SOME DEFINITE INTEGRALS OVER A POWER MULTIPLIED BY FOUR MODIFIED BESSEL FUNCTIONS

RICHARD J. MATHAR

ABSTRACT. The definite integrals $\int_0^\infty x^j I_0^s(x) I_1^t(x) K_0^u(x) K_1^v(x) dx$ are considered for non-negative integer j and four integer exponents $s + t + u + v = 4$, where I and K are Modified Bessel Functions. There are essentially 15 types of the 4-fold product of which 10 are discussed here. Partial integration of each of these types leads to correlations between these integrals. The main result are (forward) recurrences of the integrals with respect to the exponent j.

1. 15 QUADRUPLE PRODUCTS

1.1. **Classification.** The definite integrals of the form $\int_0^\infty x^j I_0^s(x) I_1^t(x) K_0^u(x) K_1^v(x) dx$ with integer exponents require $s + t \leq u + v$ to converge at the upper limit, which leads to 22 4-tuples of the four integers s, t, u, and v if we exclusively consider products of 4 Bessel Functions, $s + t + u + v = 4$. A further reduction of the set of the four integers s, t, u, and v to 15 cases is achieved, if any product $I_0(x) K_1(x)$ is substituted by [1, 9.6.15]

$$(1) \qquad I_0(x) K_1(x) = \frac{1}{x} - I_1(x) K_0(x).$$

Out of these 15, the 5 cases with exponents $(s, t, u, v) = (0, 2, 0, 2)$, $(0, 2, 1, 1)$, $(0, 2, 2, 0)$, $(1, 1, 2, 0)$, $(2, 0, 2, 0)$ are not considered here, but only the 10 cases $(0, 0, 0, 4)$, $(0, 0, 1, 3)$, $(0, 0, 2, 2)$, $(0, 0, 3, 1)$, $(0, 0, 4, 0)$, $(0, 1, 0, 3)$, $(0, 1, 1, 2)$, $(0, 1, 2, 1)$, $(0, 1, 3, 0)$, and $(1, 0, 3, 0)$.

To converge at the lower limit, the exponent of the power must be $j + t - v \geq 0$ according to the known limits.

1.2. **Reduction Strategy.** The generic approach in this manuscript to correlate integrals over powers and the Bessel Functions is to rise the exponent in the power and to differentiate the Bessel Functions by partial integration. The differentiation involves the product rule and [1, 9.6.26, 9.6.27]

$$(2) \qquad I_0'(x) = I_1(x),$$
$$(3) \qquad K_0'(x) = -K_1(x);$$

$$(4) \qquad I_1'(x) = I_0(x) - \frac{I_1(x)}{x},$$

$$(5) \qquad K_1'(x) = -K_0(x) - \frac{K_1(x)}{x}.$$

We only deal with integrals in the limits between zero and infinity: the limits 0 and ∞ are usually omitted; the pre-integrated term will be indicated by triple dots to indicate if it becomes irrelevant when it vanishes once these limits are inserted. If a product $I_0(x)K_1(x)$ appears, it will be substituted via (1) to keep the types of integrals at a minimum.

The argument of the Bessel Functions is always (x) and omitted for brevity.

2. Mixed Quadruple Products

The first type of integrals—considered in Section 2—contains factors of both types of Modified Bessel Functions, I and K.

2.1. Class $I_0K_0^3$.

$$(6) \quad \int x^j I_0 K_0^3 dx = \ldots - \int \frac{x^{j+1}}{j+1}[I_1 K_0^3 - 3I_0 K_0^2 K_1]dx$$

$$= \ldots - \int \frac{x^{j+1}}{j+1} I_1 K_0^3 dx + 3\int \frac{x^{j+1}}{j+1} I_0 K_0^2 K_1 dx$$

$$= \ldots - \int \frac{x^{j+1}}{j+1} I_1 K_0^3 dx + 3\int \frac{x^{j+1}}{j+1}(1/x - I_1 K_0)K_0^2 dx$$

$$= \ldots - \int \frac{x^{j+1}}{j+1} I_1 K_0^3 dx + 3\int \frac{x^j}{j+1} K_0^2 dx - 3\int \frac{x^{j+1}}{j+1} I_1 K_0^3 dx$$

$$= \ldots + \frac{3 \times 2^{j-2}}{j!(j+1)}\Gamma^4(\frac{1+j}{2}) - \frac{4}{1+j}\int x^{j+1} I_1 K_0^3 dx.$$

We have replaced the integral over two Bessel Functions in the last step with [5, 6.576.4]:

Definition 1.

$$(7) \qquad \int_0^\infty x^\lambda K_0(x)^2 dx = 2^{\lambda-2}\frac{1}{\Gamma(1+\lambda)}\Gamma^4(\frac{1+\lambda}{2}) \equiv \Gamma_{00}(j).$$

The first few special cases are

(8)
$$\Gamma_{00}(0) = \pi^2/4; \quad \Gamma_{00}(1) = 1/2; \quad \Gamma_{00}(2) = \pi^2/32; \quad \Gamma_{00}(3) = 1/3; \quad \Gamma_{00}(4) = 27\pi^2/512.$$

2.2. Class $I_1K_0^3$.

$$(9) \quad \int x^j I_1 K_0^3 dx = \ldots - \int \frac{x^{j+1}}{j+1}[I_0 K_0^3 - \frac{1}{x}I_1 K_0^3 - 3I_1 K_0^2 K_1]dx$$

$$= \ldots + \int \frac{x^j}{j+1} I_1 K_0^3 dx - \int \frac{x^{j+1}}{j+1}[I_0 K_0^3 - 3I_1 K_0^2 K_1]dx.$$

Whenever an integral re-appears on the right hand side we combine it with the left hand side:

$$(10) \qquad \frac{j}{j+1}\int x^j I_1 K_0^3 dx = \ldots - \int \frac{x^{j+1}}{j+1} I_0 K_0^3 dx + 3\int \frac{x^{j+1}}{j+1} I_1 K_0^2 K_1 dx.$$

$$(11) \qquad j\int x^j I_1 K_0^3 dx = \ldots - \int x^{j+1} I_0 K_0^3 dx + 3\int x^{j+1} I_1 K_0^2 K_1 dx.$$

2.3. Class $I_1 K_0^2 K_1$.

(12)
$$\int x^j I_1 K_0^2 K_1 dx = \ldots - \int \frac{x^{j+1}}{j+1}[I_0 K_0^2 K_1 - \frac{2}{x} I_1 K_0^2 K_1 - 2I_1 K_0 K_1^2 - I_1 K_0^3]dx$$

$$= \ldots - \int \frac{x^{j+1}}{j+1}[(1/x - I_1 K_0)K_0^2 - \frac{2}{x}I_1 K_0^2 K_1 - 2I_1 K_0 K_1^2 - I_1 K_0^3]dx$$

$$= \ldots - \int \frac{x^{j+1}}{j+1}[1/x K_0^2 - \frac{2}{x} I_1 K_0^2 K_1]dx - \int \frac{x^{j+1}}{j+1}[-I_1 K_0 K_0^2 - 2I_1 K_0 K_1^2 - I_1 K_0^3]dx$$

$$= \ldots - \int \frac{x^j}{j+1}K_0^2 dx + 2\int \frac{x^j}{j+1}I_1 K_0^2 K_1 dx + 2\int \frac{x^{j+1}}{j+1}[I_1 K_0^3 + I_1 K_0 K_1^2]dx.$$

Merging common integrals of both sides yields

(13) $$\frac{j-1}{j+1}\int x^j I_1 K_0^2 K_1 dx = \ldots - \int \frac{x^j}{j+1}K_0^2 dx + 2\int \frac{x^{j+1}}{j+1}[I_1 K_0^3 + I_1 K_0 K_1^2]dx.$$

Then with (7)
(14)
$$(j-1)\int x^j I_1 K_0^2 K_1 dx = \ldots - 2^{j-2}\frac{1}{j!}\Gamma^4(\frac{1+j}{2}) + 2\int x^{j+1} I_1 K_0^3 dx + 2\int x^{j+1} I_1 K_0 K_1^2 dx.$$

2.4. Class $I_1 K_0 K_1^2$.

(15)
$$\int x^j I_1 K_0 K_1^2 dx = \ldots - \int \frac{x^{j+1}}{j+1}[I_0 K_0 K_1^2 - \frac{3}{x}I_1 K_0 K_1^2 - I_1 K_1^3 - 2I_1 K_0^2 K_1]dx$$

$$= \ldots - \int \frac{x^{j+1}}{j+1}[(\frac{1}{x} - I_1 K_0)K_0 K_1 - \frac{3}{x}I_1 K_0 K_1^2 - I_1 K_1^3 - 2I_1 K_0^2 K_1]dx$$

$$= \ldots - \int \frac{x^j}{j+1}[K_0 K_1 - 3I_1 K_0 K_1^2]dx + \int \frac{x^{j+1}}{j+1}[I_1 K_1^3 + 3I_1 K_0^2 K_1]dx.$$

Merging common integrals yields
(16)
$$\frac{j-2}{j+1}\int x^j I_1 K_0 K_1^2 dx = \ldots - \int \frac{x^j}{j+1}K_0 K_1 dx + \int \frac{x^{j+1}}{j+1}[I_1 K_1^3 + 3I_1 K_0^2 K_1]dx.$$

With the notation [5, 6.576.4]

Definition 2.

(17) $$\int_0^\infty x^j K_0 K_1 dx = 2^{j-4}\frac{j}{(j-1)!}\Gamma^4(\frac{j}{2}) \equiv \Gamma_{01}(j), \quad j > 0$$

this simplifies to
(18)
$$(j-2)\int x^j I_1 K_0 K_1^2 dx = \ldots - 2^{j-4}\frac{j}{(j-1)!}\Gamma^4(\frac{j}{2}) + \int x^{j+1} I_1 K_1^3 dx + 3\int x^{j+1} I_1 K_0^2 K_1 dx.$$

The first few values of the constants defined above are

(19) $$\Gamma_{01}(1) = \pi^2/8; \quad \Gamma_{01}(2) = 1/2; \quad \Gamma_{01}(3) = 3\pi^2/64; \quad \Gamma_{01}(3) = 2/3.$$

2.5. **Class $I_1 K_1^3$.**

(20) $\quad \displaystyle\int x^j I_1 K_1^3 dx = \ldots - \int \frac{x^{j+1}}{j+1}[I_0 K_1^3 - \frac{4}{x} I_1 K_1^3 - 3 I_1 K_0 K_1^2]dx$

$\displaystyle = \ldots - \int \frac{x^{j+1}}{j+1}[(\frac{1}{x} - I_1 K_0)K_1^2 - \frac{4}{x} I_1 K_1^3 - 3 I_1 K_0 K_1^2]dx$

$\displaystyle = \ldots - \int \frac{x^{j+1}}{j+1}[\frac{1}{x}K_1^2 - \frac{4}{x} I_1 K_1^3]dx - \int \frac{x^{j+1}}{j+1}[-I_1 K_0 K_1^2 - 3 I_1 K_0 K_1^2]dx$

$\displaystyle = \ldots - \int \frac{x^j}{j+1}K_1^2 dx + 4\int \frac{x^j}{j+1} I_1 K_1^3 dx + 4\int \frac{x^{j+1}}{j+1} I_1 K_0 K_1^2 dx.$

Merging common integrals yields

(21) $\quad \displaystyle\frac{j-3}{j+1}\int x^j I_1 K_1^3 dx = \ldots - \int \frac{x^j}{j+1}K_1^2 dx + 4\int \frac{x^{j+1}}{j+1} I_1 K_0 K_1^2 dx.$

Definition 3. [5, 6.576]
(22)
$$\int x^j K_1(x)^2 dx = 2^{j-6}\frac{(j^2-1)(j-1)}{(j-2)!j}\Gamma^4(\frac{j-1}{2}) \equiv \Gamma_{11}(j) = \frac{1+j}{j-1}\Gamma_{00}(j), \quad j > 1.$$

(23)
$$(j-3)\int x^j I_1 K_1^3 dx = \ldots - 2^{j-6}\frac{(j^2-1)(j-1)}{(j-2)!j}\Gamma^4(\frac{j-1}{2}) + 4\int x^{j+1} I_1 K_0 K_1^2 dx.$$

2.6. **Synthesis.** The partial integrations of the 5 classes of the previous sections are summarized by the matrix equation
(24)
$$\begin{pmatrix} \int x^j I_0 K_0^3 \\ \int x^j I_1 K_0^3 \\ \int x^j I_1 K_0^2 K_1 \\ \int x^j I_1 K_0 K_1^2 \\ \int x^j I_1 K_1^3 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{4}{1+j} & 0 & 0 & 0 \\ -\frac{1}{j} & 0 & \frac{3}{j} & 0 & 0 \\ 0 & \frac{2}{j-1} & 0 & \frac{2}{j-1} & 0 \\ 0 & 0 & \frac{3}{j-2} & 0 & \frac{1}{j-2} \\ 0 & 0 & 0 & \frac{4}{j-3} & 0 \end{pmatrix} \cdot \begin{pmatrix} \int x^{j+1} I_0 K_0^3 \\ \int x^{j+1} I_1 K_0^3 \\ \int x^{j+1} I_1 K_0^2 K_1 \\ \int x^{j+1} I_1 K_0 K_1^2 \\ \int x^{j+1} I_1 K_1^3 \end{pmatrix} + \begin{pmatrix} \frac{3}{j+1}\Gamma_{00}(j) \\ 0 \\ -\frac{1}{j-1}\Gamma_{00}(j) \\ -\frac{1}{j-2}\Gamma_{01}(j) \\ -\frac{1}{j-3}\Gamma_{11}(j) \end{pmatrix}.$$

The matrix is singular, which inhibits a standard inversion that aims at creating a forward recurrence $j \to j+1$ at this place. We substitute $j \to j+1$ and insert the

result into the right hand side:

(25)

$$
\begin{pmatrix} \int x^j I_0 K_0^3 \\ \int x^j I_1 K_0^3 \\ \int x^j I_1 K_0^2 K_1 \\ \int x^j I_1 K_0 K_1^2 \\ \int x^j I_1 K_1^3 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{4}{1+j} & 0 & 0 & 0 \\ -\frac{1}{j} & 0 & \frac{3}{j} & 0 & 0 \\ 0 & \frac{2}{j-1} & 0 & \frac{2}{j-1} & 0 \\ 0 & 0 & \frac{3}{j-2} & 0 & \frac{1}{j-2} \\ 0 & 0 & 0 & \frac{4}{j-3} & 0 \end{pmatrix} \cdot \left[\begin{pmatrix} 0 & -\frac{4}{2+j} & 0 & 0 & 0 \\ -\frac{1}{1+j} & 0 & \frac{3}{1+j} & 0 & 0 \\ 0 & \frac{2}{j} & 0 & \frac{2}{j} & 0 \\ 0 & 0 & \frac{3}{j-1} & 0 & \frac{1}{j-1} \\ 0 & 0 & 0 & \frac{4}{j-2} & 0 \end{pmatrix} \right.
$$

$$
\begin{pmatrix} \int x^{j+2} I_0 K_0^3 \\ \int x^{j+2} I_1 K_0^3 \\ \int x^{j+2} I_1 K_0^2 K_1 \\ \int x^{j+2} I_1 K_0 K_1^2 \\ \int x^{j+2} I_1 K_1^3 \end{pmatrix} + \begin{pmatrix} \frac{3}{j+2}\Gamma_{00}(j+1) \\ 0 \\ -\frac{1}{j}\Gamma_{00}(j+1) \\ -\frac{1}{j-1}\Gamma_{01}(j+1) \\ -\frac{1}{j-2}\Gamma_{11}(j+1) \end{pmatrix} \left. \right] + \begin{pmatrix} \frac{3}{j+1}\Gamma_{00}(j) \\ 0 \\ -\frac{1}{j-1}\Gamma_{00}(j) \\ -\frac{1}{j-2}\Gamma_{01}(j) \\ -\frac{1}{j-3}\Gamma_{11}(j) \end{pmatrix}
$$

$$
= \begin{pmatrix} \frac{4}{(1+j)^2} & 0 & -\frac{12}{(1+j)^2} & 0 & 0 \\ 0 & \frac{2(5j+6)}{j^2(2+j)} & 0 & \frac{6}{j^2} & 0 \\ -\frac{2}{(j-1)(1+j)} & 0 & \frac{12j}{(j-1)^2(1+j)} & 0 & \frac{2}{(j-1)^2} \\ 0 & \frac{6}{j(j-2)} & 0 & \frac{2(5j-6)}{(j-2)^2 j} & 0 \\ 0 & 0 & \frac{12}{(j-3)(j-1)} & 0 & \frac{4}{(j-3)(j-1)} \end{pmatrix} \cdot \begin{pmatrix} \int x^{j+2} I_0 K_0^3 \\ \int x^{j+2} I_1 K_0^3 \\ \int x^{j+2} I_1 K_0^2 K_1 \\ \int x^{j+2} I_1 K_0 K_1^2 \\ \int x^{j+2} I_1 K_1^3 \end{pmatrix}
$$

$$
+ \begin{pmatrix} 3 \times 2^{j-2}\Gamma^4((1+j)/2)/(1+j)! \\ -3 \times 2^{j-4} j\Gamma^4(j/2)/[(2+j)(j-1)!] \\ -2^{j-5}(j-1)\Gamma^4((j-1)/2)/(j-2)! \\ -2^{j-4}(3j^2-3j-2)j\Gamma^4(j/2)/[(1+j)(j-2)^2(j-1)!] \\ -3 \times 2^{j-10}(j-3)^3(j-1)^2(1+j)\Gamma^4((j-3)/2)/[j(j-2)!] \end{pmatrix}.
$$

Here $0! = 1$. This iteration has separated the subspace of the second and fourth equation with a 2×2 matrix,

$$
(26) \quad \begin{pmatrix} \int x^j I_1 K_0^3 \\ \int x^j I_1 K_0 K_1^2 \end{pmatrix} = \begin{pmatrix} \frac{2(5j+6)}{j^2(2+j)} & \frac{6}{j^2} \\ \frac{6}{j(j-2)} & \frac{2(5j-6)}{(j-2)^2 j} \end{pmatrix} \cdot \begin{pmatrix} \int x^{j+2} I_1 K_0^3 \\ \int x^{j+2} I_1 K_0 K_1^2 \end{pmatrix}
$$

$$
+ \begin{pmatrix} -3 \times 2^{j-4} j\Gamma^4(j/2)/[(2+j)(j-1)!] \\ -2^{j-4}(3j^2-3j-2)j\Gamma^4(j/2)/[(1+j)(j-2)^2(j-1)!] \end{pmatrix}.
$$

That matrix is not singular and is inverted to support a calculation of integrands with a higher power x^{j+2} from integrands with a lower power x^j:

(27)

$$
\begin{pmatrix} \int x^{j+2} I_1 K_0^3 \\ \int x^{j+2} I_1 K_0 K_1^2 \end{pmatrix} = \begin{pmatrix} \frac{(5j-6)(2+j)}{32} & -\frac{3(2+j)(j-2)^2}{32j} \\ -\frac{3(2+j)(j-2)}{32} & \frac{(5j+6)(j-2)^2}{32j} \end{pmatrix} \cdot \begin{pmatrix} \int x^j I_1 K_0^3 \\ \int x^j I_1 K_0 K_1^2 \end{pmatrix}
$$

$$
+ \begin{pmatrix} 3 \times 2^{j-8}(2+j+j^3-2j^2)\Gamma^4(j/2)/[(1+j)(j-1)!] \\ 2^{j-8}(6j^2+3j^3-5j-6)\Gamma^4(j/2)/[(1+j)(j-1)!] \end{pmatrix}.
$$

Algorithm 1. (1) *For the (finite) integrals for $j \leq 2$ use* (28)–(32): [6, A255986]

$$
(28) \qquad \int_0^\infty I_1(x) K_0^3(x) dx = \zeta(3)/4 \approx 0.30051422578989857134993454038;
$$

$$
(29) \qquad \int_0^\infty x I_1(x) K_0^3(x) dx \approx 0.10116007103409665114227760576;
$$

(30) $$\int_0^\infty x^2 I_1(x) K_0^3(x) dx \approx 0.066574862465957543161422156253 4;$$

(31) $$\int_0^\infty x I_1(x) K_0(x) K_1^2(x) dx \approx 0.5473545513579486817936204237 4;$$

(32) $$\int_0^\infty x^2 I_1(x) K_0(x) K_1^2(x) dx \approx 0.18342513753404245683857784375.$$

(2) *Compute* $\int_0^\infty x^j I_1(x) K_0^3(x) dx$ *and* $\int_0^\infty x^j I_1(x) K_0(x) K_1^2(x) dx$ *for* $j \geq 3$ *recursively via* (27), *but for*

(33) $$\int x^4 I_1(x) K_0(x) K_1^2(x) dx = 1/6$$

–obtained by inserting $j = 3$ *in* (23).

The first, third and fifth line of (24) are unrolled as follows:

Algorithm 2.   (1) *The integrals* $\int_0^\infty x^j I_0 K_0^3 dx$ *are evaluated recursively with* [2, 3]

(34)
$$(k+1)^5 \int_0^\infty x^k I_0(x) K_0^3(x) dx - 4(k+2)(5k^2 + 20k + 23) \int_0^\infty x^{k+2} I_0(x) K_0^3(x) dx$$
$$+ 64(k+3) \int_0^\infty x^{k+4} I_0(x) K_0^3(x) dx = 0.$$

starting at [6, A273951]

(35) $$\int I_0(x) K_0^3(x) dx \approx 6.9975630166806323595567578269;$$

[6, A222068][4, (89)]

(36) $$\int x I_0 K_0^3 dx = \frac{1}{16} \pi^2;$$

(37) $$\int x^2 I_0(x) K_0^3(x) dx \approx 0.2179029656384928370653923596 2;$$

(38) $$\int x^3 I_0(x) K_0^3(x) dx = \frac{1}{64} \pi^2.$$

Remark 1. [6, A245058]

(39) $$\int x I_1 K_0^2 K_1 dx = \frac{1}{48} \pi^2,$$

is obviously correlated to (36) *by putting* $j = 0$ *in* (11).

Remark 2. *According to the first line of* (24), *the value in* (29) *is obtained from* (35) *by subtracting* $3\pi^2/4$ *and dividing through* -4.

Remark 3. *Recurrences equivalent to* (34) *are derived for the remaining 4 classes of integrals inserting the recurrence into* (24). *From the first line e.g.*

$$(40) \quad 24\frac{k^2(3k^3 + 14k^2 + 18k + 8)}{(k+1)(k+3)}\Gamma_{01}(k) - 4k^4(k+2)(k+4)\int_0^\infty x^k I_1 K_0^3 dx$$

$$+ 16(k+1)(k+4)(5k^2 + 10k + 8)\int_0^\infty x^{k+2}I_1 K_0^3 dx$$

$$- 256(k+2)^2\int_0^\infty x^{k+4}I_1 K_0^3 dx = 0.$$

Supposed all the values of $\int_0^\infty x^j I_0(x)K_0^3(x)dx$ are known from Algorithm 2, one can lift the degeneracy in (24) and invert the 4×4 matrix:

$$(41)$$
$$\begin{pmatrix} \int x^j I_1 K_0^3 \\ \int x^j I_1 K_0^2 K_1 \\ \int x^j I_1 K_0 K_1^2 \\ \int x^j I_1 K_1^3 \end{pmatrix} = \begin{pmatrix} 0 & \frac{3}{j} & 0 & 0 \\ \frac{2}{j-1} & 0 & \frac{2}{j-1} & 0 \\ 0 & \frac{3}{j-2} & 0 & \frac{1}{j-2} \\ 0 & 0 & \frac{4}{j-3} & 0 \end{pmatrix} \cdot \begin{pmatrix} \int x^{j+1}I_1 K_0^3 \\ \int x^{j+1}I_1 K_0^2 K_1 \\ \int x^{j+1}I_1 K_0 K_1^2 \\ \int x^{j+1}I_1 K_1^3 \end{pmatrix} + \begin{pmatrix} -\frac{1}{j}\int x^{j+1}I_0 K_0^3 \\ -\frac{1}{j-1}\Gamma_{00}(j) \\ -\frac{1}{j-2}\Gamma_{01}(j) \\ -\frac{1}{j-3}\Gamma_{11}(j) \end{pmatrix}.$$

$$(42)$$
$$\begin{pmatrix} 0 & \frac{j-1}{2} & 0 & \frac{3-j}{4} \\ \frac{j}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{j-3}{4} \\ -j & 0 & j-2 & 0 \end{pmatrix} \cdot \left[\begin{pmatrix} \int x^j I_1 K_0^3 \\ \int x^j I_1 K_0^2 K_1 \\ \int x^j I_1 K_0 K_1^2 \\ \int x^j I_1 K_1^3 \end{pmatrix} + \begin{pmatrix} \frac{1}{j}\int x^{j+1}I_0 K_0^3 \\ \frac{1}{j-1}\Gamma_{00}(j) \\ \frac{1}{j-2}\Gamma_{01}(j) \\ \frac{1}{j-3}\Gamma_{11}(j) \end{pmatrix} \right] = \begin{pmatrix} \int x^{j+1}I_1 K_0^3 \\ \int x^{j+1}I_1 K_0^2 K_1 \\ \int x^{j+1}I_1 K_0 K_1^2 \\ \int x^{j+1}I_1 K_1^3 \end{pmatrix}.$$

Algorithm 3. *Given the* $\int_0^\infty x^j I_0(x)K_0^3(x)dx$, *compute the others in this group recursively with*

$$(43) \quad \begin{pmatrix} \int x^{j+1}I_1 K_0^3 \\ \int x^{j+1}I_1 K_0^2 K_1 \\ \int x^{j+1}I_1 K_0 K_1^2 \\ \int x^{j+1}I_1 K_1^3 \end{pmatrix} = \begin{pmatrix} 0 & \frac{j-1}{2} & 0 & \frac{3-j}{4} \\ \frac{j}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{j-3}{4} \\ -j & 0 & j-2 & 0 \end{pmatrix} \cdot \begin{pmatrix} \int x^j I_1 K_0^3 \\ \int x^j I_1 K_0^2 K_1 \\ \int x^j I_1 K_0 K_1^2 \\ \int x^j I_1 K_1^3 \end{pmatrix}$$

$$+ \begin{pmatrix} \frac{1}{2}\Gamma_{00}(j) - \frac{1}{4}\Gamma_{11}(j) \\ \frac{1}{3}\int x^{j+1}I_0 K_0^3 \\ \frac{1}{4}\Gamma_{11}(j) \\ -\int x^{j+1}I_0 K_0^3 + \Gamma_{01}(j) \end{pmatrix}.$$

3. Quadruple Products of K

The other five cases of integrals discussed in this manuscript contain four factors of K functions.

3.1. **Class** K_0^4.

$$(44) \quad \int x^j K_0^4 dx = \ldots - \int \frac{x^{j+1}}{j+1}(-4K_0^3 K_1)dx = \ldots + 4\int \frac{x^{j+1}}{j+1}K_0^3 K_1 dx.$$

3.2. Class $K_0^3 K_1$.

$$(45) \quad \int x^j K_0^3 K_1 dx = \ldots - \int \frac{x^{j+1}}{j+1}[-3K_0^2 K_1^2 - K_0^4 - \frac{1}{x}K_0^3 K_1]dx$$

$$= \ldots + \int \frac{x^j}{j+1}K_0^3 K_1 dx + \int \frac{x^{j+1}}{j+1}[3K_0^2 K_1^2 + K_0^4]dx.$$

Merging common integrals yields

$$(46) \qquad \frac{j}{j+1}\int x^j K_0^3 K_1 dx = \ldots + \int \frac{x^{j+1}}{j+1}[3K_0^2 K_1^2 + K_0^4]dx.$$

$$(47) \qquad j\int x^j K_0^3 K_1 dx = \ldots + 3\int x^{j+1}K_0^2 K_1^2 dx + \int x^{j+1}K_0^4 dx$$

3.3. Class $K_0^2 K_1^2$.

$$(48) \quad \int x^j K_0^2 K_1^2 dx = \ldots - \int \frac{x^{j+1}}{j+1}[-2K_0 K_1^3 - 2K_0^3 K_1 - \frac{2}{x}K_0^2 K_1^2]dx$$

$$= \ldots - \int \frac{x^{j+1}}{j+1}[-\frac{2}{x}K_0^2 K_1^2]dx - \int \frac{x^{j+1}}{j+1}[-2K_0 K_1^3 - 2K_0^3 K_1]dx$$

$$= \ldots + 2\int \frac{x^j}{j+1}K_0^2 K_1^2 dx + 2\int \frac{x^{j+1}}{j+1}[K_0 K_1^3 + K_0^3 K_1]dx.$$

Merging common integrals yields

$$(49) \qquad \frac{j-1}{j+1}\int x^j K_0^2 K_1^2 dx = \ldots + 2\int \frac{x^{j+1}}{j+1}[K_0 K_1^3 + K_0^3 K_1]dx.$$

$$(50) \qquad (j-1)\int x^j K_0^2 K_1^2 dx = \ldots + 2\int x^{j+1}K_0 K_1^3 dx + 2\int x^{j+1}K_0^3 K_1 dx.$$

3.4. Class $K_0 K_1^3$.

$$(51) \quad \int x^j K_0 K_1^3 dx = \ldots - \int \frac{x^{j+1}}{j+1}[-K_1^4 - 3K_0^2 K_1^2 - \frac{3}{x}K_0 K_1^3]dx$$

$$= \ldots - \int \frac{x^{j+1}}{j+1}[-\frac{3}{x}K_0 K_1^3]dx - \int \frac{x^{j+1}}{j+1}[-K_1^4 - 3K_0^2 K_1^2]dx$$

$$= \ldots + 3\int \frac{x^j}{j+1}K_0 K_1^3 dx + \int \frac{x^{j+1}}{j+1}[K_1^4 + 3K_0^2 K_1^2]dx.$$

Merging common integrals yields

$$(52) \qquad \frac{j-2}{j+1}\int x^j K_0 K_1^3 dx = \ldots + \int \frac{x^{j+1}}{j+1}K_1^4 dx + 3\int \frac{x^{j+1}}{j+1}K_0^2 K_1^2 dx.$$

$$(53) \qquad (j-2)\int x^j K_0 K_1^3 dx = \ldots + \int x^{j+1}K_1^4 dx + 3\int x^{j+1}K_0^2 K_1^2 dx.$$

3.5. Class K_1^4.

(54)
$$\int x^j K_1^4 dx = \ldots - \int \frac{x^{j+1}}{j+1}[-4K_0K_1^3 - \frac{4}{x}K_1^4]dx = \ldots + 4\int \frac{x^{j+1}}{j+1}[K_0K_1^3 + \frac{1}{x}K_1^4]dx$$
$$= \ldots + 4\int \frac{x^j}{j+1}K_1^4 dx + 4\int \frac{x^{j+1}}{j+1}K_0K_1^3 dx.$$

Merging common integrals yields

(55)
$$\frac{j-3}{j+1}\int x^j K_1^4 dx = \ldots + 4\int \frac{x^{j+1}}{j+1}K_0K_1^3 dx.$$

(56)
$$(j-3)\int x^j K_1^4 dx = \ldots + 4\int x^{j+1}K_0K_1^3 dx$$

3.6. Synthesis.
Gathering these 5 partial integrations forms a homogeneous system of equations very similar to (24):

(57)
$$\begin{pmatrix} \int x^j K_0^4 \\ \int x^j K_0^3 K_1 \\ \int x^j K_0^2 K_1^2 \\ \int x^j K_0 K_1^3 \\ \int x^j K_1^4 \end{pmatrix} = \begin{pmatrix} 0 & \frac{4}{1+j} & 0 & 0 & 0 \\ \frac{1}{j} & 0 & \frac{3}{j} & 0 & 0 \\ 0 & \frac{2}{j-1} & 0 & \frac{2}{j-1} & 0 \\ 0 & 0 & \frac{3}{j-2} & 0 & \frac{1}{j-2} \\ 0 & 0 & 0 & \frac{4}{j-3} & 0 \end{pmatrix} \cdot \begin{pmatrix} \int x^{j+1} K_0^4 \\ \int x^{j+1} K_0^3 K_1 \\ \int x^{j+1} K_0^2 K_1^2 \\ \int x^{j+1} K_0 K_1^3 \\ \int x^{j+1} K_1^4 \end{pmatrix}.$$

This is also iterated:

(58)
$$\begin{pmatrix} \int x^j K_0^4 \\ \int x^j K_0^3 K_1 \\ \int x^j K_0^2 K_1^2 \\ \int x^j K_0 K_1^3 \\ \int x^j K_1^4 \end{pmatrix} = \begin{pmatrix} 0 & \frac{4}{1+j} & 0 & 0 & 0 \\ \frac{1}{j} & 0 & \frac{3}{j} & 0 & 0 \\ 0 & \frac{2}{j-1} & 0 & \frac{2}{j-1} & 0 \\ 0 & 0 & \frac{3}{j-2} & 0 & \frac{1}{j-2} \\ 0 & 0 & 0 & \frac{4}{j-3} & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & \frac{4}{2+j} & 0 & 0 & 0 \\ \frac{1}{1+j} & 0 & \frac{3}{1+j} & 0 & 0 \\ 0 & \frac{2}{j} & 0 & \frac{2}{j} & 0 \\ 0 & 0 & \frac{3}{j-1} & 0 & \frac{1}{j-1} \\ 0 & 0 & 0 & \frac{4}{j-2} & 0 \end{pmatrix} \cdot \begin{pmatrix} \int x^{j+2} K_0^4 \\ \int x^{j+2} K_0^3 K_1 \\ \int x^{j+2} K_0^2 K_1^2 \\ \int x^{j+2} K_0 K_1^3 \\ \int x^{j+2} K_1^4 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{4}{(1+j)^2} & 0 & \frac{12}{(1+j)^2} & 0 & 0 \\ 0 & \frac{2(5j+6)}{j^2(2+j)} & 0 & \frac{6}{j^2} & 0 \\ \frac{2}{(j-1)(1+j)} & 0 & \frac{12j}{(j-1)^2(1+j)} & 0 & \frac{2}{(j-1)^2} \\ 0 & \frac{6}{j(j-2)} & 0 & \frac{2(5j-6)}{(j-2)^2 j} & 0 \\ 0 & 0 & \frac{12}{(j-3)(j-1)} & 0 & \frac{4}{(j-3)(j-1)} \end{pmatrix} \cdot \begin{pmatrix} \int x^{j+2} K_0^4 \\ \int x^{j+2} K_0^3 K_1 \\ \int x^{j+2} K_0^2 K_1^2 \\ \int x^{j+2} K_0 K_1^3 \\ \int x^{j+2} K_1^4 \end{pmatrix}.$$

The subspace of lines 2 and 4 in (58) reads:

(59)
$$\begin{pmatrix} \int x^j K_0^3 K_1 \\ \int x^j K_0 K_1^3 \end{pmatrix} = \begin{pmatrix} \frac{2(5j+6)}{j^2(2+j)} & \frac{6}{j^2} \\ \frac{6}{j(j-2)} & \frac{2(5j-6)}{(j-2)^2 j} \end{pmatrix} \cdot \begin{pmatrix} \int x^{j+2} K_0^3 K_1 \\ \int x^{j+2} K_0 K_1^3 \end{pmatrix}.$$

By matrix inversion we obtain the forward recursion

(60)
$$\begin{pmatrix} \int x^{j+2} K_0^3 K_1 \\ \int x^{j+2} K_0 K_1^3 \end{pmatrix} = \begin{pmatrix} \frac{(5j-6)(2+j)}{32} & -\frac{3(j-2)^2(2+j)}{32j} \\ -\frac{3(j-2)(2+j)}{32} & \frac{(5j+6)(j-2)^2}{32j} \end{pmatrix} \cdot \begin{pmatrix} \int x^j K_0^3 K_1 \\ \int x^j K_0 K_1^3 \end{pmatrix}.$$

Algorithm 4. (1) *For the (finite) integrals $j \le 4$ use* (61)–(66):

(61)
$$\int_0^\infty x K_0^3 K_1 dx \approx 6.8103346044514933516774950661;$$

(62)
$$\int_0^\infty x^2 K_0^3 K_1 dx = 7\zeta(3)/16 \approx 0.52589989513232249986238544566;$$

(63)
$$\int_0^\infty x^3 K_0^3 K_1 dx \approx 0.14682796893546593791309356246;$$

[6, A273841],*see the first line in* (57)

(64)
$$\int_0^\infty x^4 K_0^3 K_1 dx = [7\zeta(3)/2 - 3]/16 \approx 0.075449947566161249931192728306;$$

(65)
$$\int_0^\infty x^3 K_0 K_1^3 dx \approx 0.95559936093190163449691260135;$$

(66)
$$\int_0^\infty x^4 K_0 K_1^3 dx = 1/4.$$

(2) *For $j \geq 5$ calculate $\int_0^\infty x^j K_0^3(x) K_1(x) dx$ and $\int_0^\infty x^j K_0(x) K_1(x)^3 dx$ recursively with* (60).

Remark 4. (66) *follows from the partial integration*

(67)
$$\int_0^\infty x^j K_1^{j-1}(x) K_0(x) dx = \lim_{x \to 0} \frac{1}{j}(x^j K_1^j(x)) = 1/j.$$

(62) *is half of* (70) *and that is proved by setting $j = 1$ in* (44).

Algorithm 5. *Compute the $\int_0^\infty t^j K_0^4(x) dx$ recursively with* [2, 3]

(68)
$$(k+1)^5 \int_0^\infty x^k K_0^4(x) dx - 4(k+2)(5k^2+20k+23) \int_0^\infty x^{k+2} K_0^4(x) dx + 64(k+3) \int_0^\infty x^{k+4} K_0^4(x) dx = 0$$

starting at [2, (73)][6, A273839]

(69)
$$\int_0^\infty K_0^4(x) dx = \frac{\pi^4}{4} \, {}_4F_3(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1, 1; 1) \approx 27.24133841780597340670998026;$$

[4, (90)][6, A233091]

(70)
$$\int_0^\infty x K_0^4(x) dx = 7\zeta(3)/8 \approx 1.05179979026464499972477089133;$$

[6, A273840]

(71)
$$\int_0^\infty x^2 K_0^4(x) dx \approx 0.19577062524728791721745808328;$$

[6, A273841]

(72)
$$\int_0^\infty x^3 K_0^4(x) dx = [7\zeta(3)/2 - 3]/16 \approx 0.075449947566161249931192722831.$$

To obtain $\int_0^\infty x^j K_0^2(x) K_1^2(x) dx$ and $\int_0^\infty x^j K_1^4(x) dx$ one may plug these values of Algorithm 5 into the first and last line of (58). The alternative is to eliminate the $\int_0^\infty x^j K_0^4 dx$ from (57) and to invert the remaining 4×4 matrix:

(73)
$$
\begin{pmatrix} \int x^j K_0^3 K_1 \\ \int x^j K_0^2 K_1^2 \\ \int x^j K_0 K_1^3 \\ \int x^j K_1^4 \end{pmatrix} = \begin{pmatrix} 0 & \frac{3}{j} & 0 & 0 \\ \frac{2}{j-1} & 0 & \frac{2}{j-1} & 0 \\ 0 & \frac{3}{j-2} & 0 & \frac{1}{j-2} \\ 0 & 0 & \frac{4}{j-3} & 0 \end{pmatrix} \cdot \begin{pmatrix} \int x^{j+1} K_0^3 K_1 \\ \int x^{j+1} K_0^2 K_1^2 \\ \int x^{j+1} K_0 K_1^3 \\ \int x^{j+1} K_1^4 \end{pmatrix} + \begin{pmatrix} -\frac{1}{j}\int x^{j+1} K_0^4 \\ 0 \\ 0 \\ 0 \end{pmatrix}.
$$

(74)
$$
\begin{pmatrix} 0 & \frac{j-1}{2} & 0 & \frac{3-j}{4} \\ \frac{j}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{j-3}{4} \\ -j & 0 & j-2 & 0 \end{pmatrix} \cdot \left[\begin{pmatrix} \int x^j K_0^3 K_1 \\ \int x^j K_0^2 K_1^2 \\ \int x^j K_0 K_1^3 \\ \int x^j K_1^4 \end{pmatrix} + \begin{pmatrix} \frac{1}{j}\int x^{j+1} K_0^4 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right] = \begin{pmatrix} \int x^{j+1} K_0^3 K_1 \\ \int x^{j+1} K_0^2 K_1^2 \\ \int x^{j+1} K_0 K_1^3 \\ \int x^{j+1} K_1^4 \end{pmatrix}.
$$

Algorithm 6. *Given the $\int_0^\infty x^j K_0^4(x)dx$, compute the others in this group recursively with*

(75)
$$
\begin{pmatrix} \int x^{j+1} K_0^3 K_1 \\ \int x^{j+1} K_0^2 K_1^2 \\ \int x^{j+1} K_0 K_1^3 \\ \int x^{j+1} K_1^4 \end{pmatrix} = \begin{pmatrix} 0 & \frac{j-1}{2} & 0 & \frac{3-j}{4} \\ \frac{j}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{j-3}{4} \\ -j & 0 & j-2 & 0 \end{pmatrix} \cdot \begin{pmatrix} \int x^j K_0^3 K_1 \\ \int x^j K_0^2 K_1^2 \\ \int x^j K_0 K_1^3 \\ \int x^j K_1^4 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{3}\int x^{j+1} K_0^4 \\ 0 \\ -\int x^{j+1} K_0^4 \end{pmatrix}.
$$

4. Summary

Algorithms 2, 3, 5 and 6 compute integrals over 10 different types of products of four Modified Bessel Functions multiplied by a power x^j recursively for increasing exponent j.

Appendix A. Numerical Integration

For small upper limits, a series expansion of the product of the Bessel Functions in the integrand leads to stable numerical representations. The basic series expansions are [1, 9.6.10]

(76)
$$
I_\nu(z) = \sum_{l\geq 0} \frac{1}{l!\Gamma(\nu+l+1)}(z/2)^{2l+\nu}
$$

and

(77)
$$
K_0(z) = -[\ln\frac{z}{2}+\gamma]I_0(z) + \sum_{k\geq 1} H(k)\frac{(z^2/4)^k}{(k!)^2} = -\ln\frac{z}{2}\sum_{k\geq 0}\frac{(z^2/4)^k}{(k!)^2} + \sum_{k\geq 0}(H(k)-\gamma)\frac{(z^2/4)^k}{(k!)^2},
$$

where $H(k) = \sum_{l=1}^k 1/l$, $H(0) = 0$, are the Harmonic Numbers.

The multinomial expansion is

(78)

$$(z/2)^l I_\nu(z) K_0^3(z) = -\ln^3 \frac{z}{2} \sum_{v\geq 0} \sum_{n=0}^{v} \sum_{m=0}^{n} \sum_{k=0}^{m} \frac{1}{(k!)^2 (m-k)!^2} \frac{1}{((n-m)!)^2} \frac{1}{(v-n)!\Gamma(\nu+v-n+1)} (z/2)^{2v+l+\nu}$$

$$-3\ln\frac{z}{2} \sum_{v\geq 0} \sum_{n=0}^{v} \sum_{m=0}^{n} \sum_{k=0}^{m} \frac{H(k)-\gamma}{k!^2} \frac{H(m-k)-\gamma}{(m-k)!^2} \frac{1}{((n-m)!)^2} \frac{1}{(v-n)!\Gamma(\nu+v-n+1)} (z/2)^{2v+l+\nu}$$

$$+3\ln^2\frac{z}{2} \sum_{v\geq 0} \sum_{n=0}^{v} \sum_{m=0}^{n} \sum_{k=0}^{m} \frac{1}{(k!)^2} \frac{H(m-k)-\gamma}{((m-k)!)^2} \frac{1}{((n-m)!)^2} \frac{1}{(v-n)!\Gamma(\nu+v-n+1)} (z/2)^{2v+l+\nu}$$

$$+\sum_{v\geq 0} \sum_{n=0}^{v} \sum_{m=0}^{n} \sum_{k=0}^{m} \frac{H(k)-\gamma}{k!^2} \frac{H(m-k)-\gamma}{(m-k)!^2} \frac{H(n-m)-\gamma}{((n-m)!)^2} \frac{1}{(v-n)!\Gamma(\nu+v-n+1)} (z/2)^{2v+l+\nu}.$$

The presence of the $\log(z/2)$ factor proposes to use $z = 2$ as an upper limit, [5, 2.722]

(79)
$$\int_0^2 \ln^j \frac{z}{2} (z/2)^{2v+l} dz = (-)^j \frac{2j!}{(2v+1+l)^{j+1}}, \quad j = 0, 1, \ldots.$$

(80)
$$\int_0^2 (z/2)^l I_\nu(z) K_0^3(z) dz = \sum_{v\geq 0} \frac{1}{v!(v+\nu)!} \sum_{n=0}^{v} \sum_{m=0}^{n} \sum_{k=0}^{m} \binom{m}{k}^2 \binom{n}{m}^2 \binom{v}{n} \binom{v+\nu}{n} \frac{12}{(2v+1+l+\nu)^4}$$

$$+\sum_{v\geq 0} \frac{1}{v!(v+\nu)!} \sum_{n=0}^{v} \sum_{m=0}^{n} \sum_{k=0}^{m} \binom{m}{k}^2 \binom{n}{m}^2 \binom{v}{n} \binom{v+\nu}{n} (H(k)-\gamma)(H(m-k)-\gamma) \frac{6}{(2v+1+l+\nu)^2}$$

$$+\sum_{v\geq 0} \frac{1}{v!(v+\nu)!} \sum_{n=0}^{v} \sum_{m=0}^{n} \sum_{k=0}^{m} \binom{m}{k}^2 \binom{n}{m}^2 \binom{v}{n} \binom{v+\nu}{n} (H(m-k)-\gamma) \frac{12}{(2v+1+l+\nu)^3}$$

$$+\sum_{v\geq 0} \frac{1}{v!(v+\nu)!} \sum_{n=0}^{v} \sum_{m=0}^{n} \sum_{k=0}^{m} \binom{m}{k}^2 \binom{n}{m}^2 \binom{v}{n} \binom{v+\nu}{n} (H(k)-\gamma)(H(m-k)-\gamma)(H(n-m)-\gamma) \frac{2}{2v+1+l+\nu}.$$

In an equivalent way [1, 9.6.11]
(81)
$$K_1(z) = \frac{1}{2}\frac{1}{z/2} + \frac{z}{2}\ln\frac{z}{2} \sum_{k\geq 0} \frac{1}{k!(1+k)!}(z/2)^{2k} - \frac{1}{2}\frac{z}{2} \sum_{k\geq 0} [2H(k+1) - \frac{1}{k+1} - 2\gamma] \frac{(z/2)^{2k}}{k!(k+1)!}$$

where we have replaced

(82) $\psi(k+1) + \psi(k+2) = H(k) + H(k+1) - 2\gamma = 2H(k+1) - \frac{1}{k+1} - 2\gamma, \quad k \geq 0.$

Definition 4.

(83)
$$H_2(k) \equiv H(k) + H(k+1) - 2\gamma.$$

The multinomial expansion of the product is

$$(84) \quad (z/2)^l I_\nu(z) K_1^3(z) = \frac{1}{2^3} \sum_{k \geq 0} \frac{1}{k! \Gamma(\nu + k + 1)} (z/2)^{2k+\nu-3+l}$$

$$+\ln^3 \frac{z}{2} \sum_{v \geq 0} \sum_{n=0}^{v} \sum_{m=0}^{n} \sum_{k=0}^{m} \frac{1}{k!(1+k)!} \frac{1}{(m-k)!(1+m-k)!} \frac{1}{(n-m)!(1+n-m)!} \frac{1}{(v-n)!\Gamma(\nu+v-n+1)} (z/2)^{2v+\nu+}$$

$$-\frac{1}{2^3} \sum_{v \geq 0} \sum_{n=0}^{v} \sum_{m=0}^{n} \sum_{k=0}^{m} \frac{H_2(k)}{k!(k+1)!} \frac{H_2(m-k)}{(m-k)!(m-k+1)!} \frac{H_2(n-m)}{(n-m)!(n-m+1)!} \frac{1}{(v-n)!\Gamma(\nu+v-n+1)} (z/2)^{2v+\nu+3+}$$

$$+\frac{3}{2^2} \ln \frac{z}{2} \sum_{m \geq 0} \sum_{k=0}^{m} \frac{1}{k!(1+k)!} \frac{1}{(m-k)!\Gamma(\nu+m-k+1)} (z/2)^{2m+\nu-1+l}$$

$$-\frac{3}{2^2} \frac{1}{2} \sum_{m \geq 0} \sum_{k=0}^{m} [H_2(k)] \frac{1}{k!(k+1)!} \frac{1}{(m-k)!\Gamma(\nu+m-k+1)} (z/2)^{2m+\nu-1+l}$$

$$+\frac{3}{2} \ln^2 \frac{z}{2} \sum_{n \geq 0} \sum_{m=0}^{n} \sum_{k=0}^{m} \frac{1}{k!(1+k)!} \frac{1}{(m-k)!(1+m-k)!} \frac{1}{(n-m)!\Gamma(\nu+n-m+1)} (z/2)^{2n+\nu+1+l}$$

$$+\frac{3}{2} \frac{1}{2^2} \sum_{n \geq 0} \sum_{m=0}^{n} \sum_{k=0}^{m} \frac{H_2(k)}{k!(k+1)!} \frac{H_2(m-k)}{(m-k)!(m-k+1)!} \frac{1}{(n-m)!\Gamma(\nu+n-m+1)} (z/2)^{2n+\nu+1+l}$$

$$-\frac{3}{2} \ln^2 \frac{z}{2} \sum_{v \geq 0} \sum_{n=0}^{v} \sum_{m=0}^{n} \sum_{k=0}^{m} \frac{1}{k!(k+1)!} \frac{1}{(m-k)!(m-k+1)!} \frac{H_2(n-m)}{(n-m)!(n-m+1)!} \frac{1}{(v-n)!\Gamma(\nu+v-n+1)} (z/2)^{2v+}$$

$$+\frac{3}{2^2} \ln \frac{z}{2} \sum_{v \geq 0} \sum_{n=0}^{v} \sum_{m=0}^{n} \sum_{k=0}^{m} \frac{1}{k!(k+1)!} \frac{H_2(m-k)}{(m-k)!(m-k+1)!} \frac{H_2(n-m)}{(n-m)!(n-m+1)!} \frac{1}{(v-n)!\Gamma(\nu+v-n+1)} (z/2)^{2v+}$$

$$-\frac{6}{2^2} \ln \frac{z}{2} \sum_{n \geq 0} \sum_{m=0}^{n} \sum_{k=0}^{m} \frac{1}{k!(k+1)!} \frac{H_2(m-k)}{(m-k)!(m-k+1)!} \frac{1}{(n-m)!\Gamma(\nu+n-m+1)} (z/2)^{2n+\nu+1+l}.$$

The definite integral is

$$(85) \quad \int_0^2 (z/2)^l I_\nu(z) K_1^3(z) = \frac{1}{2^2} \sum_{k \geq 0} \frac{1}{k!\Gamma(\nu + k + 1)} \frac{1}{2k + \nu - 2 + l}$$

$$-2 \cdot 6 \sum_{v \geq 0} \sum_{n=0}^{v} \sum_{m=0}^{n} \sum_{k=0}^{m} \frac{1}{k!(1+k)!} \frac{1}{(m-k)!(1+m-k)!} \frac{1}{(n-m)!(1+n-m)!} \frac{1}{(v-n)!\Gamma(\nu + v - n + 1)} \frac{1}{(2v + \nu + 4 + }$$

$$-\frac{1}{2^2} \sum_{v \geq 0} \sum_{n=0}^{v} \sum_{m=0}^{n} \sum_{k=0}^{m} \frac{H_2(k)}{k!(k+1)!} \frac{H_2(m-k)}{(m-k)!(m-k+1)!} \frac{H_2(n-m)}{(n-m)!(n-m+1)!} \frac{1}{(v-n)!\Gamma(\nu + v - n + 1)} \frac{1}{2v + \nu + 4 + }$$

$$-\frac{3}{2} \sum_{m \geq 0} \sum_{k=0}^{m} \frac{1}{k!(1+k)!} \frac{1}{(m-k)!\Gamma(\nu + m - k + 1)} \frac{1}{(2m + \nu + l)^2}$$

$$-\frac{3}{2}\frac{1}{2} \sum_{m \geq 0} \sum_{k=0}^{m} H_2(k) \frac{1}{k!(k+1)!} \frac{1}{(m-k)!\Gamma(\nu + m - k + 1)} \frac{1}{2m + \nu + l}$$

$$+3 \cdot 2 \sum_{n \geq 0} \sum_{m=0}^{n} \sum_{k=0}^{m} \frac{1}{k!(1+k)!} \frac{1}{(m-k)!(1+m-k)!} \frac{1}{(n-m)!\Gamma(\nu + n - m + 1)} \frac{1}{(2n + \nu + 2 + l)^3}$$

$$+3\frac{1}{2^2} \sum_{n \geq 0} \sum_{m=0}^{n} \sum_{k=0}^{m} \frac{H_2(k)}{k!(k+1)!} \frac{H_2(m-k)}{(m-k)!(m-k+1)!} \frac{1}{(n-m)!\Gamma(\nu + n - m + 1)} \frac{1}{2n + \nu + 2 + l}$$

$$-3 \cdot 2 \sum_{v \geq 0} \sum_{n=0}^{v} \sum_{m=0}^{n} \sum_{k=0}^{m} \frac{1}{k!(k+1)!} \frac{1}{(m-k)!(m-k+1)!} \frac{H_2(n-m)}{(n-m)!(n-m+1)!} \frac{1}{(v-n)!\Gamma(\nu + v - n + 1)} \frac{1}{(2v + \nu + 4 + }$$

$$-\frac{3}{2} \sum_{v \geq 0} \sum_{n=0}^{v} \sum_{m=0}^{n} \sum_{k=0}^{m} \frac{1}{k!(k+1)!} \frac{H_2(m-k)}{(m-k)!(m-k+1)!} \frac{H_2(n-m)}{(n-m)!(n-m+1)!} \frac{1}{(v-n)!\Gamma(\nu + v - n + 1)} \frac{1}{(2v + \nu + 4 + }$$

$$+\frac{6}{2} \sum_{n \geq 0} \sum_{m=0}^{n} \sum_{k=0}^{m} \frac{1}{k!(k+1)!} \frac{H_2(m-k)}{(m-k)!(m-k+1)!} \frac{1}{(n-m)!\Gamma(\nu + n - m + 1)} \frac{1}{(2n + \nu + 2 + l)^2}.$$

APPENDIX B. HIGHER DERIVATIVES

In conjunction with Euler-Maclaurin formulas, higher derivatives of the Modified Bessel Functions are needed. They both obey a differential equation of the form [1, 9.6.1]

$$(86) \qquad \frac{d^2}{dz^2}\mathcal{L}_\nu = (1 + \nu^2/z^2)\mathcal{L}_\nu - \frac{1}{z}\frac{d}{dz}\mathcal{L}_\nu.$$

By repeated differentiation, these derivatives may be bootstrapped from the value and *first* derivative at the desired argument z, where the prime denotes derivation with respect to the argument z:

$$(87) \qquad \frac{d^3}{dz^3}\mathcal{L}_0 = \frac{1}{z^2}[(2 + z^2)\mathcal{L}_0' - z\mathcal{L}_0].$$

$$(88) \qquad \frac{d^4}{dz^4}\mathcal{L}_0 = \frac{1}{z^3}[(-6 - 2z^2)\mathcal{L}_0' + (3z + z^3)\mathcal{L}_0].$$

$$(89) \qquad \frac{d^5}{dz^5}\mathcal{L}_0 = \frac{1}{z^4}[(24 + 7z^2 + z^4)\mathcal{L}_0' + (-12z - 2z^3)\mathcal{L}_0].$$

$$(90) \qquad \frac{d^6}{dz^6}\mathcal{L}_0 = \frac{1}{z^5}[(-120 - 33z^2 - 3z^4)\mathcal{L}_0' + (60z + 9z^3 + z^5)\mathcal{L}_0].$$

$$(91) \qquad \frac{d^7}{dz^7}\mathcal{L}_0 = \frac{1}{z^6}[(720 + 192z^2 + 15z^4 + z^6)\mathcal{L}_0' + (-360z - 51z^3 - 3z^5)\mathcal{L}_0].$$

This defines reduction polynomials p

$$(92) \qquad \frac{d^n}{dz^n}\mathcal{L}_0 = \frac{1}{z^{n-1}}[p_{01n}(z)\mathcal{L}_0' + p_{00n}(z)\mathcal{L}_0].$$

with recurrences

$$(93) \qquad p_{01n} = (1-n)p_{01n-1} + zp_{01n-1}' + zp_{00n-1};$$

$$(94) \qquad p_{00n} = (2-n)p_{00n-1} + zp_{00n-1}' + zp_{01n-1}.$$

In the equivalent manner for index $\nu = 1$

$$(95) \qquad \frac{d^3}{dz^3}\mathcal{L}_1 = \frac{1}{z^3}[(3z + z^3)\mathcal{L}_1' + (-z^2 - 3)\mathcal{L}_1].$$

$$(96) \qquad \frac{d^4}{dz^4}\mathcal{L}_1 = \frac{1}{z^4}[(-12z - 2z^3)\mathcal{L}_1' + (5z^2 + 12 + z^4)\mathcal{L}_1].$$

$$(97) \qquad \frac{d^5}{dz^5}\mathcal{L}_1 = \frac{1}{z^5}[(60z + 9z^3 + z^5)\mathcal{L}_1' + (-24z^2 - 60 - 2z^4)\mathcal{L}_1].$$

$$(98) \qquad \frac{d^6}{dz^6}\mathcal{L}_1 = \frac{1}{z^6}[(-360z - 51z^3 - 3z^5)\mathcal{L}_1' + (141z^2 + 360 - 12z^4 + z^6)\mathcal{L}_1].$$

$$(99) \quad \frac{d^7}{dz^7}\mathcal{L}_1 = \frac{1}{z^7}[(2520z + 345z^3 + 18z^5 + z^7)\mathcal{L}_1' + (-975z^2 - 2520 - 78z^4 - 3z^6)\mathcal{L}_1].$$

$$(100) \qquad \frac{d^n}{dz^n}\mathcal{L}_1 = \frac{1}{z^n}[p_{11n}(z)\mathcal{L}_1' + p_{10n}(z)\mathcal{L}_1].$$

$$(101) \qquad p_{11n} = -np_{11n-1} + zp_{11n-1}' + zp_{10n-1}.$$

$$(102) \qquad p_{10n} = (1-n)p_{10n-1} + zp_{10n-1}' + (z + \frac{1}{z})p_{11n-1}.$$

If a break point is set at $z = 2$ for a Euler-Maclaurin expansion, the starting values are [6, A070910,A096789]

$$(103) \qquad I_0(2) \approx 2.27958530233606726743720444408;$$

$$(104) \qquad I_0'(z) = I_1(2) \approx 1.59063685463732906338225444250;$$

$$(105) \qquad I_1'(z) \approx 1.48426687501740273574607722283;$$

$$(106) \qquad K_0(2) \approx 0.11389387274953343565271957493;$$

$$(107) \qquad K_1(2) = -K_0'(2) \approx 0.13986588181652242728459880703;$$

$$(108) \qquad K_1'(2) \approx -0.18382681365779464929501897845.$$

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MAX-PLANCK INSTITUTE OF ASTRONOMY, KÖNIGSTUHL 17, 69117 HEIDELBERG, GERMANY