrm{C_n} \mid \mathrm{n} \in \omega\}$.

Theorem 5.4.1 *Let* L *and* $\mathrm{K}_\infty = \{\mathrm{C_n} \mid \mathrm{n} \in \omega\}$ *be defined as above. Then there exists a purely subtle consequence operator* $C_\infty \in {}^*\mathbf{K}_\infty$ *and a *-finite* $F \in {}^*(F(\mathbf{L}))$ *such that for each* $\mathrm{B} \subset \mathrm{L}$ *and each* $\mathrm{C} \in \mathrm{K}_\infty$

(i) $\mathbf{C}(\mathbf{B}) \subset C_\infty(F)$,

(ii) $\mathbf{S_0}(\mathbf{B}) \subset C_\infty(F) \subset {}^*\mathbf{S_0}(F) \subset {}^*\mathbf{S_0}({}^*\mathbf{L})$,

(iii) ${}^*\mathbf{S_0}(F) \cap \mathbf{L} = C_\infty(F) \cap \mathbf{L}$.

Proof. (i) is but Corollary 5.3.3.1. From (i), it follows that $\bigcup\{{}^\sigma(\mathbf{C}(\mathbf{B})) \mid C \in K_\infty\} = \bigcup\{(\mathbf{C}(\mathbf{B})) \mid C \in K_\infty\} = \mathbf{S_0}(\mathbf{B}) = {}^\sigma(\mathbf{S_0}(\mathbf{B})) \subset C_\infty(F)$ and the first part of (ii) holds. By *-transfer $C_\infty < {}^*\mathbf{S_0}$ and C_∞ and ${}^*\mathbf{S_0}$ are defined on all internal subsets of ${}^*\mathbf{L}$. Hence, $C_\infty(F) \subset {}^*\mathbf{S_0}(F) \subset {}^*\mathbf{S_0}({}^*\mathbf{L})$ and this completes (ii). (iii) follows immediately from (ii) and this completes the proof. ▮

For this application, let L be a predicate type language and M any set-theoretic structure in which the predicates and constants are interpreted in the usual manner. A finite consequence operator defined on $\mathcal{P}(\mathrm{L})$ is *sound* for M if whenever $\mathrm{B} \in \mathcal{P}(\mathrm{L})$ has the property that $\mathrm{M} \models \mathrm{B}$, then $\mathrm{M} \models \mathrm{C(B)}$. As usual, $\mathrm{T(M)} = \{\mathrm{x} \mid (\mathrm{x} \in \mathrm{L}) \wedge (\mathrm{M} \models \mathrm{x})\}$. Obviously, if C is sound for M, then T(M) is a C-deductive system.

Corollary 5.3.2.2 implies that there exists *-finite $F \in {}^*(\mathbf{T}(\mathbf{M}))$ such that ${}^*\mathbf{C}(F) \cap \mathbf{L} = \mathbf{T}(\mathbf{M})$. Notice that F being *-finite implies that F is *-recursive. Moreover, F is a *-axiom system for ${}^*\mathbf{C}(F)$ and we do not lack knowledge about the behavior of F since any formal property about C or recursive sets, among others, must hold true for ${}^*\mathbf{C}$ or F when properly interpreted. If L is a first-order language with at least one predicate, then its associated consequence operator S_1 is sound for first-order structures. Theorem 5.4.1 not only yields a *-finite F_1 but a purely subtle consequence operator C_1 such that F_1 is a *-axiom system for $C_1(F_1)$ and ${}^*\mathbf{S_1}(F_1)$. In this case, we have that ${}^*\mathbf{S_1}(F_1) \cap \mathbf{L} = \mathbf{T}(\mathbf{M}) = C_1(F_1) \cap \mathbf{L}$. As strange as it may appear, by use of internal and external objects, the nonstandard logics $\{{}^*\mathbf{C}, {}^*\mathbf{L}\}$, $\{C_1, {}^*\mathbf{L}\}$, $\{{}^*\mathbf{S_1}, {}^*\mathbf{L}\}$ technically bypass a portion of Gödel's first incompleteness theorem. Of course, this incompleteness theorem still holds under an internal interpretation.

By definition $\mathrm{b} \in \mathrm{S}_0(\mathrm{B}), \mathrm{B} \subset \mathrm{L}$ iff there is a finite length proof of b from the premises B. Thus for each $b \in {}^*(\mathbf{T}(\mathbf{M}))$ there exists a *-finite length proof of b from the *-finite F_1. If we let ${}^*\mathcal{M}$ be an enlargement with the $\aleph_1$-isomorphism property, among others, then each *-finite length proof is either externally finite or externally infinite. Further, all externally infinite proof lengths would be of the same cardinality.

{Remark: Using the customary notation in this chapter, the relation $\leq$ has not been starred in ${}^*\mathcal{N}$. If this omission is confusing, the * can be easily inserted. When these two different order relations are compared, the * notation becomes necessary. For example, the relation ${}^*\leq$ in ${}^*\mathcal{N}$ is NOT an extension, in the usual sense, of the relation $\leq$ as defined in $\mathcal{N}$, although it is an extension of ${}^\sigma\leq$. Also notice that if we had restricted our attention to $\mathcal{C}_f$, then the partial order $\leq$ is characterized totally by the finite subsets of A. This is useful since that ${}^*\leq$ is characterized by the *-finite subsets of *A. It's clear that our concept of a consequence-type operator must be generalized slightly. Let $\mathcal{B}$ and $\mathcal{B}_0$ be two families of sets. Then if $f\colon \mathcal{B} \to \mathcal{B}_0$ satisfies axioms (2)(3)(4) or (2)(3)(5) or the *-transform of these axiom systems, then f is a subtle consequence operator. I also point out that, unfortunately, there are many typographical errors in reference [24].

Chapter 6

ASSOCIATED MATERIAL

6.1 Perception

In this section, the theory of ultralogics is applied to one aspect of *subliminal perception.* What is needed is an interpretation scheme. When subsets of ${}^*\mathcal{E}$ are concerned the *conscious objects* are subsets [resp. elements of] ${}^\sigma\mathcal{E} = \mathcal{E}$. The *subconscious objects* are nonstandard internal subsets [resp. elements of] ${}^*\mathcal{E}$. Moreover, subconscious objects can contain conscious objects and the union of a subconscious set and a finite conscious set is a subconscious set. The *unconscious objects* are external nonstandard subsets of ${}^*\mathcal{E}$. Like definitions apply to members of ${}^*\mathcal{E} \times {}^*\mathcal{E}$ and so forth. In what follows, only strong reasoning from the perfect is considered. You may assume that it is defined on a natural language "very,|||" or a formal language "V$\wedge$" and the like.

As to some sort of interpretation procedure the following seems adequate. Let $\lceil\ \rceil$ denote an interpretation symbol. First, we have *subperception and the better than ordering.* Let $A \subset {}^*\mathcal{E}$ be one of the above defined objects in the domain of ${}^*\mathbf{\Pi}$. Let internal $D \subset {}^*\mathbf{\Pi}(A)$ and standard ${}^\sigma E \subset {}^*\mathbf{\Pi}(A)$. Assume that ${}^\sigma E \leq_B D$ and that each member of $i^{-1}[E]$ is a sentence which is distinctly comparable by the "very,|||" symbol string. [Note I am not differentiating between the object and a constant representing that object.] One might interpret the following: $\lceil \forall x((x \in {}^\sigma E) \to \exists y((y \in D) \wedge (x \leq_B y)))\rceil$: "You are (I am, we are, etc) subperceptibly aware that for each conscious (known) object (element, member) of (in) $\lceil {}^\sigma E\rceil$ there exists an object (element, member) of (in) $\lceil D\rceil$ which is better than that conscious object of (in) $\lceil {}^\sigma E\rceil$. "

Let $a \in {}^\sigma E$. Then: $\lceil \exists y((y \in D) \wedge (a \leq_B y))\rceil$: "You are (I am, we are) subperceptibly aware that there exists a conscious (known) object (element, member) of (in) $\lceil D\rceil$ which is better than $\lceil a\rceil$." Note that $a = [f_0] \in \mathbf{BP}$ and $\lceil a\rceil = i^{-1}(f_0(0))$. Another example is $\lceil {}^\sigma E \leq_B D\rceil$: "You are (I am, we are) subperceptibly aware that $\lceil D\rceil$ is better than $\lceil {}^\sigma E\rceil$. "

For another example, let ${}^\sigma F \subset {}^*\mathbf{\Pi}(A)$. Then $\lceil (|{}^\sigma E| < |{}^\sigma F|) \wedge ({}^\sigma E \leq_B D) \wedge ({}^\sigma F \leq_B D)\rceil$: "The result that $\lceil D\rceil$ is better than $\lceil {}^\sigma F\rceil$ is a stronger subperceptible property than $\lceil D\rceil$ is better than $\lceil {}^\sigma E\rceil$."

We also have the idea of *general subperception.* In this case, we use some of the meaningful set-theoretic terminology. Let internal nonstandard $A,\ B \subset {}^*\mathbf{BP}$. Now any elementary set-theoretic relation existing between A and B can be subperceptibly interpreted as $\lceil A \subset B\rceil$: "You are (I am, we are) subperceptible aware of the following: $\lceil A\rceil$ is contained in $\lceil B\rceil$. " Also you might interpret relations between standard objects as a complete awareness.

6.2 Existence

Some philosophers of science differentiate between theoretical entities and those that are assumed to exist in objective reality. In the original work in ultralogics, these two concepts were disjointly modeled. This was done as follows: Consider ${}^\sigma[f_0] = [f_0] \in {}^*\mathcal{E}$ to be the unique partial sequence with the property that $i^{-1}(f_0(0)) =$ externally|||exists|||in|||reality $= \lceil [f_0]\rceil$. For $A \subset {}^*\mathbf{BP}$, let $(A)_R = \{(x, [f_0]) \mid x \in A\}$. Then for any $E \subset {}^*\mathbf{BP}$, define $RR(E, A) = \{(x, y) \mid (x \in E) \land (y \in (A)_R\}$ to be the *realism relation.* These definitions are then extended to ${}^*\mathbf{BP} \times {}^*\mathbf{BP}$ in such a manner that $(A)_R \times (B)_R$ is considered to be isomorphic to $(A \times B)_R$. I now believe that this is a waste of effort. The difference lies in the interpretation and not in the mathematical structure. Thus, under the interpretation, if one wishes to differentiate between these two concepts, one simply includes "existence in objective reality" as a part of the interpretation for some entities and the statement "theoretical entities" for other distinct entities.

6.3 An Alternate Approach

What is presented in this section is mainly of historical interest although this author's first research into nonstandard analysis used this alternate approach. This approach utilizes a pseudo-set theory and has essentially been replaced by the superstructure approach. Some years ago, certain applications employ this alternate approach due to its use of a basic language that is somewhat more expressive than the $\in, =$ language. However, what might be gained in an additional freedom of expression will lead to a more complex array of extensions, definitions and the requirement that extreme care be exercised.

All of our constructions are within **ZFC**. We utilize the *transitive closure operator,* denoted by TC. Let V be a set. (Note: This definition also applies to atoms and sets containing atoms.) The transitive closure of V is obtained by an inductive construction using the union operation. Let $V_0 = V$ and for each $i \in \mathbb{N}$, let $V_{i+1} = \cup V_i$. Then the transitive set $TC(V) = \bigcup\{V_i \mid i \in \mathbb{N}\}$. The set $TC(V)$ for a set V has the property that if W is another transitive set such that $V \subset W$, then $V \subset TC(V) \subset W$. Define the *superstructure operator*, denoted by SS, on V as $SS(V) = \bigcup\{U_i \mid i \in \mathbb{N}\}$, where

$U_0 = V,\ U_{i+1} = U_i \cup \mathcal{P}(U_i), i \in \mathbb{N}$. Recall that this is the first type of superstructure defined in Chapter 2. To correspond to our previous investigation, let $V = \mathcal{W} \cup \mathbb{N}$ and $\mathcal{N}_1 = SS(TC(V))$. (If V is a set of individuals $TC(V) = V$.) Let the structure $\mathcal{M}_1 = \langle \mathcal{N}_1, \in, =, \mathrm{ap}, \mathrm{pr}\rangle$, where $\in,\ =$ are the usual set-theoretic membership and set equality relations restricted to $\mathcal{N}_1$ and ap, pr are two ternary relations, the "applying a function to its argument" and "ordered pair creation", respectively. (Of course, $=$, ap, pr can all be defined in terms of $\in$.) Notice that $\mathcal{M}_1$ is a fragment of our **ZFC** model.

Consider a κ-adequate ultrafilter, where $\kappa > |\mathcal{N}_1|$. By Theorem 7.5.2 in [19] or 1.5.1 in [9], such an ultrafilter $\mathcal{U}$ exists in our **ZFC** model and is determined by the indexing set $J = F(\mathcal{P}(\kappa))$. By the ultrapower or ultralimit construction, a first-order structure $\mathcal{M}_2 = (\mathcal{N}^J, \in_{\mathcal{U}}, =_{\mathcal{U}}, \mathrm{ap}_{\mathcal{U}}, \mathrm{pr}_{\mathcal{U}})$ is obtained of the same type as is $\mathcal{M}_1$ but $\mathcal{M}_2$ is a nonstandard model for the set of all sentences, K_0, in our first-order language L, with predicates $\in, =$, ap, pr which hold in $\mathcal{M}_1$. (Theorem 3.8.3 in [19]) Note that the cardinality of the set of constants in L $\geq |\mathcal{M}_1|$. Further, the members in $\mathcal{N}^J$ are interpreted by constants in an extended language L$'$. By the axioms of our **ZFC** set-theory, the relation "=" is an equivalence relation with substitution for $\in$, ap, pr and, hence, $=_{\mathcal{U}}$ has these properties for $\in_{\mathcal{U}}, \mathrm{ap}_{\mathcal{U}}, \mathrm{pr}_{\mathcal{U}}$. Consequently, we shift the structure $\mathcal{M}_2$ (i. e. contract it) [13, p. 83] and obtain a structure ${}'\mathcal{M} = \langle {}'\mathcal{N}_1, \epsilon, =, {}'\mathrm{ap}, {}'\mathrm{pr}\rangle$, where $=$ is the original equality in our **ZFC** model. Note that members of ${}'\mathcal{N}$ are still interpreted by constants in L$'$ as before. [In [4] and [11], a structure isomorphic to ${}'\mathcal{M}$ is obtained by application of the compactness theorem for a first-order language.]

We next isomorphicly embed $\mathcal{M}_1$ into ${}'\mathcal{M}$ in the same manner as outlined in [4, p. 22]. However, please notice that the following notation differs from that used in this reference. First, let I be the the original interpretation map from L onto $\mathcal{M}_1$ and let ${}'I$ by the composition of the extended ultrapower interpretation map and the contraction interpretation map restricted to the constants in L.

Now for each "a" that is a constant in L, define ${}^\sigma(I(a)) = {}'I(a) \in {}'\mathcal{N}$. Assume that a, b are constants in L that represent the same element and consider the well-formed formula (a = b). Then $\mathcal{M}_1 \models (\mathrm{a} = \mathrm{b})$ iff ${}'\mathcal{M} \models (\mathrm{a} = \mathrm{b})$ implies that ${}^\sigma(I(a)) = {}^\sigma(I(b))$. Thus the map σ is well-defined. Again let a, b be constants in L. Then $\mathcal{M}_1 \models \neg(\mathrm{a} = \mathrm{b})$ iff ${}'\mathcal{M} \models \neg(\mathrm{a} = \mathrm{b})$ implies that ${}^\sigma(I(\mathrm{a})) \neq {}^\sigma(\mathrm{I(b)})$ iff $I(\mathrm{a}) \neq \mathrm{I(b)}$. Thus σ is injective. It is immediately clear that $\mathcal{M}_1 \models (\mathrm{a} \in \mathrm{b})$ iff ${}'\mathcal{M} \models (\mathrm{a} \in \mathrm{b})$ implies that $I(\mathrm{a}) \in \mathrm{I(b)}$ iff ${}^\sigma(I(\mathrm{a}))\ \epsilon\ {}^\sigma(\mathrm{I(b)})$ and, in like manner, for the relations "ap" and "pr". Consequently,

σ is an isomorphic embedding of $\mathcal{M}_1$ into $'\mathcal{M}$. For convenience in all that follows, we suppress the interpretation map notation and simply use the constants of the language L (and the extended language L$'$) to represent members in $\mathcal{N} = \{x \mid \exists \mathrm{a}(\mathrm{a} \in \mathrm{L}) \wedge ({}^{\sigma}(\mathrm{I}(\mathrm{a})) = {}'\mathrm{I}(\mathrm{a})) \wedge (\mathrm{x} = {}'\mathrm{I}(\mathrm{a}))\}$. Of course, $\mathcal{N} \subset {}'\mathcal{N}$ and $\mathcal{M}_1$ is isomorphic to $\mathcal{M} = \langle \mathcal{N}, (\epsilon)/\sigma, =, ({}'\mathrm{ap})/\sigma, ({}'\mathrm{pr})/\sigma \rangle$. [Note: $(\epsilon)/\sigma$ is the relation restricted to members of $\mathcal{N}$, etc.]

Now to continue this construction. First, by Theorem 1.5.2 in [9], $'\mathcal{M}$ is an enlargement. For each $p \in {}'\mathcal{N}$, let ${}^*p = \{x \mid (x \in {}'\mathcal{N}) \wedge (x \,\epsilon\, p)\}$ and ${}^*\mathcal{N} = \{{}^*p \mid p \in {}'\mathcal{N}\}$. Notationally, let ${}^{**}\mathcal{N} = {}'\mathcal{N} \cup \mathcal{N}_1 \cup {}^*\mathcal{N}$ and define ${}^0\mathcal{N} = SS(TC({}^{**}\mathcal{N}))$. Obviously, $\mathcal{N} \cup \mathcal{N}_1 \cup {}'\mathcal{N} \cup {}^*\mathcal{N} \cup {}^{**}\mathcal{N} \subset TC({}^{**}\mathcal{N})$. Since $TC({}^{**}\mathcal{N})$ is a member of $SS(TC({}^{**}\mathcal{N}))$, then this implies, by Theorem 2.6 (iii) [4], that $\mathcal{N}, \mathcal{N}_1, {}'\mathcal{N}, {}^*\mathcal{N}, {}^{**}\mathcal{N} \in {}^0\mathcal{N}$. Finally, applying Theorem 2.6 (ii) and 2.10 in [4] to the transitive set ${}^0\mathcal{N}$, one obtains that ${}^0\mathcal{N}$ is closed under finite power set iteration and finite Cartesian products. The structure ${}^0\mathcal{M} = \langle {}^0\mathcal{N}, \in, =, \mathrm{ap}, \mathrm{pr} \rangle$ that is a fragment of our **ZFC** model is the G-structure in this alternative approach.

The model ${}^0\mathcal{M}$ contains all of the set-theoretic objects needed for this investigation. The fact that we are only interested in semantic consistency allows us to consider all of the structure $\mathcal{M}$ as the standard model in which sentences from L are interpreted and $'\mathcal{M}$ the nonstandard model for sentences from L. Of course, we can always return to $\mathcal{M}_1$ by application of σ^{-1}.

There is one important notational convention that is continually employed. The σ map is suppressed when considering members of $\mathcal{N}$. That is to say that for each constant $\mathrm{a} \in \mathrm{L}$, ${}^{\sigma}(I(\mathrm{a})) = {}'\mathrm{I}(\mathrm{a}) = \mathrm{a}$. The use of $\mathcal{M}$ can be made more efficient since there should be no great difficulty if you consider $(\epsilon)/\sigma, ({}'\mathrm{pr})/\sigma, ({}'\mathrm{ap})/\sigma$ to be the same as the **ZFC** model relations $\in, \mathrm{ap}, \mathrm{pr}$ for there is no first-order differences between these structures.

The fact that we are actually working with the restricted ϵ, $'\mathrm{ap}$, $'\mathrm{pr}$ can be determined by the additional result that the only objects to which the restriction of these relations apply are members of $\mathcal{N}$. The only other objects to which nonrestricted ϵ, $'\mathrm{ap}$, $'\mathrm{pr}$ apply are elements of $'\mathcal{N}$ that are not members of $\mathcal{N}$. The actual $\in, \mathrm{ap}, \mathrm{pr}$ are used in all other contexts such as the following important definition as previously stated. For each $A \in {}'\mathcal{N}$, ${}^*A = \{(x \in {}'\mathcal{N}) \wedge (x \,\epsilon\, A)\}$. This is the beginning of certain technical features for this model. What is significant as we define some of these technical terms is that all of the objects within this and other nonstandard investigations are set-theoretic members of ${}^0\mathcal{M}$ and, of course, ${}^0\mathcal{M}$ is in our **ZFC** model.

Rather than force the reader to seek out references [4] or [11], I reproduce here the more significant definitions required to relate many of our results to the $\in, \mathrm{ap}, \mathrm{pr}$ operations within the structure ${}^0\mathcal{M}$. Let $p \in\ {}'\mathcal{N}$. If $p \notin \mathcal{N}$, then p is called a *nonstandard object or entity.* If $p \in \mathcal{N}$, then p is called a *standard object.* If $S \subset {}^*P$ and there exists some $Q \in\ {}'\mathcal{N}$ such that $S = {}^*Q$, then S is called an *internal object or set.* Observe that $P \in {}'\mathcal{N}$ that is not a ${}'$atom is an internal subset of itself. Also it is often the case that each element of ${}^*P$ is called *internal* for if $P \in {}'\mathcal{N}$, then there exists some X_n such that $P \epsilon X_n$ and X_n is ${}'$-transitive. Thus if $p \epsilon P$, then $p \epsilon X_n$ implies that $p \in\ {}'\mathcal{N}$. Intuitively internal means that there exists a symbolic name in L$'$ for the object that generates, under the given definitions, the second corresponding object.

This generation of the second corresponding object is of a special nature. Let $f \in {}'\mathcal{N}$ be an ${}'$n-ary relation where $n > 1$. Thus f satisfies in L$'$ the appropriate sentence that defines such a object. Extend f in the following manner. Let $f^\star = \{(a_1, \ldots, a_n) \mid (a_1 \in {}'\mathcal{N}) \wedge \cdots \wedge (a_n \in {}'\mathcal{N}) \wedge ((a_1{}'\mathrm{pr} \cdots {}'\mathrm{pr}\ a_n)\ \epsilon\ f)\}$. In general, ${}^*f \neq f^\star$. For the many properties associated with this definition, refer to references [4] [11]. I note that in [4] one of the important properties for such an extension of f relative to the i'th projection [Theorem 4.5 (vii)] is stated on one side of the equation incorrectly. However, the proof goes through correctly and one should correct the statement of that small portion of the theorem to show that the i'th projection of the n-ary relation $f^\star = {}^*$(of the i'th projection of f). [Note: there are two theorems in [4] that are proved incorrectly, even though the theorem statement is correct. The proofs were corrected when these results were published.] Any n-ary relation that is produced by an extension that has the * on the right is called *an internal n-ary relation.* Notice that what this actually means is that there is a name for the ${}'$n-ary relation in the extended language L$'$.

Our major interest and application for this model will be confined to objects in $\mathcal{E}$ as well as in ${}^*\mathcal{E} - \mathcal{E}$, and a fixed power set iteration or Cartesian products of these objects. The use of the *-ing process is different in this model than it is in the model utilized in the previous sections of this chapter and previous chapters of this book. For example, it is important to realize that the set $f \in \mathcal{E}$ is a finite set of functions. Thus $\forall x(x \in [f] \leftrightarrow (x = a_1) \vee \ldots \vee (x = a_n))$ holds in $\mathcal{M}$. Therefore, $\forall x(x \in {}^*[f] \leftrightarrow (x = a_1) \vee \ldots \vee (x = a_n))$ implies that ${}^*[f] = [f]$ and $[f]$ is internal. Observe that the symbols $a_1, \ldots, a_n$ do not carry the * notation as would be necessary, prior to our identification process, in the previous sections of this chapter and previous chapters. Further, each $g \in [f]$ is a finite set of ordered pairs, as previously. Thus ${}^*g = g$ and g is internal.

Even though the above property seems to be a nice property, the nonstarring of standard objects, it turns out that the partition concept must be handled differently. Indeed, Theorem 3.2.3 is not true in this model. If $A,\ B \in \mathcal{N}$ and B is a partition of A, then ${}^*B$ is not a partition of ${}^*A$. What is needed is to consider the set $D = \{{}^*x \mid x \in {}^*B\}$. The D is a partition for ${}^*A$, but D is not in general an internal set. On the other hand, each element of ${}^*B$ is an internal set as is each finite subset. It is interesting to note that we require a different extension definition for the consequence operators when ${}^0\mathcal{M}$.

Let $C\colon \mathcal{P}(A) \to \mathcal{P}(B)$ be a standard set-valued map. One must be more careful with the extensions of such set-valued maps than the other maps since the types of objects contained in the ordered pairs are of significance. Observe that $C^\star$ is composed of ordinary ordered pairs of $'$sets and as such these sets are in ${}^0\mathcal{M}$ and contain $'$elements. However, these sets may also contain ordinary elements as well. Assume that $(P,Q) \in C^\star$, and that $P,\ Q$ are not finite standard sets, then $(P,Q) \neq ({}^*P,\ {}^*Q)$. Consequently, in general, a map such as C can be extended to a map ${}^*\underline{C} = \{({}^*P,\ {}^*Q) \mid (P,Q) \in C^\star\}$. In this case, ${}^*P,\ {}^*Q$ contain only $'$elements from P and Q. This different interpretation occurs because the "starring" process in the first model used in this analysis is distinct from the "starring" process as employed with respect to ${}^0\mathcal{M}$. In fact, the $*$ process in the first model is a renaming of the standard language objects as they are interpreted within that model and is a member of the extended language L$'$. The other members of L$'$ are restricted to internal members of our model.

More importantly, with the first model all of the relations ϵ, $'$pr, $'$ap have been replaced by the ordinary $\in$, pr ap within the **ZFH** model and the standard model has been embedded into the structure that what would have been the ϵ, $'$pr, $'$ap defined objects denoted by members of L are so altered that they become the original relations restricted to entities that are isomorphicly related to the original standard objects. One can say that the first model alters the objects with a "minimal" language change. This alternate approach requires a much larger language change but is more expressive in character.

The above extension processes lead to three distinct objects ${}^*C$, $C^\star$ and ${}^*\underline{C}$ within ${}^0\mathcal{M}$. If $C\colon \mathcal{P}(A) \to \mathcal{P}(B)$ is a consequence operator, then $C^\star$ and ${}^*\underline{C}$ are interesting but distinct member of ${}^0\mathcal{M}$. They both satisfy (extended) Tarski type axioms. If we were to continue this development, then the map $^*\underline{C}$ appears to be the most appropriate for such an investigation. However, $^*\underline{C}$ is, in general, an external object. It does not satisfy the internal defining method for $'$n-ary relations that requires the variables to vary over $'\mathcal{N}$. Observe that $^*\underline{C}$ is defined for all of the internal subsets (within $^0\mathcal{M}$) of $^*A$. With respect to the first model, $^*C$ is restricted to its internal subsets of $^*A$ as well. Now $C^\star$ is an internal map in $^0\mathcal{M}$ that is defined on "internal entities" that are $'$subsets of A and it yields $'$subsets of B which, when viewed from the structure $^0\mathcal{M}$, could contain many non-$'$elements. Notice that we cannot obtain any information about these other objects by simply transferring, by the *-transfer method, information from the standard model. These objects could be investigated by a more careful analysis of the exact construction of $^0\mathcal{M}$.

The definition of the "standard restriction" for $^0\mathcal{M}$ would depend upon which type of extension $^*C$, $C^\star$, or $^*\underline{C}$ is used. It is clear that all of this as well as the appropriate definitions for human, subtle and purely subtle entities can be successfully accomplished.

CHAPTERS 1—6 REFERENCES

1 Barwise J. (ed.) *Handbook of Mathematical Logic,* North-Holland, Amsterdam, 1977.

2 Birkhoff, G. D., A set of postulates for plane geometry based on scale and protractor, Annals of Math. 33(1932), 329—345.

3 Hamilton, S. G., *Logic for Mathematicians,* Cambridge University Press, New York, 1978.

4 Herrmann, R. A., Nonstandard Topology. Ph. D. Dissertation, American University, 1973. (University Micro Film # 73-28,762)

5 Jech, T. J., *Lectures in Set Theory,* Lecture Notes in Mathematics Vol. 217, Springer-Verlag, Berlin, 1971.

6 Jech, T. J., *The Axiom of Choice,* North-Holland, Amsterdam, 1973.

7 Kleene, S. C., *Introduction to Metamathematics,* D. Van Nostrand Co., Princeton, 1950.

8 Kleene, S. C., *Mathematical Logic,* John Wiley and Sons, Inc., New York, 1967.

9 Luxemburg, W. A. J., A general theory of monads. in *Applications of Model Theory to Algebra, Analysis and Probability,* (ed. Luxemburg), Holt, Rinehart and Winston, New York, (1969), 18—86.

10 Luxemburg, W. A. J., What is nonstandard analysis, in *Papers in the Foundations of Mathematics* No. 13 Slaught Memorial Papers, Amer. Math. Monthly, (June-July 1973), 38—67.

11 Machover, M. and J. Hirschfeld, *Lectures on Non-standard Analysis,* Lecture Notes in Mathematics, Vol. 94, Springer-Verlag, Berlin, 1969.

12 Markov, A. A., Theory of algorithms, Amer. Math. Soc. Transl., Ser. 2, 15(1960), 1—14.

13 Mendelson, E., *Introduction to Mathematical Logic*, 2'nd ed., D. Van Nostrand Co., New York, 1979.

14 Rasiowa, H. and R. Sikorski, *The Mathematics of Metamathematics,* Polska Akademia Nauk, Monografie Methematyczne, Tom 41, Warsaw, 1963.

15 Robinson, A., On languages which are based on non-standard arithmetic, Nagoya Math., 22(1963), 83—118.

16 Robinson A., *Non-Standard Analysis*, (2'nd ed.) North-Holland, 1974.

17 Robinson, A., and E. Zakon, A set-theoretic characterization of enlargements, in *Applications of Model Theory to Algebra, Analysis* *and Probability,* (ed. Luxemburg), Holt, Rinehart and Winston, New York, (1969),109—122.

18 Stoll, R., *Set Theory and Logic,* W. H. Freeman and Co., San Francisco, 1963.

19 Stroyan, K. D. and W. A. J. Luxemburg, *Introduction to the Theory of Infinitesimals,* Academic Press, New York, 1976.

20 Suppes, P. *Axiomatic Set Theory,* Van Nostrand, 1960, (Reprint Dover, 1972.)

21 Tarski, A., *Logic*, *Semantics*, *Metamathematics,* (Papers from 1923—1938), Oxford University Press, New York, 1956.

22 Thue, A. Probleme über Veränderunger von Zeichenreihen nach gegebenen Regeln, Skrifter utgit av Videnskapsselskapet i Kristiania, I. Metematisk—naturvidenskabelig Klasse 1914.10.

Additional References

23 Hurd, A. E. and P. A. Loeb, *An Introduction to Nonstandard Analysis,* Academic Press, Orlando, 1985.

24 Herrmann, R. A., Nonstandard consequence operators, Kobe J. Math. 4(1987), 1—14.

NOTES

Chapter 7

DEVELOPMENTAL PARADIGMS

7.1 Introduction.

Consider the real line. If you believe that time is the ordinary continuum, then the entire real line can be your time line. Otherwise, you may consider only a subset of the real line as a time line. In the original version of this section, the time concept for the MA-model was presented in a unnecessarily complex form. As shown in [3], one can assume an absolute substratum time within the NSP-world. It is the infinitesimal light-clock time measures that may be altered by physical processes. In my view, the theory of quantum electrodynamics would not exist without such a NSP-world time concept.

Consider a small interval $[a,b),\ a<b$ as our basic time interval where as the real numbers increase the time is intuitively considered to be increasing. In the following approach, one may apply the concept of the persistence of mental version relative to descriptions for the behavior of a Natural system at a moment of time within this interval. An exceptionally small subinterval can be chosen within $[a,b)$ as a maximum subinterval length $= M$. "Time" and the size of a "time" interval as they are used in this and the following sections refer to an intuitive concept used to aid in comprehending the notation of an event sequence - an event ordering concept. First, let $a = t_0$. Then choose t_1 such that $a < t_1 < b$. There is a partition $t_1, \ldots, t_m$ of $[a,b)$ such that $t_0 < t_1 < \cdots < t_m < b$ and $t_{j+1} - t_j \leq M$. The final subinterval $[t_m, b)$ is now separated, by induction, say be taking midpoints, into an increasing sequence of times $\{t_q\}$ such that $t_m < t_q < b$ for each q and $\lim_{q\to\infty}\ t_q = b$.

Assume the prototype $[a,b)$ with the time subintervals as defined above. Let $[t_j, t_{j+1})$ be any of the time subintervals in $[a,b)$. For each such subinterval, let W_i denote the readable sentence

This|||frozen|||segment|||gives|||a|||description|||for|||the|||
time|||interval|||that|||has|||as|||its|||leftmost|||endpoint|||the
|||time|||$\lceil t_i \rceil$|||that|||corresponds|||to|||the|||natural|||number|||i.

Let $\mathrm{T}_i = \{\mathrm{xW}_i \mid \mathrm{x} \in \mathcal{W}\}$. The set T_i is called a *totality* and each member of any such T_i is called a *frozen segment.* Notice that since the empty word is not a member of $\mathcal{W}$, then the cardinality of each member of T_i is greater than that of W_i. Each T_i is a (Dedekind) denumerable set, and if $i \neq j$, then $\mathrm{T}_i \cap \mathrm{T}_j = \emptyset$. (See note [1] on p. 82.)

I point out two minor aspects of the above constructions. First, within certain descriptions there are often "symbols" used for real, complex, natural numbers etc. These objects also exist as abstract objects within the structure $\mathcal{M}$. No inconsistent interpretations should occur when these objects are specifically modeled within $\mathcal{M}$ since to my knowledge all of the usual mathematical objects used within physical analysis are disjoint from $\mathcal{E}$ as well as disjoint from any finite Cartesian product of $\mathcal{E}$ with itself. If for future research within physical applications finite partial sequences of natural numbers and the finite equivalence classes that appear in $\mathcal{E}$ are needed and are combined into one model for different purposes than the study of descriptions, then certain modifications would need to be made so that interpretations would remain consistent. Secondly, I have tried whenever intuitive strings are used or sets of such strings are defined to use Roman letter notation for such objects. This only applies for the intuitive model. Also W_i is only an identifier and may be altered.

7.2 Developmental Paradigms

It is clear that if one considers a time interval of the type $(-\infty, +\infty)$, $(-\infty, b)$ or $[a, +\infty)$, then each of these may be considered as the union of a denumerable collection of time intervals of the type $[a, b)$ with common endpoint names displayed. Further, although $[a, b)$ is to be considered as subdivided into denumerably many subintervals, it is not necessary that each of the time intervals $[t_j, t_{j+1}) \subset [a, b)$ be accorded a corresponding description for the appearance of a specific Natural system that is distinct from all others that occur throughout the time subinterval. Repeated descriptions only containing a different last natural number i in the next to last position will suffice. Each basic developmental paradigm will be restricted, at present, to such a time interval $[a, b)$.

Where human perception and descriptive ability is concerned, the least controversial approach would be to consider only finitely many descriptive choices as appropriate. A finite set is recursive and such a choice, since the result is such a set, would be considered to be the simplest type of algorithm. You "simply" check to see if an expression is a member of such a finite set.

If we limited ourselves to finitely many human choices for Natural system descriptions from the set of all totalities and did not allow a denumerable or a continuum set to be chosen, then the next result establishes that within the Nonstandard Physical world (i.e. NSP-world) such a finite-type of choice can be applied and a continuum of descriptions obtained.

The following theorem is not insignificant even if we are willing to accept a denumerable set of distinct descriptions — descriptions that are not only distinct in the next to the last symbol, but are also distinctly different in other aspects as well. For, if this is the case, the results of Theorem 7.2.1 still apply. The same finite-type of process in the NSP-world yields such a denumerable set as well.

The term "NSP-world" will signify a certain second type of interpretation for nonstandard entities. In particular, the subtle logics, unreadable sentences, etc. This interpretation will be developed throughout the remainder of this book. One important aspect of how descriptions are to be interpreted is that a description correlates directly to an assumed or observed real Natural phenomenon, and conversely. In these investigations, the phenomenon is called an *event.*

In order to simplify matters a bit, the following notation is employed. Let $\mathcal{T} = \{\mathrm{T}_i \mid i \in \mathbb{N}\}$. Let $F(\mathcal{T})$ be the set of all **nonempty** and *finite* subsets of $\mathcal{T}$. This symbol has been used previously to include the empty set, this set is now excluded. Now let $\mathrm{A} \in F(\mathcal{T})$. Then there exists a finite choice set s such that $\mathrm{x} \in \mathrm{s}$ iff there exists a unique $\mathrm{T}_i \in \mathrm{A}$ and $\mathrm{x} \in \mathrm{T}_i$. Now let the set $\mathcal{C}$ denote the set of all such finite choice sets. As to interpreting these results within the NSP-world, the following is essential. Within nonstandard analysis the term "hyper" is often used for the result of the * map. For example, you have ${}^*\mathbb{R}$ as the hyperreals since $\mathbb{R}$ is termed the real numbers. For certain, but not all concepts, the term "hyper" or the corresponding * notation will be universally replaced by the term "ultra." Thus, certain purely subtle words or *-words become "ultrawords" within the developmental paradigm interpretation. [Note: such a word was previously called a superword.] Of course, for other scientific or philosophical systems, such abstract mathematical objects can be reinterpreted by an appropriate technical term taken from those disciplines.

As usual, we are working within any enlargement and all of the above intuitive objects are embedded into the G-structure. Recall, that to simplify expressions, we often suppress within our first-order statements a specific superstructure element that bounds a specific quantifier. The alphabet $\mathcal{A}$ is now assumed to be countable.

[Note 2 MAY 1998: The material between the [[and the]] has been altered from the original that appears in the 1993 revision.] [[Although theorem 7.2.1 may be significant, it is no longer used for the other portions of this research. The set of all developmental paradigms corresponds to the set of all choice functions define on $\mathcal{T}$.

Theorem 7.2.1 *Let $\emptyset \neq \gamma \subset \mathbb{N}$ and $\widetilde{\mathcal{T}} = \{\mathbf{T}_i \mid i \in \gamma\}$. There exists a set of sets $\mathcal{S}$ determined by hyperfinite set Q and hyper finite choice defined on Q such that:*

(i) $s' \in \mathcal{S}$ iff for each $\mathbf{T} \in \widetilde{\mathcal{T}}$ there is one and only one $[g] \in {}^*\mathbf{T}$ such that $[g] \in s'$, and if $x \in s'$, then there is some $\mathbf{T} \in \widetilde{\mathcal{T}}$ and some $[g] \in {}^*\mathbf{T}$ such that $x = [g]$. (If ${}^*[g] \in {}^\sigma\mathbf{T}$, then $[g] = [f] \in \mathbf{T}$.)

Proof. (i) Let $A \in F(\widetilde{\mathcal{T}})$. Then from the definition of $\widetilde{\mathcal{T}}$, there exists some $n \in \mathbb{N}$ such that $A = \{\mathbf{T}_{j_i} \mid i = 0, \ldots, n \wedge j_i \in \mathbb{N}\}$. From the definition of $\mathbf{T}_k$, each $\mathbf{T}_k$ is denumerable. Notice that any $[f] \in \mathbf{T}_k$ is associated with a unique member of $A_1 = i[\mathcal{W}]$. Simply consider the unique $f_0 \in [f]$. The unique member of A_1 is by definition $f_0(0)$. Thus each member of $\mathbf{T}_k$ can be specifically identified. Hence, for each $\mathbf{T}_i$ there is a denumerable $M_i \subset \mathbb{N}$ and a bijection $h_i \colon M_i \to \mathbf{T}_i$ such that $a_i \in \mathbf{T}_i$ iff there is a $k_i \in M_i$ and $h_i(k_i) = a_i$. Consequently, for each $i = 0, \ldots, n$ and $a_{j_i} \in \mathbf{T}_{j_i}$, we have that $h_{j_i}(k_{j_i}) = a_{j_i}$, and conversely for each $i = 0, \ldots, n$ and $k_{j_i} \in M_{j_i},\ h_{j_i}(k_{j_i}) \in \mathbf{T}_{j_i}$. Obviously, $\{h_{j_i}(k_{j_i}) \mid i = 0, \ldots, n\}$ is a finite choice set. All of the above may be translated into the following sentence that holds in $\mathcal{M}$. (Note: Choice sets are usually considered as the range of choice functions. Further, "bounded formula simplification" has been used.)

$$\begin{aligned} &\forall y(y \in F(\widetilde{\mathcal{T}}) \to \exists s((s \in \mathcal{P}(\mathcal{E})) \wedge \forall x((x \in y) \to \exists z((z \in x) \wedge (z \in s) \wedge \\ &\qquad \forall w(w \in \mathcal{E} \to ((w \in s) \wedge (w \in x) \leftrightarrow (w = z))))) \wedge \\ &\qquad \forall u(u \in \mathcal{E} \to ((u \in s) \leftrightarrow \exists x_1((x_1 \in y) \wedge (u \in x_1)))))) \end{aligned} \tag{7.2.1}$$

For each $A \in F(\widetilde{\mathcal{T}})$, let S_A be the set of all such choice sets generated by the predicate that follows the first $\to$ formed from (7.2.1) by deleting the $\exists s$ and letting $y = A$. Of course, this set exists within our set theory. Now let $\mathcal{C} = \{S_A \mid A \in F(\widetilde{\mathcal{T}})\}$.

Consider ${}^*\mathcal{C}$ and ${}^*(S_A)$. Then $s \in {}^*(S_A)$ iff s satisfies (7.2.1) as interpreted in ${}^*\mathcal{M}$. Since we are working in an enlargement, there exists an internal $Q \in {}^*(F(\widetilde{\mathcal{T}}))$ such that ${}^\sigma\widetilde{\mathcal{T}} \subset Q \subset {}^*\widetilde{\mathcal{T}}$. Recall that ${}^\sigma\widetilde{\mathcal{T}} = \{{}^*\mathbf{T} \mid \mathbf{T} \in \widetilde{\mathcal{T}}\}$. Also ${}^\sigma\mathbf{T} \subset {}^*\mathbf{T}$ for each $\mathbf{T} \in \widetilde{\mathcal{T}}$. From the definition of ${}^*\mathcal{C}$, there is an internal set S_Q and $s \in S_Q$ iff s satisfies the internal defining predicate for members of S_Q and this set is the set of all such s. ($\Rightarrow$) Consequently, since for each $\mathbf{T} \in \widetilde{\mathcal{T}},\ {}^*\mathbf{T} \in Q$, then the generally external $s' = \{s \cap {}^*\mathbf{T} \mid \mathbf{T} \in \widetilde{\mathcal{T}}\}$ satisfies the $\Rightarrow$ for (i). Note, however, that for ${}^*\mathbf{T},\ \mathbf{T} \in \widetilde{\mathcal{T}}$, it is possible that $s \cap {}^*\mathbf{T} = \{{}^*[f]\}$ and ${}^*[f] \in {}^\sigma\mathbf{T}$. In this case, by the finiteness of $[f]$ it follows that $[f] = {}^*[f]$ implies that $s \cap {}^*\mathbf{T} = \{[f]\}$. Now let $\mathcal{S} = \{s' \mid s \in S_Q\}$. In general, $\mathcal{S}$ is an external object.

($\Leftarrow$) Consider the internal set S_Q. Let s' be the set as defined by the right-hand side of (i). For each internal $x \in s'$ and applying, if necessary, the *-axiom of choice for *-finite sets, we have the internal set $A_x = \{y \mid (y \in S_Q) \wedge (x \in y)\}$ is nonempty. The set $\{A_x \mid x \in s'\}$ has the finite intersection property. For, let nonempty internal $B = \{x_1, \ldots, x_n\}$. Then the set $A_B = \{y \mid (y \in S_Q) \wedge (x_1 \in y) \cdots \wedge (x_n \in y)\}$ is internal and nonempty by the *-axiom of choice for *-finite sets. Since we are in an enlargement and s' is countable, then $D = \bigcap\{A_x \mid x \in s'\} \neq \emptyset$. Now take any $s \in D$. Then $s \in S_Q$ and from the definition of $\mathcal{S}$, $s' \in \mathcal{S}$. This completes the proof.∎

[Note: Theorem 7.2.1 may be used to model physical developmental paradigms associated with event sequences.]

Although it is not necessary, for this particular investigation, the set $\mathcal{S}$ may be considered *a set of all developmental paradigms.* Apparently, $\mathcal{S}$ contains every possible developmental paradigm for all possible frozen segments and $\mathcal{S}$ contains paradigms for any *-totality ${}^*\mathbf{T}$. There are *-frozen segments contained in various s' that can be assumed to be unreadable sentences since ${}^\sigma\mathbf{T} \neq {}^*\mathbf{T}$.]]

Let $A \in F(\widetilde{\mathcal{T}})$ and $M(A)$ be a subset of S_A for which there exists a written set of rules that selects some specific member of S_A. Obviously, this may be modeled by means of functional relations. First, $M(A) \subset S_A$ and it follows, from the difference in cardinalities, that there are infinitely many members of ${}^*(S_A)$ for which there does not exist a readable rule that will select such members. However, this does not preclude the possibility that there is a set of purely unreadable sentences that do determine a specific member of ${}^*S_A - {}^\sigma M(A)$. This might come about in the following manner. Suppose that H is an infinite set of formal sentences that is interpreted to be a set of rules for the selection of distinct members of $M(A)$. Suppose we have a bijection $h\colon M(A) \to \mathbf{H}$ that represents this selection process. Let ${}^*\mathcal{M}$ be at least a polysaturated enlargement of $\mathcal{M}$, and consider ${}^\sigma f : {}^\sigma(M(A)) \to {}^\sigma\mathbf{H}$. The map ${}^\sigma f$ is also a bijection and ${}^\sigma f : {}^\sigma(M(A)) \to {}^*\mathbf{H}$. Since $|{}^\sigma(M(A))| < |\mathcal{M}|$, it is well-known that there exists an internal map $h\colon A' \to {}^*\mathbf{H}$ such that $h \mid {}^\sigma(M(A)) = {}^\sigma f$, and A', $h[A']$ are internal. Further, for internal $A' \cap {}^*(S_A) = B$, ${}^\sigma(M(A)) \subset B$. However, ${}^\sigma(M(A))$ is external. This yields that h is defined on B and $B \cap ({}^*S_A - {}^\sigma(M(A))) \neq \emptyset$. Also, ${}^\sigma\mathbf{H} \subset h[B] \subset {}^*\mathbf{H}$ implies, since $h[B]$ is internal, that ${}^\sigma H \neq h[B]$. Consequently, in this case, $h[B]$ may be interpreted as a set of *-rules that determine the selection of members of B. That is to say that there is some $[g] \in h[B] - {}^\sigma H$ and a $[k] \in {}^*S_A - {}^\sigma(M(A))$ such that $([k],[g]) \in h$. As it will be shown in the next section, the set H can be so constructed that if $[g] \in h[B] - {}^\sigma H$, then $[g]$ is unreadable.

7.3 Ultrawords

Ordinary propositional logic is not compatible with deductive quantum logic, intuitionistic logic, among others. In this section, a subsystem of propositional logic is investigated which rectifies this incompatibility. I remark that when a standard propositional language L or an informal language P isomorphic to L is considered, it will always be the case that the L or P is minimal relative to its applications. This signifies that if L or P is employed in our investigation for a developmental paradigm, then L or P is constructed only from those distinct propositional atoms that correspond to distinct members of d, etc. The same minimizing process is always assumed for the following constructions.

Let B be a formal or, informal nonempty set of propositions. Construct the language P_0 in the usual manner from B (with superfluous parentheses removed) so that P_0 forms the smallest set of formulas that contains B and such that P_0 is closed under the two binary operations $\land$ and $\to$ as they are formally or informally expressed. Of course, this language may be constructed inductively or by letting P_0 be the intersection of all collections of such formula closed under $\land$ and $\to$.

We now define the deductive system S. Assume substitutivity, parenthesis reduction and the like. Let $\mathrm{d} = \{\mathrm{F}_i \mid i \in \mathbb{N}\} = \mathrm{B}$ be a development paradigm, where each F_i is a readable frozen segment and describes the behavior of a Natural system over a time subinterval. Let the set of axioms be the schemata

$$(\mathcal{A} \land \mathcal{B}) \to \mathcal{A},\ \mathcal{A} \in \mathrm{B} \tag{1}$$

$$(\mathcal{A} \land \mathcal{B}) \to \mathcal{B} \tag{2}$$

$$\mathcal{A} \land (\mathcal{B} \land \mathcal{C}) \to (\mathcal{A} \land \mathcal{B}) \land \mathcal{C}, \tag{3}$$

$$(\mathcal{A} \land \mathcal{B}) \land \mathcal{C} \to \mathcal{A} \land (\mathcal{B} \land \mathcal{C}). \tag{4}$$

If P_0 is considered as informal, which appears to be necessary for some applications, where the parentheses are replaced by the concept of symbol strings being to the "left" or "right" of other symbol strings and the concept of strengths of connectives is used (i.e. $\mathrm{A} \land \mathrm{B} \to \mathrm{C}$ means $((\mathrm{A} \land \mathrm{B}) \to \mathrm{C})$, then axioms 3 — 4 and the parentheses in (1) and (2) may be omitted. The one rule of inference is Modus Ponens (MP). Proofs or demonstrations from hypotheses Γ contain finitely many steps, hypotheses may be inserted as steps and the last step in the proof is either a theorem if $\Gamma = \emptyset$ or if $\Gamma \neq \emptyset$, then the last step is a consequence of (a deduction from) Γ. Notice that repeated application of (4) along with (MP) will allow all left parentheses to be shifted to the right with the exception of the (suppressed) outermost left one. Thus this leads to the concept of left to right ordering of a formula. This allows for the suppression of such parentheses. In all the following, this suppression will be done and replaced with formula left to right ordering.

For each $\Gamma \subset \mathrm{P}_0$, let $S(\Gamma)$ denote the set of all formal theorems and consequences obtained from the above defined system S. Since hypotheses may be inserted, for each $\Gamma \subset \mathrm{P}_0$, $\Gamma \subset S(\Gamma) \subset \mathrm{P}_0$. This implies that $S(\Gamma) \subset S(S(\Gamma))$. So, let $\mathrm{A} \in S(S(\Gamma))$. The general concept of combining together finitely many steps from various proofs to yield another formal proof leads to the result that $\mathrm{A} \in S(\Gamma)$. Therefore, $S(\Gamma) = S(S(\Gamma))$. Finally, the finite step requirement also yields the result that if $\mathrm{A} \in S(\Gamma)$, then there exists a finite $\mathrm{F} \subset \Gamma$ such that $\mathrm{A} \in S(\mathrm{F})$. Consequently, S is a finitary consequence operator and observe that if C is the propositional consequence operator, then $S(\Gamma) \underset{\neq}{\subset} C(\Gamma)$. Of course, we may now apply the nonstandard theory of consequence operators to S.

It is well-known that the axiom schemata chosen for S are theorems in intuitionistic logic. Now consider quantum logic with the Mittelstaedt conditional $i_1(\mathrm{A}, \mathrm{B}) = \mathrm{A}^{\perp} \vee (\mathrm{A} \wedge \mathrm{B})$. [1] Notice that $i_1(\mathrm{A} \wedge \mathrm{B}, \mathrm{B}) = (\mathrm{A} \wedge \mathrm{B})^{\perp} \vee ((\mathrm{A} \wedge \mathrm{B}) \wedge \mathrm{B}) = (\mathrm{A} \wedge \mathrm{B})^{\perp} \vee (\mathrm{A} \wedge \mathrm{B}) = I$ (the upper unit.) Then $i_1((\mathrm{A} \wedge \mathrm{B}), \mathrm{A}) = (\mathrm{A} \wedge \mathrm{B})^{\perp} \vee ((\mathrm{A} \wedge \mathrm{B}) \wedge \mathrm{A}) = (\mathrm{A} \wedge \mathrm{B})^{\perp} \vee (\mathrm{A} \wedge \mathrm{B}) = I$; $i_1((\mathrm{A} \wedge \mathrm{B}) \wedge \mathrm{C}, \mathrm{A} \wedge (\mathrm{B} \wedge \mathrm{C})) = ((\mathrm{A} \wedge \mathrm{B}) \wedge \mathrm{C})^{\perp} \vee (\mathrm{A} \wedge (\mathrm{B} \wedge \mathrm{C})) = I = i_1(\mathrm{A} \wedge (\mathrm{B} \wedge \mathrm{C}), (\mathrm{A} \wedge \mathrm{B}) \wedge \mathrm{C})$. Thus with respect to the interpretation of $\mathcal{A} \to \mathcal{B}$ as conditional i_1 the axiom schemata for the system S are theorems and the system S is compatible with deductive quantum logic under the Mittelstaedt conditional.

Recall that $\mathrm{d} = \{\mathrm{F}_i \mid i \in \mathbb{N}\}$ is a development paradigm, where each F_i is a readable frozen segment, and describes the behavior of a Natural system at each moment of a time interval. For the next construction a formal language that is, of course, isomorphic to the informal language is employed. Each $\wedge$ [resp. F_i] corresponds to a specific |||and||| [resp. a propositional atom that corresponds to a specific word] when embedded. This eliminates confusion when |||and||| appears in the F_i. Let $\mathrm{M}_0 = \mathrm{d}$. Define $\mathrm{M}_1 = \{\mathrm{F}_0|||\mathrm{and}|||\mathrm{F}_1\}$. Assume that M_n is defined. Define $\mathrm{M}_{n+1} = \{\mathrm{x}|||\mathrm{and}|||\mathrm{F}_{n+1} \mid \mathrm{x} \in \mathrm{M}_n\}$. From the fact that d is a developmental paradigm, where the last two symbols in each member of d is the time indicator "i.", it follows that no member of d is a member of M_n for $n > 0$. Now let $\mathrm{M}_\mathrm{d} = \bigcup\{\mathrm{M}_n \mid n \in \mathbb{N}\}$. Intuitively, |||and||| behaves as a conjunction and each F_i as an atom within our language. Notice the important formal demonstration fact that for an hypothesis consisting of any member of $\mathrm{M}_n,\ n > 0$, repeated applications of (1), (MP), (2), (MP) will lead to the members of d appearing in the proper time ordering at increasing (formal) demonstration step numbers.

Theorem 7.3.1 *For* $\mathbf{d} = \{\mathbf{F}_i \mid i \in \mathbb{N}\}$, *there exists an ultraword* $w \in {}^*\mathbf{M_d} - {}^*\mathbf{d}$ *such that* $\mathbf{F}_i \in {}^*\mathbf{S}(\{w\})$ *(i.e.* $w\ {}^*\!\vdash_S \mathbf{F}_i$*) for each* $i \in \mathbb{N}$.

Proof. Consider the binary relation $G = \{(x,y) \mid (x \in \mathbf{d}) \wedge (y \in \mathbf{M_d} - \mathbf{d}) \wedge (x \in \mathbf{S}(\{y\})\}$. Suppose that $\{(x_1,y_1),\ldots(x_n,y_n)\} \subset G$. For each $i = 1,\ldots,n$ there is a unique $k_i \in \mathbb{N}$ such that $x_i = \mathbf{F}_{k_i}$. Let $m = \max\{k_i \mid (x_i = \mathbf{F}_{k_i}) \wedge (i = 1,\ldots,n)\}$. Let $b \in \mathbf{M}_{m+1}$. It follows immediately that $x_i \in \mathbf{S}(\{b\})$ for each $i = 1,\ldots,n$ and, from the construction of d, $b \notin \mathbf{d}$. Thus $\{(x_1,b),\ldots,(x_n,b)\} \subset G$. Consequently, G is a concurrent relation. Hence, there exists some $w \in {}^*\mathbf{M_d} - {}^*\mathbf{d}$ such that ${}^\sigma\mathbf{F}_i = \mathbf{F}_i \in {}^*\mathbf{S}(\{w\})$ for each $i \in \mathbb{N}$. This completes the proof. [See note 5.] ▮

Observe that w in Theorem 7.3.1 has all of the formally expressible properties of a readable word. For example, w has a hyperfinite length, among other properties. However, since d is a denumerable set, each ultraword has a very special property.

Recall that for each $[g] \in \mathcal{E}$ there exists a unique $m \in \mathbb{N}$ and $f' \in T^m$ such that $[f'] = [g]$ and for each k such that $m < k \in \mathbb{N}$, there does not exist $g' \in T^k$ such that $[g'] = [g]$. The function $f' \in T^m$ determines all of the alphabet symbols, the symbol used for the blank space, and the like, and determines there position within the intuitive word being represented by $[g]$. Also for each j such that $0 \leq j \leq m,\ f'(j) = i(\mathrm{a}) \in i[\mathcal{W}] = T$, where $i(\mathrm{a})$ is the "encoding" in T of the symbol "a". For each $m \in \mathbb{N}$, let $P_m = \{f \mid (f \in T^m) \wedge (\exists z((z \in \mathcal{E}) \wedge (f \in z) \wedge \forall x((x \in \mathbb{N}) \wedge (x > m) \to \neg\exists y((y \in T^x) \wedge (y \in z)))))\}$. An element $n \in {}^*T$ is a *subtle alphabet symbol* if there exists $m \in \mathbb{N}$ and $f \in {}^*(P_m) = ({}^*P)({}^*m) = ({}^*P)_m = {}^*P_m\ (m = {}^*m)$ or if $\delta \in \mathbb{N}_\infty$ and $f \in {}^*P_\delta$, and some $j \in {}^*\mathbb{N}$ such that $f(j) = n$. A symbol is a *pure subtle alphabet symbol* if $f(j) = n \notin i[\mathcal{W}]$. Subtle alphabet symbols can be characterized in ${}^*\mathcal{E}$ for they are singleton objects. A $[g] \in {}^*\mathcal{E}$ represents a subtle alphabet symbol iff there exists some $f \in ({}^*T)^0$ such that $[f] = [g] = [(0,f(0))],\ f = \{(0,f(0))\}$.

Theorem 7.3.2 *Let* $\mathrm{d} = \{\mathrm{F}_i \mid i \in \mathbb{N}\}$ *be a denumerable developmental paradigm and use* $\mathcal{M}_1$ *of 9.1. For each ultraword* w*, that yields* $\mathbf{d}$ *via* ${}^*\mathbf{S}$*, as in Theorem 7.3.1, the external cardinality of the collection of all pure subtle alphabet symbols represented in each* w *is greater than or equal to* $2^{\aleph_0}$*.*

Proof. Consider the conceptual Kleene "tick" notation for the natural numbers (i.e $|, ||\cdot|||, \ldots$). For this proof, let $|$ correspond to 0. Every member of $\mathrm{d} = \{\mathrm{F}_i \mid i \in \mathbb{N}\}$ contains a distinct symbol-string b_i that represents the natural number followed by the "period" symbol that appears as the last two symbols in a member of d. Consider the single $\mathrm{W}_n \in \mathrm{M}_n,\ n > 0$. Then $n+2$ of these distinct symbol-strings, the Kleene symbols and a "period" symbol, appear in W_n along with other alphabet systems. Hence, in W_n, there are more than $n+2$ alphabet symbols.

For the embedding $\mathcal{E}$, there is a W_n representation $[g] \in \mathcal{E}$ and two unique mappings $f_k \sim f_0 \sim g$, where the inverse of the embedding i yields the entire word for f_0 and, for $f_k \in P_k$, yields the entire word as it is join constructed from individual symbols (eq. 1.2.4). In this case, $k > n+2$.

Consider the *-transform. Let $w = [g]$ be an ultraword such that for each $i \in \mathbb{N}$, $\mathbf{F}_i \in {}^*\mathbf{S}(\{w\})$. Theorems 7.3.1 shows that such ultrawords exist. From the definition of S, $w \in {}^*\mathbf{M_d} - {}^\sigma\mathbf{M_d}$. Hence, there is a $\nu,\ \delta \in \mathbb{N}_\infty$ and ${}^*\mathbf{M}_\nu \in \{{}^*\mathbf{M}_x \mid x \in {}^*\mathbb{N}\}$ such that $[f_\delta] = [g] \in {}^*\mathbf{M}_\nu,\ \delta > \nu + 2,\ f_\delta \in {}^*P_\delta$

Let $\mathrm{K} = \{[1, \mathrm{n}+2] \mid \mathrm{n} \in \mathbb{N}\}$. Then there exists a mapping $\mathrm{C}\colon \mathrm{K} \to \mathbb{N}$ such that $\mathrm{C}([1, \mathrm{n}+2]) = \mathrm{n}+2$. The mapping C is considered as yielding the intuitive cardinality of $[1, n+2]$. Hence, ${}^*\mathbf{C}([1, \nu+1]) = \nu + 2$. To get an idea as to the external cardinality $|[1, \nu+2]|$ of $[1, \nu+2]$, consider Theorem 3.1 in [16, p. 201], where it is shown that $|[1, \nu+2]| \geq 2^{\aleph_0}$. Since $|[1, \delta]| \geq |[1, \nu+2]|$, and the set of all subtle alphabet symbols that yields members of $\mathcal{W}$ is denumerable, then it follows that for w the set of all pure subtle alphabet symbols also has an external cardinality great than or equal to $2^{\aleph_0}$. This completes the proof. ▮

With respect to the proof of Theorem 7.3.2, the function f_δ determines the alphabet composition of the ultraword w. The word w is unreadable not only due to its infinite length but also due to the fact that it is composed of infinitely many purely subtle alphabet symbols.

The developmental paradigm d utilized for the two previous theorems is composed entirely of readable sentences. We now investigate what happens if a developmental paradigm contains countably many unreadable sentences. Let the nonempty developmental paradigm d' be composed of at most countably many members of ${}^*\mathcal{E} - \mathcal{E}$ and, for countable B, let $d' \subset {}^*\mathbf{B} \subset {}^*\mathbf{P}_0$. Construct, as previously, the set $\mathrm{M_B}$ from B, rather than from d and suppose that $\mathrm{B} \cap \mathrm{M}_i = \emptyset,\ i \neq 0$. [See Note [2] on page 82.] Let $\neq \lambda \subset \mathbb{N}$.

Theorem 7.3.3 *Let $d' = \{[g_i] \mid i \in \lambda\}$. Then there exists an ultraword $w \in {}^*\mathbf{M_B} - {}^*\mathbf{B}$ such that for each $i \in \mathbb{N}$, $[g_i] \in {}^*\mathbf{S}(\{w\})$.*

Proof. Consider the internal binary relation $G = \{(x,y) \mid (x \in {}^*\mathbf{B}) \wedge (y \in {}^*\mathbf{M_B} - {}^*\mathbf{B}) \wedge (x \in {}^*\mathbf{S}(\{y\})\}$. Note that members of d' are members of ${}^\sigma\mathcal{E}$ or, at the most, denumerably many members of ${}^*\mathcal{E} - {}^\sigma\mathcal{E}$. From the analysis in the proof of Theorem 7.3.1, for a finite $\mathrm{F} \subset \mathrm{B}$, there exists some $\mathrm{y} \in \mathrm{M_B} - \mathrm{B}$ such that $\mathrm{F} \subset S(\{\mathrm{y}\})$. It follows by *-transfer that if F is a finite or *-finite subset of ${}^*\mathbf{B}$, then there exists some $y \in {}^*\mathbf{M_B} - {}^*\mathbf{B}$ such that $F \subset {}^*\mathbf{S}(\{y\})$. As in the proof of Theorem 7.3.1, this yields that G is at least concurrent on ${}^*\mathbf{B}$. However, $d' \subset {}^*\mathbf{B}$ and $|d'| \leq \aleph_0$. From $\aleph_1$-saturation, there exists some $w \in {}^*\mathbf{M_B} - {}^*\mathbf{B}$ such that for each $[g_i] \in d',\ [g_i] \in {}^*\mathbf{S}(\{w\})$. This completes the proof. ▮

Let nonempty $\gamma, \lambda \subset \mathbb{N}$, $j \in \gamma$, $\mathcal{D}_j = \{d_{ij} \mid i \in \lambda\}$, and for each $j \in \gamma$, $i \in \lambda$, $d_{ij} \subset {}^*\mathbf{B}$ is considered to be a developmental paradigm either of type d or type d' and $\mathrm{B} \cap \mathrm{M}_i = \emptyset,\ i \neq 0$. Notice that $\mathcal{D}_j$ is finite or denumerable. Theorem 7.3.1 holds for the case that $\mathrm{d} \subset \mathrm{B}$, where $w \in {}^*\mathbf{M_B} - {}^*\mathbf{B}$. For each $d_{ij} \in \mathcal{D}_j$, use the Axiom of Choice to select an ultraword $w_{ij} \in {}^*\mathbf{M_B} - {}^*\mathbf{B}$ that exists by Theorems 7.3.1 (extended) or 7.3.3. Let $\{w_{ij} \mid i \in \lambda\}$ be such a set of ultrawords.

Theorem 7.3.4 *For, $j \in \gamma$, there exists an ultimate ultraword $w'_j \in {}^*\mathbf{M_B} - {}^*\mathbf{B}$ such that for each $i \in \lambda$, $w_{ij} \in {}^*\mathbf{S}(\{w'_j\})$ and, hence, for each $d_{ij} \in \mathcal{D}_j$, $d_{ij} \subset {}^*\mathbf{S}(\{w_{ij}\}) \subset {}^*\mathbf{S}(\{w'_j\})$.*

Proof. For each finite $\{\mathrm{F}_1, \ldots, \mathrm{F_n}\} \subset \mathrm{M_B} - \mathrm{B}$ there is a natural number, say m, such that for each $i = 1, \ldots, n$, $\mathrm{F}_i \in \mathrm{M}_j$ for some $j \leq m$. Hence, taking $\mathrm{b} \in \mathrm{M}_{m+1}$, we obtain that each $\mathrm{F}_i \in S(\{\mathrm{b}\})$. Observe that $\mathrm{b} \notin \mathrm{B}$. By *-transfer, it follows that the internal relation $G = \{(x,y) \mid (x \in {}^*\mathbf{M_B} - {}^*\mathbf{B}) \wedge (y \in {}^*\mathbf{M_B} - {}^*\mathbf{B}) \wedge (x \in {}^*\mathbf{S}(\{y\})\}$ is concurrent on internal ${}^*\mathbf{M_B} - {}^*\mathbf{B}$ and $\{w_{ij} \mid i \in \lambda\} \subset {}^*\mathbf{M_B} - {}^*\mathbf{B}$. Again $\aleph_1$-saturation yields that there is some $w'_j \in {}^*\mathbf{M_B} - {}^*\mathbf{B}$ such that for each $i \in \lambda$, $w_{ij} \in {}^*\mathbf{S}(\{w'_j\})$. The last property is obtained from $d_{ij} \subset {}^*\mathbf{S}(\{w_{ij}\}) \subset {}^*\mathbf{S}({}^*\mathbf{S}(\{w'_j\})) = {}^*\mathbf{S}(\{w'_j\})$ since finite $\{w_{ij}\}$ is an internal subset of ${}^*\mathbf{P}_0$. This completes the proof.▮

Corollary 7.3.4.1 *There exists an ultimate ultraword* $w' \in {}^*\mathbf{M_B} - {}^*\mathbf{B}$ *such that for each* $j \in \gamma$, $w'_j \in {}^*\mathbf{S}(\{w'\})$ *and, hence, for each* $d_{ij} \in \bigcup \mathcal{D}_j$, $d_{ij} \subset {}^*\mathbf{S}(\{w'_j\}) \subset {}^*\mathbf{S}(\{w'\})$.

The same analysis used to obtain Theorem 7.3.2 can be applied to the ultrawords of Theorems 7.3.3 and 7.3.4. (See note [6].)

7.4 Ultracontinuous Deduction

In 1968, a special topology on the set of all nonempty subsets of a given set X was constructed and investigated by your author. We apply a similar topology to subsets of $\mathcal{E}$.

Suppose that nonempty $X \subset \mathcal{E}$. Let τ be the discrete topology on X. In order to topologize $\mathcal{P}(X)$, proceed as follows: for each $G \in \tau$, let $N(G) = \{A \mid (A \subset X) \land (A \subset G)\} = \mathcal{P}(G)$.

Consider $\mathcal{B} = \{N(G) \mid G \in \tau\}$ to be a base for a topology τ_1 on $\mathcal{P}(X)$. Let $A \in N(G_1) \cap N(G_1)$. The discrete topology implies that $N(A)$ is a base element and that $N(A) \subset N(G_1) \cap N(G_2)$. There is only one member of $\mathcal{B}$ that contains X and this is $\mathcal{P}(X)$. Thus if $\mathcal{P}(X)$ is covered by members of $\mathcal{B}$, then $N(X) = \mathcal{P}(X)$ is one of these covering objects. Thus $(\mathcal{P}(X), \tau_1)$ is a compact space. Further, since $N(\emptyset) \subset N(G)$ for each $G \in \tau$, the space $(\mathcal{P}(X), \tau_1)$ is connected. The topology τ_1 is a special case of a more general topology with the same properties. [2] Suppose that $D \subset X$. Let $D \in N(G) = \mathcal{P}(G), G \in \tau$. Then $D \in N(D) \subset N(G)$. This yields that the nonstandard monad is $\mu(D) = \bigcap\{{}^*N(G) \mid N(G) \in \mathcal{B}\} = {}^*(\mathcal{P}(D)) = {}^*\mathcal{P}({}^*D)$.

Theorem 7.4.1 *Any consequence operator* $C\colon (\mathcal{P}(X), \tau_1) \to (\mathcal{P}(X), \tau_1)$ *is continuous.*

Proof. Let $A \in \mathcal{P}(X)$ and $H \in {}^*\mathbf{C}[\mu(A)]$. Then there exists some $B \in \mu(A)$ such that ${}^*\mathbf{C}(B) = H$. Hence, $B \in {}^*\mathcal{P}({}^*A)$. By *-transfer of a basic property of our consequence operators, ${}^*\mathbf{C}(B) \subset {}^*\mathbf{C}({}^*A) = {}^*(\mathbf{C}(A))$. Thus ${}^*(\mathbf{C}(B)) \in {}^*(\mathcal{P}(\mathbf{C}(A)))$ implies that ${}^*\mathbf{C}(B) \in \mu(A)$. Therefore, ${}^*\mathbf{C}[\mu(A)] \subset \mu([\mathbf{C}(A)])$. Consequently, $\mathbf{C}$ is continuous. ▮

Corollary 7.4.1.1 *For any* $X \subset \mathcal{E}$, *and any consequence operator* $\mathbf{C}\colon \mathcal{P}(X) \to \mathcal{P}(X)$, *the map* ${}^*\mathbf{C}\colon {}^*(\mathcal{P}(X)) \to {}^*(\mathcal{P}(X))$ *is ultracontinuous.*

Corollary 7.4.1.2 *Let* d [*resp.* d', d *or* d'] *be a developmental paradigm as defined for Theorem 7.3.1* [*resp. Theorem 7.3.3*, *7.3.4*]. *Let* w *be a ultraword that exists by Theorem 7.3.1* [*resp Theorem 7.3.3*, *7.3.4*]. *Then* d [*resp.* d', d *or* d'] *is obtained by means of a ultracontinuous subtle deductive process applied to* $\{w\}$.

Recall that in the real valued case, a function $f\colon [a,b] \to \mathbb{R}$ is uniformly continuous on $[a,b]$ iff for each $p,\ q \in {}^*[a,b]$ such that $p - q \in \mu(0)$, then $f(p) - f(q) \in \mu(0)$. If $D \subset [a,b]$ is compact, then $p,\ q \in {}^*D$ and $p - q \in \mu(0)$ imply that there is a standard $r \in D$ such that $p,\ q \in \mu(r)$. Also, for each $r \in D$ and any $p,\ q \in \mu(r)$, it follows that $p - q \in \mu(r)$. Thus, if compact $D \subset [a,b]$, then $f : D \to \mathbb{R}$ is uniformly continuous iff for every $r \in D$ and each $p,\ q \in \mu(r),\ \ {}^*f(p),\ \ {}^*f(q) \in \mu(f(r))$. With this characterization in mind, it is clear that any consequence operator $\mathbf{C}\colon \mathcal{P}(X) \to \mathcal{P}(X)$ satisfies the following statement. For each $A \in \mathcal{P}(X)$ and each $p,\ q \in \mu(A),\ \ {}^*\mathbf{C}(p),\ \ {}^*\mathbf{C}(q) \in \mu(\mathbf{C}(A))$.

From the above discussion, one can think of ultracontinuity as being a type of ultrauniform continuity.

7.5 Hypercontinuous Gluing

There are various methods that can be used to investigate the behavior of adjacent frozen segments. All of these methods depend upon a significant result relative to discrete real or vector valued functions. The major goal in this section is to present a complete proof of this major result and to indicate how it is applied.

First, as our standard structure, consider either the intuitive real numbers as atoms or axiomatically a standard structure with atoms $\mathbf{ZFR} = \mathbf{ZF} + \mathbf{AC} + A_1(\text{atoms}) + A(\text{atoms}) + |A| = \mathrm{c}$, where A is isomorphic to the real numbers and $A_1 \cap A = \emptyset$. Then, as done previously, there is a model $\langle C, \in, = \rangle$ within our $\mathbf{ZF}+\mathbf{AC}$ model for $\mathbf{ZFR}$, where A has all of the ordered field properties as the real numbers. A superstructure $\langle \mathcal{R}, \in, = \rangle$ is constructed in the usual manner, where the superstructure $\langle \mathcal{N}, \in, = \rangle$ is a substructure. Proceeding as in Chapter 2, construct ${}^*\mathcal{M}_1 = \langle {}^*\mathcal{R}, \in = \rangle$ and $\mathcal{Y}_1$. The structure $\mathcal{Y}_1$ is called the *Extended Grundlegend Structure* — the EGS. The Grundlegend Structure is a substructure of $\mathcal{Y}_1$.

It is important to realized in what follows that the objects utilized for the G-structure *interpretations* are nonempty finite equivalence classes of partial sequences. Due to this fact, the following results should not lead to ambiguous interpretations.

As a preliminary to the technical aspects of this final section, we introduce the following definition. A function $f\colon [a,b] \to \mathbb{R}^m$ is *differentiable-C* on $[a,b]$ if it is continuously differentiable on (a,b) except at finitely many removable discontinuities. This definition is extended to the end points $\{a,b\}$ by application of one-sided derivatives. For any $[a,b]$, consider a partition $P = \{a_0, a_1, \cdots, a_n, a_{n+1}\},\ n \geq$

$1,\ a = a_0,\ b = a_{n+1}$ and $a_{j-1} < a_j,\ 1 \leq j \leq n+1$. For any such partition P, let the real valued function g be defined on the set $D = [a_0, a_1) \cup (a_1, a_2) \cup \cdots \cup (a_n, a_{n+1}]$ as follows: for each $x \in [a_0, a_1)$, let $g(x) = r_1 \in \mathbb{R}$; for each $x \in (a_{j-1}, a_j)$, let $g(x) = r_j \in \mathbb{R},\ 1 < i \leq n$; for each $x \in (a_n, b]$, let $g(x) = r_{n+1} \in \mathbb{R}$. It is obvious that g is a type of simple step function. Notationally, let $\mathcal{F}(A, B)$ denote the set of all functions with domain A and codomain B.

Theorem 7.5.1 *There exists a function $G \in {}^*(\mathcal{F}([a,b], \mathbb{R}))$ with the following properties.*

*(i) The function G is *-continuously *-differentiable and *-uniformly *-continuous on ${}^*[a,b]$,*

*(ii) for each odd $n \in {}^*\mathbb{N},\ (n \geq 3),\ G$ is *- differentiable-C of order n on ${}^*[a,b]$,*

*(iii) for each even $n \in {}^*\mathbb{N},\ G$ is *-continuously *-differentiable of order n in ${}^*[a,b]$ except at finitely many points,*

(iv) if $c = \min\{r_1, \cdots, r_{n+1}\},\ d = \max\{r_1, \cdots, r_{n+1}\}$, then the range of $G = {}^*[c,d]$, $\mathtt{st}(G)$ at least maps D into $[c,d]$ and $(\mathtt{st}(G))|D = g$.

Proof. First, for any real c, d, where $d \neq 0$, consider the finite set of functions

$$h_j(x,c,d) = (1/2)(r_{j+1} - r_j)\Big(\sin\big((x-c)\pi/(2d)\big) + 1\Big) + r_j, \quad (7.5.1)$$

$1 \leq j \leq n$. Each h_j is continuously differentiable for any order at each $x \in \mathbb{R}$. Observe that for each odd $m \in \mathbb{N}$, each m'th derivative $h_j^{(m)}$ is continuous at $(c+d)$ and $(c-d)$ and $h_j^{(m)}(c+d) = h_j^{(m)}(c-d) = 0$ for each j.

Let positive $\delta \in \mu(0)$. Consider the finite set of internal intervals $\{[a_0, a_1 - \delta), (a_1 + \delta, a_2 - \delta), \cdots, (a_n + \delta, b]\}$ obtained from the partition P. Denote these intervals in the expressed order by $I_j,\ 1 \leq j \leq n+1$. Define the internal function

$$G_1 = \{(x, r_1) | x \in I_1\} \cup \cdots \cup \{(x, r_{n+1}) | x \in I_{n+1}\}. \quad (7.5.2)$$

Let internal $I_j^\dagger = [a_j - \delta, a_j + \delta],\ 1 \leq j \leq n$, and for each $x \in I_j^\dagger$, let internal

$$G_j(x) = (1/2)(r_{j+1} - r_j)\Big({}^*\!\sin\big((x - a_j)\pi/(2\delta)\big) + 1\Big) + r_j. \quad (7.5.3)$$

Define the internal function

$$G_2 = \{(x, G_1(x)) | x \in I_1^\dagger\} \cup \cdots \cup \{(x, G_n(x)) | x \in I_n^\dagger\}. \quad (7.5.4)$$

The final step is to define $G = G_1 \cup G_2$. Then $G \in {}^*(\mathcal{F}([a,b], \mathbb{R}))$.

By *-transfer, the function G_1 has an internal *-continuous *-derivative $G_1^{(1)}$ such that $G_1^{(1)}(x) = 0$ for each $x \in I_1 \cup \cdots \cup I_{n+1}$. Applying *-transfer to the properties of the functions $h_j(x, c, d)$, it follows that G_2 has a unique internal *-continuous *-derivative

$$G_2^{(1)} = (1/(4\delta))(r_{j+1} - r_j)\pi\Big({}^*\!\cos\big((x - a_j)\pi/(2\delta)\big)\Big) \tag{7.5.5}$$

for each $x \in I_1^\dagger \cup \cdots \cup I_n^\dagger$. The results that the *-left limit for the internal $G_1^{(1)}$ and the *-right limit for internal $G_2^{(1)}$ at each $a_j - \delta$ as well as the *-left limit of $G_2^{(1)}$ and *-right limit of $G_1^{(1)}$ at each $a_j + \delta$ are equal to 0 and $0 = G_2^{(1)}(a_j - \delta) = G_2^{(1)}(a_j + \delta)$ imply that internal G has a *-continuous *- derivative $G^{(1)} = G_1^{(1)} \cup G_2^{(1)}$ defined on ${}^*[a,b]$.

A similar analysis and *-transfer yield that for each $m \in {}^*\mathbb{N}$, $m \geq 2$, G has an internal *-continuous *-derivative $G^{(m)}$ defined at each $x \in {}^*[a,b]$ except at the points $a_j \pm \delta$ whenever $r_{j+1} \neq r_j$. However, it is obvious from the definition of the functions h_j that for each odd $m \in {}^*\mathbb{N}$, $m \geq 3$, each internal $G^{(m)}$ can be made *-continuous at each $a_j \pm \delta$ by simply defining $G^{(m)}(a_j \pm \delta) = 0$ and with this parts (i), (ii), and (iii) are established.

For part (iv), assume that $r_j \leq r_{j+1}$. From the definition of the functions h_j, it follows that for each $x \in I_j \cup I_j^\dagger \cup I_{j+1}$, $r_j \leq G(x) \leq r_{j+1}$. The nonstandard intermediate value theorem implies that $G\big[{}^*[a_j, a_{j+1}]\big] = {}^*[r_j, r_{j+1}]$ and in like manner for the case that $r_{j+1} < r_j$. Hence, $G\big[{}^*[a,b]\big] = {}^*[c,d]$. Clearly, $\mathtt{st}({}^*D) = [a,b]$. If $p \in D$ and $x \in \mu(p) \cap {}^*D$, then $G(x) = r_j = g(p)$ for some j such that $1 \leq j \leq n+1$. This completes the proof. ∎

The nonstandard approximation theorem 7.5.1 can be extended easily to functions that map D into $\mathbb{R}^m$. For example, assume that $F: D \to \mathbb{R}^3$, the component functions F_1, F_2 are continuously differentiable on $[a,b]$; but that F_3 is a g type step function on D. Then letting $H = ({}^*F_1, {}^*F_2, G)$, on ${}^*[a,b]$, where G is defined in Theorem 4.1, we have an internal *-continuously *-differentiable function $H: {}^*[a,b] \to {}^*\mathbb{R}^3$, with the property that $\mathtt{st}(H)|D = F$.

With respect to Theorem 7.5.1, it is interesting to note that if h_j is defined on $\mathbb{R}$, then for even orders $n \in \mathbb{N}$,

$$\mid h_j^{(n)}(c \pm d) \mid = \left|\frac{(r_{j+1} - r_j)\pi^n}{2^{n+1}d^n}\right| = 0 \tag{7.5.6}$$

for $r_{j+1} = r_j$ but not 0 otherwise. If $r_{j+1} - r_j \neq 0$, then $G_2^{(n)}(a_j \pm \delta)$ is an infinite nonstandard real number. Indeed, if m_i is an increasing sequence of even numbers in ${}^*\mathbb{N}$ and $r_{j+1} \neq r_j$, then $|G_2^{(m_i)}(a_j \pm \delta)|$ forms a decreasing sequence of nonstandard infinite numbers. The next result is obvious from the previous result.

Corollary 7.5.1.1 *For each $n \in {}^*\mathbb{N}$, then internal $G^{(n)} = G_1^{(n)} \cup G_2^{(n)}$ is *-bounded on ${}^*[a, b]$.*

Let $D(a, b)$ be the set of all bounded and piecewise continuously differentiable functions defined on $[a, b]$. By considering all of the possible (finitely many) subintervals, where $f \in D(a, b)$, it follows from the Riemann sum approach that for each real $\nu > 0$, there exists a real $\nu_1 > 0$ such that for each real ν_i, $0 < \nu_i < \nu_1$, a sequence of partitions $P_i = \{a = b_0^i < \cdots < b_{k_i}^i = b\}$ can be selected such that the mesh$(P_i) \leq \nu_i$ and

$$|(f(b) - f(a)) - \sum_{n=1}^{k_i} f'(t_n)(b_n^i - b_{n-1}^i)| < \nu \tag{7.5.7}$$

for any $t_n \in (b_{n-1}^i, b_n^i)$.

Moreover, for any given number M, the sequence of partitions can be so constructed such that there exists a j such that for each $i > j$, $k_i > M$, where P_i and P_j are partitions within the sequence of partitions. By *-transfer of these facts and by application of Theorem 7.5.1 and its corollary we have the next result.

Corollary 7.5.1.2 *For each $n \in \mathbb{N}$ and each internal $G^{(n)}$, the difference $G^{(n)}(b) - G^{(n)}(a)$ is infinitesimally close to an (externally) infinity *-finite sum of infinitesimals.*

A developmental paradigm is a very general object and, therefore, can be used for numerous applications. At present, developmental paradigms are still being viewed from the *substratum* or external world. For what follows, it is assumed that a developmental paradigm d traces the evolutionary history of a specifically named natural system or systems. In this first application, let each $\mathrm{F}_i \in$ d have the following property (**P**).

> F_i describes "the general behavior and characteristics of the named natural system S_1 as well as the behavior and characteristics of named constituents contained within S_1 at time t_i."

Recall that for F_i, $\mathrm{F}_{i+1} \in$ d, there exist unique functions $f_0 \in \mathbf{F}_i = [f]$, $g_0 \in \mathbf{F}_{i+1} = [g]$ such that f_0, $g_0 \in T^0$ and

$\{(0, f_0(0))\} \in [f]$, $\{(0, g_0(0))\} \in [g]$. Thus, to each $\mathrm{F}_j \in \mathrm{d}$, correspond the unique natural number $f_0(0)$. Let $D = [t_{i-1}, t_i) \cup (t_i, t_{i+1}]$ and define $f_1 \colon D \to \mathbb{N}$ as follows: for each $x \in [t_{i-1}, t_i)$, let $f_1(x) = f_0(0)$; for each $x \in (t_i, t_{i+1}]$, let $f_1(x) = g_0(0)$. Application of theorem 7.5.1 yields the internal function G such that $G|D = f_1$. For these physical applications, utilize the term "substratum" in the place of the technical terms "pure nonstandard." [Note: Of course, elsewhere, the term "pure NSP-world" or simply the "NSP-world" is used as a specific name for what has here been declared as the substratum.] This yields the following statements, where the symbols F_i and F_{i+1} are defined and characterized by the expression inside the quotation marks in property (**P**).

> (**A**): There exists a substratum hypercontinuous, hypersmooth, hyperuniform process G that binds together F_i and F_{i+1}.
>
> (**B**): There exists a substratum hypercontinuous, hypersmooth, hyperuniform alteration process G that transforms F_i into F_{i+1}.
>
> (**C**): There exists an ultracontinuous subtle force-like (i.e. deductive) process that yields F_i for each time t_i within the development of the natural system.

In order to justify (**A**) and (**B**), specific measures of physical properties associated with constituents may be coupled together. Assume that for a subword $\mathrm{r}_i \in F_i \in \mathrm{d}$, the symbols r_i denote a numerical quantity that aids in characterizing the behavior of an object in a system S_1 or the system itself. Let ($\mathbf{M}_1$) be the statement:

> "There exists a substratum hypercontinuous, hypersmooth, hyperuniform functional process G_i such that G_i when restricted to the standard mathematical domain it is f_i and such that G_i hypercontinuously changes r_i for system S_1 at time t_i into r_{i+1} for system S_1 at time t_{i+1}."

This modeling procedure yields the following interpretation:

> (**D**) If there exists a continuous or uniform [resp. discrete] functional process f_i that changes r_i for S_1 at time t_i into r_{i+1} for S_1 at time t_{i+1}, then ($\mathbf{M}_1$).

At a particular moment t_i, two natural systems S_1 and S_2 may interface. More generally, two very distinct developmental paradigms may exist one d_1 at times prior to t_i (in the t_i past) and one d_2 at time after t_i (in the t_i future). We might refer to the time t_i as a *standard time fracture.* Consider the developmental paradigm $\mathrm{d}_3 = \mathrm{d}_1 \cup \mathrm{d}_2$. In this case, the paradigms may be either of type d or d'. For the type d', the corresponding system need not be considered a natural system but could be a pure substratum system.

At t_i an $\mathrm{F}_i \in \mathrm{d}_3$ can be characterized by statement (**P**) (with the term natural removed if F_i is a member of a d'). In like manner, F_{i+1} at time t_{i+1} can be characterized by (**P**). Statements (**A**), (**B**), (**C**) can now be applied to d_3 and a modified statement (**D**), where the second symbol string S_1 is changed to S_2. Notice that this modeling applies to the actual human ability that only allows for two discrete descriptions to be given, one for the interval $[t_{i-1}, t_i)$ and one for the interval $(t_i, t_{i+1}]$. From the modeling viewpoint, this is often sufficient since the length of the time intervals can be made smaller than Planck time.

Recall that an analysis of the scientific method used in the investigation of natural system should take place exterior to the language used to describe the specific system development. Suppose that $\mathcal{D}$ is the language accepted for a scientific discipline and that within $\mathcal{D}$ various expressions from mathematical theories are used. Further, suppose that enough of the modern theory of sets is employed so that the EGS can be constructed. The following statement would hold true for $\mathcal{D}$.

> *If by application of first-order logic to a set of non-mathematical premises taken from $\mathcal{D}$ it is claimed that it is not logically possible for statements such as* (**A**), (**B**), (**C**) *and* (**D**) *to hold, then the set of premises is inconsistent.*

CHAPTER 7 REFERENCES

1 Beltrametti, E. G.and G. Cassinelli, The logic of quantum mechanics, in *Encyclopedia of Mathematics and its Application*, Vol. 15, Addison-Wesley, Reading, 1981.

2 Herrmann, R. A., *Some Characteristics of Topologies on Subsets of a Power Set*, University Microfilm, M-1469, 1968.

3 Herrmann, R. A., *Einstein Corrected,* (1993) I. M. P., P. O. Box 3268, Annapolis, MD 21403-0268

NOTES

[1] It is obvious that the concept of "time" need not be the underlying interpretation for these intervals. Time simply refers to an external event ordering concept. For other purposes, simply call these

intervals "event intervals." In the above descriptions for W_i, simply replace "time|||interval" with "event|||interval" and replace the second instance of the word "time" with the word "event." If this interpretation is made, then other compatible interpretations would be necessary when applying a few of the following results. Also, the partition points t_i can be notationally refined if more then one interval is being considered.

[2] Note that this last requirement for B can be achieved as follows: construct a special symbol not originally in $\mathcal{A}$. Then this symbol along with $\mathcal{A}$ is considered the alphabet. Next only consider a B that does not contain this special symbol within any of its members. Now using this special symbol in place of the $\wedge$, consistently construct $\mathrm{M}_i,\ i \neq 0$. Of course, $\wedge$ is interpreted as this special symbol in the axiom system S.

[3] The actual members, F_i, of a developmental paradigm d need not be unique. However, the specific information contained in each readable word used for a specific $\mathrm{F}_i \in \mathbb{N}$ is unique. Other readable sentences can be used in place of a specific F_i as long as they are "equivalent" in the sense that the specific information being displayed by each is the same information.

[4] Depending upon the application, a single standard word may also be termed as an ultraword.

[5] For a new more refined method to obtain an ultraword for a refined developmental paradigm, see pages 4-7 at http://arxiv.org/abs/math/0605120

[6] (Added 9/20/2009.) A concurrent relation is not needed to obtain important "ultrawords."

Theorem 7.3.5 *For* $\mathbf{d} = \{\mathbf{F}_n \mid n \in \mathbb{N}\}$ *and each infinite* $\lambda \in \mathbb{N}_\infty$, *there exists one and only one* $w_\lambda \in {}^*\mathbf{M}_\lambda$ *and hyperfinite* d_λ *such that* $\mathbf{d} \subset d_\lambda \subset {}^*\mathbf{S}(\{w_\lambda\})$, *and* $d_\lambda \subset {}^*\mathbf{d}$.

Proof. For each, $n \in \mathbb{N}$, let $G(n) = \{\mathbf{F}_i \mid 0 \leq i \leq n\} \subset \mathbf{d}$. Thus, $G\colon \mathbb{N} \to \mathcal{F}(\mathbf{d})$ the set of all finite subsets of $\mathbf{d}$. Let $n > 0$. Then $\mathbf{M}_n$ has one and only one member and by definition $\mathbf{w}_n \in \mathbf{M}_n$ has the property that $G(n) \subset \mathbf{S}(\{\mathbf{w}_n\})$. Hence, by *-transfer, for the function ${}^*\mathbf{M}$, and each $\lambda \in \mathbb{N}_\infty$, there is one and only one $w_\lambda \in {}^*\mathbf{M}_\lambda$ such that hyperfinite ${}^*G(\lambda) \subset {}^*\mathbf{S}(\{w_\lambda\})$. Finally, by defintion of G, $\mathbf{d} \subset {}^*G(\lambda) \subset {}^*\mathbf{d}$. ▮

Note that theorems that generate or use ultrawords may need to be trivially modified depending upon the definition for d.

Chapter 8

A SPECIAL APPLICATION

8.1 A Neutron Altering Process.

The purpose of this chapter is to justify the interpretations utilized in reference [1]. Let B be the set of all nondecreasing bounded real valued functions defined on $D = [a, t') \cup (t', T]$. Let $\mathcal{Q} \in B$ and $\mathcal{Q}(t) = 2$ for each $t \in [a, t')$; $\mathcal{Q}(t) = 3$ for each $t \in (t', T]$ be the *discrete neutron altering process*. Application of Theorem 7.5.1 implies that there exists internal $G\colon {}^*[a, T] \to {}^*\mathbb{R}$ such that $\mathtt{st}(G)|D = G|D = \mathcal{Q}$, and G is hypercontinuous, hypersmooth, hyperaltering process defined on the hyperinterval ${}^*[a, T]$. Hence, G satisfies statement (A) in section (2) of [1]. Theorem 7.5.1 also implies that G is hyperuniformly continuous on ${}^*[a, T]$.

Recall how a *-special partition for ${}^*[a, T]$ is generated. Let $0 < \Delta t \in \mathbb{R}^+$. Then $P(\Delta t) = \{a = t_0 \leq \cdots t_n \leq t_{n+1} = T\}$, where n is the largest natural number such that $a + n(\Delta t) \leq T$ and for $i = 0, \ldots, (n-1)$; $t_{i+1} - t_i = \Delta t$, and $t_{n+1} - t_n = b - (a + n(\Delta t)) < \Delta t$. It is possible that $t_{n+1} = t_n$. If $\mathcal{P}$ is the set of all special partitions, then letting $dt \in \mu(0)^+$ (the set of all positive infinitesimals) it follows that $P(dt) \in {}^*\mathcal{P}$ and $P(dt)$ has the same first-order properties as does $P(\Delta t)$.

Theorem 8.1.1 *Let internal G be hypercontinuous on ${}^*C = {}^*[a, T]$ and $z_j \in (a, T)$, $j = 1, \ldots, m$. For any $dx \in \mu(0)^+$, there exists a $dy \in \mu(0)^+$ such that for each $x,\ y \in {}^*D$ such that $|x - y| < dy$ it follows that $|G(x) - G(y)| < dx$, and there is a hyperfinite partition $\{a = t'_0 < \cdots < t'_{\nu+1} = T\}$ such that for $i = 0, \ldots, \nu + 1$; $j = 1, \ldots, m$ we have $t'_j \neq z_j$, $G(t'_{i+1}) - G(t'_i) \in \mu(0)$, $t'_{i+1} - t'_i \in \mu(0)$ and $G(T) - G(a) = \sum_0^{\nu} (G(t'_{i+1}) - G(t'_i))$.*

Proof. Since internal G is *-uniformly continuous, it follows that for any $dy \in \mu(0)^+$ there exists some δ such that $0 < \delta \in {}^*\mathbb{R}$ and for each $x,\ y \in {}^*[a, T]$ such that $|x - y| < \delta$, it follows that $|G(x) - G(y)| < dx$. Now let $dy < \delta$ and $dy \in \mu(0)^+$ and consider the *-special partition $P(dy/3)$. Let $y \in [t_i, t_{i+1}]$, $x \in [t_{i+1}, t_{i+2}]$, $i = 0, \ldots, \nu - 1$ and $x,\ y \neq t'$. Then $|y - x| < dy$ and each *-closed interval is nonempty. By means of internal first-order statements that imply the existence of certain objects and the choice axiom, select $t'_0 = a$, $t'_{\nu+1} = T$ and if $t_{\nu+1} = t_\nu$, then for $i = 1, \ldots, \nu - 2$ select some $t'_i \in [t_i, t_{i+1}]$ such that $t'_i \neq z_j$ for $j = 1, \ldots, m$; or if $t_{\nu+1} \neq t_\nu$, then for $i = 1, \ldots, \nu - 1$ select some $t'_i \in [t_i, t_{i+1}]$ such that $t'_i \neq z_j$ for $j = 1, \ldots, m$. This yields a hyperfinite internal partition with the properties listed in the hypothesis and by

*-transfer of the properties of a finite telescoping series, we have that $G(T) - G(a) = \sum_0^{\nu}((G(t'_{i+1}) - G(t'_i))$, and $|G(t'_{i+1}) - G(t'_i)| < dx$ for $i = 0, \ldots, \nu$ implies that $G(t'_{i+1}) - G(t'_i) \in \mu(0)$ and each $t'_i \in {}^*D$ has the property that $|t'_{i+1} - t'_i| < dy$. This complete the proof. ▮

We now apply Theorem 8.1.1 to the discrete altering function $\mathcal{Q}$. Let Q be the set of all finite partitions of D. Then, for $n > 0$ and the partition $\{a = t_0 < t_1 < \cdots < t_n \leq t_{n+1} = T\}$, consider the partial sequence $S\colon [0, n+1] \to \mathbb{R}$ defined by $S(i) = t_i,\ i = 0, \ldots, n+1$. Define $T_i = [t_i, t_{i+1}],\ i = 0, \ldots, n$. Consider the set $H = \{T_i \mid i = 0, \ldots, n\}$. Then $H \in \mathcal{P}(C)$, where $C = [a, T]$. There is an $N \in \mathbb{R}^{\mathcal{P}(C) - \emptyset}$ such that $N(T_i) = \mathcal{Q}(t_{i+1)} - \mathcal{Q}(t_i),\ \mathcal{Q}(t_i) = r_i,\ i = 0, \ldots, n$. The function N is a *resolving process* for the function $\mathcal{Q}$ and each r_i is a degree for the *constituent* $N(T_i)$. Let $M(\mathcal{Q})$ be the set of all such resolving processes generated by the infinite set of finite partitions of D for a fixed $\mathcal{Q}$. Consider the *-finite partition $P(dy/3)$ of D generated in the proof of Theorem 8.1.1. Now modify this *-finite partition in the following manner. Consider the standard finite partition generated by $S\colon [0, n+1] \to \mathbb{R}$. Let $T_i = [t_i, t_{i+1}] = [S(i), S(i+1)],\ t_i \neq t',\ i = 0, \ldots, n;\ H = \{T_i \mid i = 0, \ldots, n\}$ and, for the fixed $\mathcal{Q}$, $N(t_i) = \mathcal{Q}(t_{i+1}) - \mathcal{Q}(t_i) = G(t_{i+1}) - G(t_i),\ i = 0, \ldots, n$, where G is in the statement of Theorem 8.1.1. This sequence S extends in the usual manner to ${}^*S\colon [0, \nu + 1] \to {}^*\mathbb{R}$.

Since $0 < |t_i - t_j|$, where $i \neq j$, for each t_i there exists a *-closed interval $[v_j, v_{j+1}]$ generated by $P(dy/3)$ such that $t_i \in (v_j, v_{j+1})$, or $t_i = v_j$ or v_{j+1} not both. In the case that $t_i \in (v_j, v_{j+1})$, the interval is unique. Moreover, there are only finitely many such t_i in the standard partition. Hence for these finitely many real number cases, where $t_i \in (v_j, v_{j+1})$, modify the partition by subdividing $[v_j, v_{j+1}]$ into two intervals $[v_j, t_i] \cup [t_i, v_{j+1}]$. This process can, obviously, be defined by a finite set of first-order statements. This adds an additional finite number of intervals to our hyperfinite partition and yields a partition number $\lambda \in {}^*\mathbb{N}$ to replace ν. Since the infinitesimal length of these adjoined intervals is $< dy/3$, Theorem 8.1.1 still holds with λ replacing ν. This yields an internal sequence $S'\colon [0, \lambda+1] \to {}^*\mathbb{R}$ such that $S'(i) = t'_i$ as defined in the proof of Theorem 8.1.1 with the condition that finitely many of the t'_j correspond to the standard partition elements t_i. (Notice that, for all of this construction, the assumed n is fixed.)

From the above, we have the *-closed intervals $T'_i = [t'_i, t'_{i+1}],\ i = 0, \ldots, \lambda$ as well as the internal $H' = \{z \mid (z \subset {}^*[a, T]) \wedge (\exists i(i \in [0, \lambda]) \wedge \forall x(x \in z \leftrightarrow S'(i) \leq x \leq S'(i+1)))\}$. Obviously, $H' \in {}^*(\mathcal{P}(C))$. Thus there is in ${}^*M(\mathcal{Q})$ an internal hyperresolving process N' such that $N'(T'_i) = {}^*\mathcal{Q}(t'_{i+1}) - {}^*\mathcal{Q}(t'_{i+1}) = G(t'_{i+1}) - G(t_i)$, where

$T' = [S'(i), S'(i+1)] \in H'$ and $i = 0, \ldots, \lambda$.

Technically, it is not true that $H \subset H'$. Thus define the *standard restriction* of N' to N, where N is generated by the standard sequence $S\colon [0, n+1] \to \mathbb{R}$ that is obtained as follows: consider the set $\{S'(i) \mid i = 0, \ldots, \lambda+1\} \cap [a, T] = P_0$. Since P_0 is a finite standard set, it can be ordered by the $<$ of the reals and let $P_0 = \{a = t_0 < t_i < \cdots t_n \leq t_{n+1} = T.\}$ This yields a sequence $S"\colon [0, n+1] \to \mathbb{R},\ S"(i) = t_i,\ i = 0, \ldots, n+1$. Let $S" = S$. Utilizing $S"$, generate the original resolving process N from N'.

Application of Theorem 8.1.1 yields the following description. There exists a hyperpartition (generated by) S' for the hyperinterval ${}^*D$ (since $t' \in$ range S') and S' (generates) the hyperresolution N' for the hyperaltering process G. The hyperresolution N' is defined on the hyperfinitely many internal subintervals of ${}^*[a, T]$ and the range of N' is composed of hyperfinitely many hyperconstitutents $G(t'_{i+1}) - G(t'_i)$ that, by *-transfer of the standard supremum function defined on nonempty finite sets of real numbers, yields a maximum degree among all of the degrees of the hyperconstitutents. This maximum degree is infinitesimal, by Theorem 8.1.1, and since $G(T) - G(a) \in \mathbb{R}^+$ and taking G as nondecreasing, this maximum degree is a positive infinitesimal. By the above restriction process, N is the restriction of N' to the standard world. Consequently, N' satisfies statement (B) in section 4 of [1].

Finally, the length function L defined on the set of all closed intervals extends to the set of all *-closed intervals that are subsets of ${}^*\mathbb{R}$. Then ${}^*L({}^*[a, T]) = T - a = L([a, T])$. Thus (C) of section 4 in [1] holds. (D) in section 4 of [1] follows from the unused conclusions that appear in Theorem 8.1.1, among others.

For the nondecreasing bounded classical neutron altering process $\mathcal{CQ}$, there is assumed to exist a standard smooth function f defined on $[a, t]$ such that $f|D = \mathcal{CQ}$. Now define standard $G\colon [a, T] \to \mathbb{R}$ as follows: let $G_0(t) = f(t),\ t \in [a, t');\ G_1(t) = f(t),\ t \in (t', T]$. Then since $G_0(t_0) \leq G_1(t_1)$, for $t_0 \in [a, t')$ and $t_1 \in (t', T]$, it follows that $h = \sup\{G_0(t) \mid t \in [a, t')\}$ exists, and we can let $G = G_0 \cup G_1 \cup \{(t', h)\}$. Obviously, $G(t_0) \leq G(t') \leq G(t_1),\ t_0 \in [a, t'),\ t_1 \in (t', T]$, and $G|D = \mathcal{CQ}$. It follows from left and right limit considerations that $G = f$. (Note: G is defined in this manner only to conform to the discrete case.) Theorem 8.1.1 holds for ${}^*G$ and, in this case, we simply repeat the entire discussion that appears after that statement of Theorem 8.1.1 and replace the G that appears in that discussion with ${}^*G = {}^*f$. This yields a model for statements (E), (F), (G) and (H) in section 4 of reference [1].

CHAPTER 8 REFERENCE

1 Herrmann, R. A. Mathematical philosophy and developmental processes, *Nature and System,* 5(1/2) (1983), 17—36.

Chapter 9

9. NSP-WORLD ALPHABETS

9.1 An Extension.

Although it is often not necessary, we assume when its useful that we are working within the EGS. Further, this structure is assumed to be $|\mathcal{M}_1|^+$-saturated, and a polyenlargement [4, p. 35], where $\mathcal{M}_1 = \langle \mathcal{N}, \in, = \rangle$, where the ground set is $\mathcal{W}' \cup \mathbb{R}$ or $\mathcal{M}_1 = \langle \mathcal{N}, \in, = \rangle$, where the ground set is $\mathcal{W}' \cup \mathcal{Q}$ and $\mathcal{Q}$ is the set of rational numbers). The set $\mathcal{W}'$ is an extended language. Referring to the paragraph prior to Theorem 7.3.3, it can be assumed that the developmental paradigm $d' \subset {}^*\mathbf{B} \subset {}^*\mathbf{P}_0$. It is not assumed that such a developmental paradigm is obtained from the process discussed in Theorem 7.2.1, although a modification of the proof of Theorem 7.2.1 appears possible in order to allow this method of selection.

Theorem 9.1.1 *Let $d' = \{[g_i] \mid i \in \lambda\},\ |\lambda| < |\mathcal{M}_1|^+$. There exists an ultraword $w \in {}^*\mathbf{M}_B - {}^*\mathbf{B}$ such that for each $i \in \lambda,\ [g_i] \in {}^*\mathbf{S}(\{w\})$.*

Proof. The same as Theorem 7.3.3 with the change in saturation.▮

Let $\mathcal{D} = \{d_i \mid i \in \lambda\},\ |\lambda| < |\mathcal{M}_1|^+,\ |d_i| < |\mathcal{M}_1|^+$ and each $d_i \subset {}^*\mathbf{B}$ is considered to be a developmental paradigm either of type d or type d'. For each $d_i \in \mathcal{D}$, use the Axiom of Choice to select an ultraword $w_i \in {}^*\mathbf{M}_B - {}^*\mathbf{B}$ that exists by Theorems 9.1.1. Let $\{w_i \mid i \in \lambda\}$ be such a set of ultrawords.

Theorem 9.1.2 *There exists an ultraword $w' \in {}^*\mathbf{M}_B - {}^*\mathbf{B}$ such that for each $i \in \lambda,\ w_i \in {}^*\mathbf{S}(\{w'\})$ and, hence, for each $d_i \in \mathcal{D},\ d_i \subset {}^*\mathbf{S}(\{w'\})$.*

Proof. The same as Theorem 7.3.4 with the change in saturation.▮

9.2 NSP-World Alphabets.

First, recall the following definition. $P_m = \{f \mid (f \in T^m) \land (\exists z((z \in \mathcal{E}) \land (f \in z) \land \forall x((x \in \mathbb{N}) \land (x > m) \to \neg \exists y((y \in T^x) \land (y \in z)))))\}$. The set $T = i[\mathcal{W}]$. The set P_m determines the unique partial sequence $f \in [g] \in \mathcal{E}$ that yields, for each $j \in \mathbb{N}$ such that $0 \leq j \leq m,\ f(j) = i(\mathrm{a})$, where $i(\mathrm{a})$ is an "encoding" in $i[\mathcal{W}]$ of the alphabet symbol "a" used to construct our intuitive language $\mathcal{W}$. The set $[g]$ represents an intuitive word constructed from such an alphabet of symbols.

Within the discipline of Mathematical Logic, it is assumed that there exists symbols — a sequence of variables — each one of which corresponds, in a one-to-one manner, to a natural number. Further, under the subject matter of generalized first-order theories [2], it isassumed that the cardinality of the set of constants is greater than $\aleph_0$. In the forthcoming investigation, it may be useful to consider an alphabet that injectively corresponds to the real numbers $\mathbb{R}$. This yields a new alphabet $\mathcal{A}'$ containing our original alphabet. A new collection of words $\mathcal{W}'$ composed of nonempty finite strings of such alphabet symbols may be constructed. It may also be useful to well-order $\mathbb{R}$. The set $\mathcal{E}$ also exists with respect to the set of words $\mathcal{W}'$. Using the ESG, many previous results in this book now hold with respect to $\mathcal{W}'$ and for the case that we are working in a $|\mathcal{M}_1|^+$-saturated polyenlargement.

With respect to this extended language, if you wish to except the possibility, a definition as to what constitutes a purely subtle alphabet symbol would need to be altered in the obvious fashion. Indeed, for T in the definition of P_m, we need to substitute $T' = i[\mathcal{W}']$. Then the altered definition would read that $r \in {}^*i[\mathcal{W}']$ is a *pure subtle alphabet symbol* if there exists an $m \in \mathbb{N}$ and $f \in {}^*(P_m)$, or if $m \in {}^*\mathbb{N} - \mathbb{N}$ an $f \in P_m$, and some $j \in {}^*\mathbb{R}$ such that $f(j) = r \notin i[\mathcal{W}']$. Notice that if one chooses to use $\mathcal{W}'$, then r corresponds to an $r' \in {}^*\mathcal{W}'$. Further, some of the previous theorems also hold when the proofs are modified.

Although these extended languages are of interest to the mathematician, most of science is content with approximating a real number by means of a rational number. In all that follows, the cardinality of our language, if not denumerable, will be specified. All theorems from this book that are used to establish a result relative to a denumerable language will be stated without qualification. If a theorem has not been reestablished for a higher language but can be so reestablished, then the theorem will be termed an *extended* theorem.

9.3 General Paradigms.

There is the developmental paradigm, and for nondetailed descriptions the *general developmental paradigm.* But now we have something totally new — the *general paradigm.* It is important to note that the general paradigm is considered to be distinct from developmental paradigms, although certain results that hold for general paradigms will hold for developmental paradigms and conversely. For example, associated with each general paradigm $\mathrm{G_A}$ is an ultraword w_g such that the set $\mathbf{G_A} \subset {}^*\mathbf{S}(\{w_g\})$ and all other theorems relative to such ultrawords hold for general paradigms. The general paradigm is a collection of words that discuss, in general, the behavior of entities and other constituents of a natural system. They, usually, do not contain a time statement W_i as it appears in section 7.1 for developmental paradigm descriptions. Our interest in this section is relative to only two such general paradigms. The reader can easily generate many other general paradigms.

The formal language is the usual first-order set-theoretic language with variables and constants. And, as used throughout, $\mathcal{W}'$ is a set of words formed by an alphabet, where if $w \in \mathcal{W}'$ there is no set a in our structure such that $a \in w$. Obviously, if infinite $\mathcal{W}'$ is not denumerable, then the modified Robinson approach is the most appropriate, relative to the nonstandard language. Depending upon the application, the alphabet is assumed to have symbols for informal mathematical entities. Thus, there are mostly two mathematically styled languages, the symbolic language N, which is part of the "object language" that denotes the informal natural numbers considered as constants and the formal natural number $\mathbb{N}$ used to analyze the language. In the form of constants, members of N, only have, as previously defined, intuitive meaning. This allows one, as done by Robinson, to consider formal relations that tend to characterize the intuitive meanings.

Consider the symbol c' and let $\mathrm{n}' \in \mathrm{N}$. These symbols form a denumerable subset of $\mathcal{W}'$. The symbol $0' \notin \mathrm{N}$. These symbols are considered as alphabet members and correspond to constants that further correspond to the nonzero natural numbers. Hence, as set-theoretic entities $\mathrm{N} \subset \mathcal{W}'$. In what follows, the intended alphabet symbols are employed as constants of the formal first-order language. The formal mathematical structure also has the usual array of constants that denote the members. [Note: Within some of my papers on this subject you may find the notation $\underline{\mathcal{W}}$ or $\underline{\mathcal{W}'}$. Although these symbols usually indicate the set of equivalences cases without the coding i, this notation may also be used to represent either of the equivalence class representations.]

Now consider the following informally defined set of words. Of course, in the extended case, it can be assumed that the cardinality of $\mathcal{W}'$ is no greater than that of $\mathbb{R}$. It should be noted that the members of $\mathrm{G_A}$ are but linguistic forms that do, at least, partially have meaning when interpreted physically. Due to the possible non-countable cardinality of $\mathcal{W}'$, the modified Robinson approach is employed in what follows.

$$\begin{aligned}\mathrm{G_A} = \{&\mathrm{An}|||\mathrm{elementary}|||\mathrm{particle}|||\alpha(\mathrm{n}')|||\mathrm{with}|||\\ &\mathrm{kinetic}|||\mathrm{energy}|||\mathrm{c}'{+}1/(\mathrm{n}').\mid n' \in \mathrm{N}\}\end{aligned} \tag{9.3.1}$$

Of particular interest is the composition of members of

${}^*\mathbf{G_A} - \mathbf{G_A}$.

Theorem 9.3.1 *A set* $[g] \in {}^*\mathbf{G_A} - \mathbf{G_A}$ *if and only if there exists a* $f \in {}^*(P_{55})$ *and a nonstandard* $\nu \in {}^*\mathrm{N} - \mathrm{N}$ *such that* $f \in [g]$, *and* $f(55) = \mathrm{A},\ f(54) = \mathrm{n},\ f(53) = |||, \cdots, f(30) = f(2), \cdots, f(3) = (, f(2) = \nu,\ f(1) =),\ f(0) = .$

Proof. From the definition of $\mathrm{G_A}$ the sentences

$$\forall z((z \in \mathbf{G_A}) \to \exists x \exists w((w \in \mathrm{N}) \land (x \in P_{55}) \land (x \in z) \land$$
$$((55, \mathrm{A}) \in x) \land ((54, \mathrm{n}) \in x) \ \land \cdots \land (x(30) = x(2)) \land \cdots \land$$
$$((3, () \in x) \land (x(2) = w) \land$$
$$((1,)) \in x) \land ((0, .) \in x))). \tag{9.3.2}$$
$$\forall x \forall w((x \in P_{55}) \land (w \in \mathrm{N}) \land$$
$$((55, \mathrm{A}) \in x) \land ((54, \mathrm{n}) \in x) \ \land \cdots \land (x(30) = x(2)) \land \cdots \land$$
$$((3, () \in x) \land (x(2) = w) \land ((1,)) \in x) \land ((0, .) \in x) \to$$
$$\exists z((z \in \mathbf{G_A}) \land (x \in z))).$$

hold in $\mathcal{M}$, hence in ${}^*\mathcal{M}$. There is in the standard structure bijection $j[\mathrm{N}] = \mathbb{N}'$. Hence, bijection ${}^*j[{}^*\mathrm{N}] = {}^*\mathbb{N}'$. Consequently ${}^*j[{}^*\mathrm{N} - \mathrm{N}] = {}^*\mathbb{N} - \mathbb{N}$. Since ${}^*j[\mathrm{N}] = j[\mathrm{N}]$ under our notational convention, where, for atoms a, ${}^*a = a$, then there is a nonstandard $\nu \in {}^*\mathrm{N} - \mathrm{N}$ that satisfies the *-transformed statements 9.3.2 for a $[g] \in {}^*\mathbf{G_A} - \mathbf{G_A}$, where internal partial sequence $f \in [g]$ is the member that characterizes the alphabet members and, thus, also varies over members of ${}^*\mathrm{N} - \mathrm{N}$ for $f(2)$ and $f(2) = f(30)$. ▮

Using Theorem 9.3.1, each member of ${}^*\mathbf{G_A} - \mathbf{G_A}$, when interpreted, has only two positions with a single missing standard object since positions 30 and 2 do not correspond to any symbol string in our language $\mathcal{W}'$. This interpretation still retains a vast amount of content, however. The members of ${}^*\mathbb{N} - \mathbb{N} = \mathbb{N}_\infty$ correspond to the infinite Robinson numbers. Thus, considering "new" constant symbols not used in our language, such as $\wr$, and let them denote infinite numbers, we have symbolic forms such as

$$\mathrm{G'_A} = \mathrm{An}|||\mathrm{elementary}|||\mathrm{particle}|||\alpha(\wr)|||\mathrm{with}|||$$
$$\mathrm{kinetic}|||\mathrm{energy}|||\mathrm{c}'{+}1/(\wr). \tag{9.3.3}$$

9.4 Interpretations

Recall that the Natural world portion of the NSP-world model may contain *undetectable* objects, where "undetectable" means that there does not appear to exist human, or humanly constructible machine sensors that directly detect the objects or directly measure any of the objects physical properties. The rules of the scientific method utilized within the micro-world of subatomic physics allow all such undetectable Natural objects to be accepted as existing in reality.[1] The properties of such objects are indirectly deduced from the observed properties of gross matter. In order to have indirect evidence of the objectively real existence of such objects, such indirectly obtained behavior will usually satisfy a specifically accepted model.

Although the numerical quantities associated with these undetectable Natural (i.e. standard) world objects, if they really do exist, cannot be directly and exactly measured via any known instrumentation, these quantities are still represented by standard mathematical entities. By the rules of correspondence for interpreting pure NSP-world entities, such entities with a property being described by G'_{A} must be considered as undetectable pure NSP-world objects, assuming any of them exist in this background world. On the other hand, physical entities could satisfy this behavior, when viewed from the substratum. The G'_{A} type statements are actually being **predicted by the mathematical method employed**. Consequently, some such measures *may be* assumed to have an indirect affect within the Natural world. The predicted measure $1/\wr$ is that of an infinitesimal. From a substratum viewpoint, when c' is interpreted as the $0 \in \mathbb{N}$, it rationally verifies a stance original held by Newton that such measures are "real" as well as a remark by Robinson that such measures may be of significance in the world of particle physics.

The concept of *realism* often dictates that all interpreted members of a mathematical model be considered as existing in reality. The philosophy of science that accepts only *partial realism* allows for the following technique. One can stop at any point within a mathematically generated physical interpretation. Then proceed from that point to deduce an intuitive physical theory, but only using other not interpreted mathematical formalism as auxiliary constructs or as catalysts. Entities having such infinitesimal measures could be restricted to the substratum. Or, as mentioned, they could be physical entities that exhibit such behavior only when viewed from the substratum. With respect to the NSP-world, another aspect of this interpretation enters the picture. Assuming realism, then the question remains which, if any, entity with infinitesimal behavior actually indirectly influences Natural world processes? Partial realism allows for the possibility that none of these pure NSP-world measures has any affect upon the standard world. These ideas should always be kept in mind.

If you accept that such particle measures as described by G_{A} can exist in reality, then the philosophy of realism leads to the next interpretation.

(1) *If there exists an elementary particle with Natural system behavior described by* $\mathrm{G_A}$, *then there exists an entity that displays the behavior described by statement* $\mathrm{G'_A}$.

The concept of absolute realism would require that the acceptance of entities with behavior described by $\mathrm{G_A}$ is indirect evidence for the existence of the $\mathrm{G'_A}$ described behavior. I caution the reader that the interpretation we apply to such sets of sentences as $\mathrm{G_A}$ are only to be applied to such sets of sentences.

The EGS may, of course, be interpreted in infinitely many different ways. Indeed, the NSP-world model with its physical-type language can also be applied in infinitely many ways to infinitely many scenarios. I have applied it to such models as the MA-model and the GGU-model among others. In this section, I consider another possible interpretation relative to those Big Bang cosmologies that postulate real objects at or near infinite temperature, energy or pressure. These theories incorporate the concept of the *initial singularity(ies).*

One of the great difficulties with many Big Bang cosmologies is that no meaningful physical interpretation for formation of the initial singularity is forthcoming from the theory itself. The fact that a proper and acceptable theory for creation of the universe requires that consideration not only be given to the moment of zero cosmic time but to what might have occurred "prior" to that moment in the nontime period is what partially influenced Wheeler to consider the concept of a *pregeometry.*[3], [3] It is totally unsatisfactory to dismiss such questions as "unmeaningful" simply because they cannot be discussed in your favorite theory. Scientists must search for a broader theory to include not only the question but a possible answer.

Although the initial singularity for a Big Bang type of state of affairs apparently cannot be discussed in a meaningful manner by many standard physical theories, unless one adjoins to the theory an ad hoc quantum field, it can be discussed by application of our NSP-world language. Let c' be a symbol that represents any fixed real number. Define

$$\begin{aligned}\mathrm{G_B} = \{&\mathrm{An|||elementary|||particle|||}\alpha(\mathrm{n'})\mathrm{|||with|||}\\ &\mathrm{total|||energy|||c'{+}n'}.\mid n\in \mathbb{N}\},\end{aligned}\tag{9.4.1}$$

Application of Theorem 9.3.1 to $\mathrm{G_B}$ yields the form

$$\begin{aligned}\mathrm{G'_B} = &\mathrm{An|||elementary|||particle|||}\alpha(\zeta')\mathrm{|||with|||}\\ &\mathrm{total|||energy|||c'{+}}\wr\end{aligned}\tag{9.4.2}$$

(2) *If there exist an elementary particle with Natural*

system behavior described by $\mathrm{G_B}$, *then there exist an entity that displays behavior described by* $\mathrm{G'_B}$.

The entities being described by $\mathrm{G'_B}$ have infinite energy. This infinite energy **does not** behave in the same manner as would the real number energy measures discussed in $\mathrm{G_B}$. As is usual when a metalanguage physical theory is generated from a formalism, we can further extend and investigate the properties of $\mathrm{G'_B}$ described entities by imposing upon them the corresponding behavior of the positive infinite hyperreal numbers. This produces some interesting propositions. Hence, we are able to use a nonstandard physical world language in order to give further insight into the state of affairs at or near a cosmic initial singularity. This gives *one* solution to a portion of the pregeometry problem. I point out that there are other NSP-world models for the beginnings of our universe, if there was such a beginning. Of course, the statement $\mathrm{G'_B}$ need not be related at all to any Natural world physical scenario, but could refer only to the behavior of pure NSP-world entities.

Notice that Theorems such as 7.3.1 and 7.3.4 relative to the generation of developmental paradigms by ultrawords, also apply to general paradigms, where $\mathrm{M}, \mathrm{M_B}, \mathrm{P_0}$ are defined appropriately. The following is a slight extension of Theorem 7.3.2 for general paradigms. Theorem 9.4.1 will also hold for developmental paradigms.

Theorem 9.4.1 *Let* $\mathrm{G_C}$ *be any denumerable general paradigm. Then there exists an ultraword* $w \in {}^*\mathbf{P}_0$ *such that for each* $\mathbf{F} \in \mathbf{G_C}$, $\mathbf{F} \in {}^*\mathbf{S}(\{w\})$ *and there exist infinitely many* $[g] \in {}^*\mathbf{G_C} - \mathbf{G_C}$ *such that* $[g] \in {}^*\mathbf{S}(\{w\})$.

Proof. In the proof of Theorem 7.3.2, it is shown that there exists some $\nu \in {}^*\mathbb{N} - \mathbb{N}$ and a bijection h such that ${}^*h[[0,\nu]] \subset {}^*\mathbf{S}(\{w\})$ and ${}^*h[[0,\nu]] \subset {}^*\mathbf{G_C}$. Since $|{}^*h[[0,\nu]]| \geq |\mathcal{M}_1|^+$, then $|{}^*h[[0,\nu]] - h[\mathbb{N}]| \geq |\mathcal{M}_1|^+$. This completes the proof. ▮

Corollary 9.4.1.1 *Theorem 9.4.1 holds, where* $\mathrm{G_C}$ *is replaced by a developmental paradigm.*

(3) *Let* $\mathrm{G_C}$ *be a denumerable general paradigm. There exists an intrinsic ultranatural process*, ${}^*\mathbf{S}$, *such that objects described by members of* $\mathrm{G_C}$ *are produced by* ${}^*\mathbf{S}$. *During this production, numerously many pure NSP-objects as described by statements in* ${}^*\mathbf{G_C} - \mathbf{G_C}$ *are produced.*

9.5 A Barrier To Knowledge.

Our final discussion in this chapter deals with the use of $|\mathcal{M}|^+$-saturated models and our ability to analyze sets of sentences such as

G'_{A}.

Each of our previous investigations is done with respect to a specific NSP-world structure ${}^*\mathcal{M}$ based upon a infinite standard set H (with a cardinality usually equal to $\aleph_0$) into which is mapped the symbols and words for all languages. The requirement that H be a standard set is relative to the standard universe in which we function. Although there are infinitely many distinct nonisomorphic NSP-world structures, each of our results is with respect to members of a subclass of the class of all such structures. In particular, $|\mathcal{M}|^+$-saturated polyenlargement , where $\mathcal{M}$ is based upon a standard set H, where $\mathbb{N} \subset H$.

In order to analyze general paradigms $\mathrm{G}'_{\mathrm{A}},\ \mathrm{G}'_{\mathrm{B}}$ and the like, we need to start, I believe, with a comprehensible set of sentences, such as $\mathrm{G}_{\mathrm{A}},\ \mathrm{G}_{\mathrm{B}}$, with nonempty content and insert new symbols but retain some of the content of the original sentences. What is shown next is that if we use any of our models based on H and require them to be $|\mathcal{M}|^+$-saturation polyenlargement, then we cannot embed our new alphabet into the standard set H and, thus, we cannot fully analyze sets of sentences such as $\mathrm{G}'_{\mathrm{A}},\ \mathrm{G}'_{\mathrm{B}}$ using our embedding procedures.

Theorem 9.5.1 *Let Γ' be a set of symbols adjoined to a countable alphabet $\mathcal{A}$, which is disjoint from $\mathcal{A}$, and such that it is used to obtain the set of sentences in G'_{B}. Let ${}^*\mathcal{M} = \langle {}^*\mathcal{H}, \in, = \rangle$ be any $|\mathcal{M}|^+$-saturated polyenlargement of a superstructure based on the ground set H, where here $\mathbb{N} \subset H \subset \mathbb{R}$. There does does not exist an injection from $\Gamma' \cup \mathcal{A}$ into $\mathbf{L}$, where $\mathbf{L} \in \mathcal{H}$.*

Proof. Suppose that there exists an injection $i\colon (\Gamma' \cup \mathcal{A}) \to \mathbf{L}$. Since the model is a polyenlargement, then $|\Gamma' \cup \mathcal{A}| \geq |\mathcal{M}|^+$. However, $|\mathbf{L}| < |\mathcal{M}|^+$. But under the assumption $|\Gamma' \cup \mathcal{A}| \leq |\mathbf{L}|$. This contradiction implies that the injection does not exist and this completes the proof. ▮

CHAPTER 9 REFERENCES

1 Evans, R., *The Atomic Nucleus*, McGraw-Hill, New York, 1955.

2 Mendelson, E., *Introduction to Mathematical Logic*, Ed. 2, D. Van Nostrand Co., New York, 1979.

3 Patton C. and A. Wheeler, Is physics legislated by cosmogony? in *Quantum Gravity*, ed. Isham, Penrose and Sciama, Oxford University Press, Oxford, 1977, 538—605.

4 Stroyan, K. D. and J. M. Bayod, *Foundations of Infinitesimal Stochastic Analysis*, North Holland, New York, 1986.