

Entropy Function for the Teleparallel Kaluza-Klein Reduction

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Abstract

This work is devoted to examine the galactic behavior of entropy by checking the validity of thermodynamics laws for the teleparallel reduction of Kaluza-Klein theory in thermal equilibrium state. We assume a flat Friedmann-Lemaître-Robertson-Walker type space-time model in which dark matter is interacting with dark energy. With this scenario, calculating the variation of galactic entropy function for each dark component and for the dynamical apparent horizon itself, we prove that the laws of thermodynamics are independent of the specific interaction term and dependent of the selected Kaluza-Klein model. Finally, some limiting conditions and their implications have been presented.

Keywords: Thermodynamics; 1st and 2nd laws; dark energy.

1. Introduction

The cosmological evidences through different schemes indicate that our universe has an accelerated expansion. Supernovae Cosmology Project Collaboration [1-3] has given the first clue about this expansionary behavior and subsequent attempts [4,5] also favor this mysterious phenomenon. Entering in the accelerated expansion phase is one of the well-established notions of our universe in modern cosmology. It is commonly believed [1-5] that this enigmatic property of the universe is raised by three exotic dark contents (matter, energy and radiation).

Although there are plenty of suggestions to be a candidate for the dark contents, the dark behavior of the universe is still unknown [6]. Recently, Li et al. [7] have presented a good summary for the galactic dark effect including a survey of some theoretical proposals. The equation-of-state parameter ω_e indicates three different phases of dark contents such as the vacuum case ($\omega_e=1$), phantom era ($\omega_e<-1$) and quintessence sector ($\omega_e>-1$) [8]. The best and simplest theoretical instrument to identify the dark universe is the cosmological constant, actually it gives the earliest explanation for the dark behavior. Besides, it also causes some problematic issues like the fine-tuning and cosmic-coincidence puzzle [9, 10, 11]. Additionally, tachyon [12], K-essence [13], quintom [14], phantom [15], quintessence [16], Chaplygin gas [17], Polytrropic gas [18] and modified gravity [19, 20] are also dark energy proposals which have been given to investigate the dark nature of universe. Furthermore, some researchers believe that extra dimensions of space are also responsible for the mysterious accelerated expansion phenomenon. The recent observational data indicate that multi-dimensional theories of gravity may help to find a satisfactory answer for such problems of modern cosmology and astrophysics. The most interesting multi-dimensional theory is suggested firstly by Kaluza [21] and Klein [22]. They assumed an additional dimension in Einstein's theory of general relativity to define five-dimensional (5D) Kaluza-Klein theory. This

extra dimension was mainly introduced to investigate the gravitational electromagnetism.

Galactic thermodynamics is another significant issue in modern cosmology. Jacobson's paper [23] is the first work that introduces a gravity-thermodynamics relation. Using the first thermodynamics law in black hole physics and the Bekenstein-Hawking entropy-area formulation [24,25], i.e. $S_h = A(4G_4)^{-1}$ where A denotes the area of the horizon while G_4 shows the gravitational constant, Jacobson rewritten the Einstein field equations on the local Rindler horizons [23]. Subsequently, this attractive phenomenon has been generalized to cosmological point of view. It was shown [26], in the general theory of relativity, that the Friedmann equation can be derived by making use of the first law of galactic thermodynamics

$$-dE = \hat{T}_h dS \quad (1)$$

Here dE , $\hat{T}_h = (2\pi R_h)^{-1}$, R_h and $S = \pi R_h^2 (G)^{-1}$ shows the amount of internal energy flow through dynamical horizon, Hawking temperature, apparent horizon and horizon entropy, respectively [23]. After this important conclusion, Akbar and Cai [27] obtained another form of the Friedman equations at dynamical apparent horizon

$$dE_I = \hat{T}_h dS + WdV, \quad (2)$$

where $dE_I = \rho V$, $W = 0.5(\rho - p)$ and V denote the internal energy, work density and volume, respectively. The problem of finding a relation between thermodynamics and gravity has been studied in several gravity theories such as the Gauss-Bonnet theory [27], Lovelock model [28], Braneworld theory [29], $f(R)$ gravity [30], $f(T)$ theory of gravity [31] and $f(R, \theta)$ model [32].

Encouragement comes from above studies guides us to investigate whether the first and second laws of galactic thermodynamics are satisfied in the framework of

teleparallel Kaluza-Klein theory. This investigation may offer some specific discussions and conclusions which would discriminate the teleparallel Kaluza-Klein reduction from various theories of modified gravity. Therefore, we assume that the teleparallel Kaluza-Klein universe is in thermal equilibrium state and composed of interacting dark matter and dark energy. To do so, in the next section, we review the teleparallel Kaluza-Klein cosmology. In the third section, we introduce the dark energy scenario that we use in further calculations. The fourth section is devoted to investigate thermodynamic properties of the teleparallel Kaluza-Klein theory and their theoretical implications. In the fifth section we give summary and final discussions.

2. Teleparallel Theory in Kaluza-Klein Scenario

The main concept in teleparallel theory of gravity is to assume a geometry with vanishing curvature and surviving torsion [33]. In this theory, we assume an orthonormal frame $v^i = h^i_\mu dx^\mu$ with a given metric $g_{\mu\nu} = \eta_{ij} h^i_\mu h^j_\nu$ and η_{ij} denotes the Minkowski metric. Using the tetrad field h^i_μ , the torsion tensor is defined as below [34]

$$T^i_{\mu\nu} = \partial_\mu h^i_\nu - \partial_\nu h^i_\mu, \quad (3)$$

and it is also related to the Weitzenböck connection which is given in the following form

$$\Gamma^{\lambda}_{\mu\nu} = h^{\lambda}_i \partial_\mu h^i_\nu. \quad (4)$$

In the teleparallel framework, the Lagrangian density is described as

$$L = \frac{1}{2\kappa} T - L_{matter}, \quad (5)$$

where $\kappa = 8\pi G$ denotes the gravitational coupling constant. Note that $T_a = T^b_{ba}$ [35] and the 4D torsion scalar is constructed as [36]

$$T = \frac{1}{4} T^{\mu\nu} T_{\mu\nu} + \frac{1}{2} T^{\mu\nu} T_{\mu\nu} - T^a T_a. \quad (6)$$

In the 5D teleparallel theory, the corresponding metric is given by

$$\bar{g}_{MN} = \bar{\eta}_{IJ} h^I_M h^J_N, \quad (7)$$

here we have $\bar{\eta}_{IJ} = \text{diag}(+1, -1, -1, -1, \varepsilon)$ with $\varepsilon = \mp 1$. The surviving torsion 2-form components in the orthonormal frame are \bar{T}^c_{ab} , \bar{T}^c_{5b} , \bar{T}^5_{a5} and the 5D torsion scalar ${}^{(5)}T$ can be written in a similar method given the Eq. (4) in five dimensions

$${}^{(5)}T = \bar{T} + \frac{1}{2} (\bar{T}^c_{k5j} \bar{T}^{k5j} + \bar{T}^c_{k5j} \bar{T}^{j5k}) + 2\bar{T}^k_{i5} \bar{T}^5_{i5} - \bar{T}^j_{5j} \bar{T}^{k5}, \quad (8)$$

where \bar{T} is the induced 4D torsion scalar [37]. In the original Kaluza-Klein theory, the 5D space-time bulk $V = H \times C^1$ is a product space of a hypersurface H and a circle C^1 with small radius r defining the compactified

extra dimension [37]. Because of the embedding, the 5D metric tensor \bar{g} can be defined in the coordinate system as

$$\bar{g}_{MN} = \delta_M^1 \delta_N^1 g_{\mu\nu}(x^\mu, y) + \delta_M^2 \delta_N^2 \varepsilon \phi^2(x^\mu, y), \quad (9)$$

with $y = x^5$ and $\varepsilon = -1$. Furthermore, in the orthonormal frame, the tetrad fields are h^i_μ and $h^5_5 = \phi$. Thence, in the coordinate frame, the 5D torsion scalar is written as

$${}^{(5)}T = \bar{T} + \frac{\bar{T}^{\mu 5\nu} \bar{T}^{\mu 5\nu} + \bar{T}^{\mu 5\nu} \bar{T}^{\nu 5\mu}}{2} + 2\bar{T}^{\nu\mu} \bar{T}^5_{\mu 5} - \bar{T}^{\nu 5} \bar{T}^{\mu 5}, \quad (10)$$

where $\bar{T} = T$ is a pure 4D object due to $\bar{T}^{\sigma}_{\mu\nu} = T^{\sigma}_{\mu\nu}$, and it is important to mention here that we also have $g_{\mu\nu,5} = 0$ which means all fields are free from the fifth-dimensional component (the Kaluza-Klein ansatz) [37,38]. The Kaluza-Klein ansatz reduces the metric (7) to the following form:

$$\bar{g}_{MN} = \delta_M^1 \delta_N^1 g_{\mu\nu}(x^\mu) - \delta_M^2 \delta_N^2 \phi^2(x^\mu). \quad (11)$$

Hence, the residual non-zero components of torsion tensor are obtained as $T^{\sigma}_{\mu\sigma}$ and $\bar{T}^5_{\mu 5} = \frac{1}{\phi} \partial_\mu \phi$. Next, we also need the new form of 5D torsion scalar to construct five dimensional action:

$${}^{(5)}T = T + 2T^{\nu\mu} \bar{T}^5_{\mu 5}. \quad (12)$$

The extra dimension $y = r\theta$ is the ground state of \bar{g}_{55} with radius r and the invariant volume element is defined as

$${}^{(5)}hd^5x = h\phi r d\theta d^4x \quad (13)$$

using the dimensional reduction ${}^{(5)}h = \det(h^i_M)$ and $h = \det(h^i_\mu)$ [37]. From this point of view, the 4D effective Lagrangian density can be written as

$$L_{\text{eff}} = \frac{1}{2\kappa_4} (\phi T + 2T^{\mu\nu} \partial_\mu \phi) - L_{matter}, \quad (14)$$

where $\kappa_4 := \frac{\kappa_5}{2\pi r}$ (with $\kappa_5 = 8\pi G_5$) and $T^\mu := T^{\nu\mu}$ are the effective coupling and the torsion trace vector, respectively. We see here that the form of effective Lagrangian density is not a scalar-tensor-like because of the extra non-minimal coupling $2T^{\mu\nu} \partial_\mu \phi$, which is different from the reduction Lagrangian density in general relativity [37]. By considering this Lagrangian density, one can get the following equation-of-motion for the 5D teleparallel gravity [37]:

$$\begin{aligned} & \frac{1}{2} h_i^\mu (\phi T + 2T^\sigma \partial_\sigma \phi) - h_i^\rho (\phi T^j_{\rho\nu} S_j^{\mu\nu}) \\ & + \frac{1}{h} \partial_\nu [h(\phi S_i^{\mu\nu} + h_i^\mu \partial^\nu \phi - h_i^\nu \partial^\mu \phi)] \\ & - h_i^\nu (\partial_\sigma \phi T^{\mu\sigma}_\nu + \partial_\nu \phi T^\mu + \partial^\mu \phi T_\nu) = \kappa_4 \Theta_i^\mu, \end{aligned} \quad (15)$$

where

$$\Theta_i^\mu = \frac{-1}{h} \frac{\delta L_{matter}}{\delta h_\mu^i}, \quad (16)$$

$$S_\rho^{\mu\nu} \equiv K_\rho^{\mu\nu} + \delta_\rho^\mu T_\sigma^{\sigma\nu} - \delta_\rho^\nu T_\sigma^{\sigma\mu} = -S_\rho^{\nu\mu}, \quad (17)$$

$$K = -\frac{1}{2}(T_{\mu\nu}^\rho - T_{\mu\nu}^\nu - T_{\nu\mu}^\rho) = -K_{\mu\nu}^\rho. \quad (18)$$

3. Dark Energy Scenario

The flat Friedmann-Lemaitre-Robertson-Walker universe is represented by the following metric tensor

$$g_{\mu\nu} = \delta_\mu^0 \delta_\nu^0 - a^2(t)[\delta_\mu^1 \delta_\nu^1 + \delta_\mu^2 \delta_\nu^2 + \delta_\mu^3 \delta_\nu^3], \quad (19)$$

and the non-vanishing tetrad components for the selected space-time model are

$$h_\mu^i = \delta_\mu^0 \delta_0^i + a(t)[\delta_\mu^1 \delta_1^i + \delta_\mu^2 \delta_2^i + \delta_\mu^3 \delta_3^i]. \quad (20)$$

Suppose that the teleparallel Kaluza-Klein space-time is filled with perfect dark fluid introduced by the next relation

$$\Theta_\nu^\mu = \text{diag}(\rho, -p, -p, -p), \quad (21)$$

where $\rho = \rho_e + \rho_m$ and $p = p_e + p_m$ represent the total energy and pressure densities, respectively (here the subscripts e and m denote dark energy and dark matter respectively). In such coordinates, the Friedmann equations are given by [37]

$$3\phi H^2 + 3H\dot{\phi} = \kappa_4 \rho, \quad (22)$$

$$3\phi H^2 + 2H\dot{\phi} + 2\phi\dot{H} + \ddot{\phi} = -\kappa_4 p, \quad (23)$$

where $H = \frac{\dot{a}}{a}$ denotes the Hubble parameter. It is easy to see that taking $\phi = 1$ reduces Eqs. (17) and (18) to the original Friedmann equations of the teleparallel gravity. Next, the continuity equation gives

$$\dot{\rho} + 3H(\rho + p) = 0. \quad (24)$$

Considering dark energy which is in interact with the dark matter, the continuity equations for the dark components can be written in the following form:

$$\dot{\rho}_m + 3H(\rho_m + p_m) = \Delta, \quad (25)$$

$$\dot{\rho}_e + 3H(\rho_e + p_e) = -\Delta. \quad (26)$$

Here we introduced Δ term due to the mutual interaction [39-43]. Cai and Su [44] obtained that the interaction term may pass through the non-interacting line ($\Delta_4 = 0$) and the sign of this term changes around $\frac{1}{2}$. Negative interaction term values describe the energy transfer from dark matter sector to dark energy one, while positive values of Δ mean there is an energy transfer flow from dark energy sector to the other dark territories [39]. Recently, it has been reported that Abell Cluster A586 observations show such transitions [45, 46]. Furthermore, this interesting behavior may appear effectively as a self-

conserved dark energy component as in the Λ XCDM scenario [39, 47-49]. Howbeit, the importance of this phenomena is not clearly explained [14], but it is consistent with the generalized second law of galactic thermodynamics both for the early time when $\hat{T}_m > \hat{T}_e$ and for the late time when $\hat{T}_m < \hat{T}_e$ (here \hat{T}_m and \hat{T}_e describe the temperature of dark contents) [50,51]. To be general, we assume the following representation of interaction:

$$\Delta = 3b^2 H(\rho_m + \rho_e) = 3b^2 H\rho_e(1 + \lambda), \quad (27)$$

where b is a coupling parameter for the interaction while $\lambda = \frac{\rho_m}{\rho_e}$ describes the energy ratio. It is obvious that signs of b^2 indicate the directions of energy transitions. Additionally, $b = 0$ case defines the non-interacting model and sometimes b^2 is chosen in the range [0,1] [52]. Galactic clusters and CMB observational evidences indicate that the interaction parameter has a small positive value of the order of the unity, i.e. $b^2 < 0.025$ [53,54] and the negative coupling parameter cases are avoided due to the violation of laws of galactic thermodynamics.

The equations of state are written as

$$p_m = \omega_m \rho_m, \quad p_e = \omega_e \rho_e. \quad (28)$$

And, the continuity Eqs. (20) and (21) in effective theory are given in the following forms

$$\dot{\rho}_m + 3H(1 + \omega_m^{\text{eff}})\rho_m = 0, \quad (29)$$

$$\dot{\rho}_e + 3H(1 + \omega_e^{\text{eff}})\rho_e = 0, \quad (30)$$

where

$$\omega_m^{\text{eff}} = \omega_m - \frac{\Delta}{3H\rho_m} = \omega_m - b^2(1 + \frac{1}{\lambda}), \quad (31)$$

$$\omega_e^{\text{eff}} = \omega_e + \frac{\Delta}{3H\rho_e} = \omega_e + b^2(1 + \lambda). \quad (32)$$

4. Thermodynamics Implications

It is seen in Refs [55, 56] that the temperature is connected with the surface gravity at the dynamical horizon while the entropy depends on the horizon area in black hole physics. The quantities such as the entropy and temperature satisfy the first law of galactic thermodynamics for a black hole [57]. This interesting behavior motivates several physicists to define a connection between Einstein's field equations and the black hole thermodynamics [24, 58, 59]. In Ref [24], Bekenstein constructed a relation between the black hole thermodynamics and event horizon. On the other hand, in 2006, Wang et al. [58] published very significant conclusions for the laws of galactic thermodynamics. They mainly found that our universe should be non-static and the usual thermodynamic descriptions may be more complicated than in the static space-time model.

The Gibb's law [59] of thermodynamics is

$$\hat{T}dS_I = PdV + dE_I. \quad (33)$$

From this point of view, we can write

$$dS = \frac{PdV + dE}{\hat{T}}, \quad (34)$$

hence, the time derivation of this relation gives

$$\dot{S}_I = \frac{P\dot{V} + \dot{E}_I}{\hat{T}}, \quad (35)$$

and using the above equation and assuming that the system is in equilibrium which indicates that all contents feel equal temperature value [24] yields the following results

$$\dot{S}_m = \frac{p_m\dot{V} + \dot{E}_m}{\hat{T}}, \quad \dot{S}_e = \frac{p_e\dot{V} + \dot{E}_e}{\hat{T}}. \quad (36)$$

Additionally, it is known that $E_m = \rho_m V$, $E_e = \rho_e V$.

4.1 The First Law of Thermodynamics

In this subsection, we investigate the first law of gravitational thermodynamics in the teleparallelKaluza-Klein universe which is bounded by a dynamical apparent horizon with size R_h . Note that, in the flat geometry, the dynamical apparent horizon coincides with the Hubble horizon, i.e. $R_h = \frac{1}{H}$.

Now, we consider the following form of the first law of thermodynamics

$$-dE_I = \hat{T}_h dS_I, \quad (37)$$

where the internal entropy is defined as $S_I = S_m + S_e$. Simple arguments show that all the fluids in the universe need equal temperature values after establishing of the equilibrium [60], because the energy flow may disfigure the space-time geometry [39, 59]. It is also known that the horizon temperature \hat{T}_h is related to its radius R_h , i.e. $\hat{T}_h = (2\pi R_h)^{-1}$ [23, 39, 61]. The measure of energy flowing through dynamical horizon is given by the next formulation [62]

$$-dE_I = 4\pi R_h^3 H \Theta_{\mu\nu} U^\mu U^\nu dt, \quad (38)$$

where U^μ is the four velocity vector such that $U^\mu U_\mu = 1$. Thence we find

$$-dE_I = 4\pi R_h^3 H (\rho + p) dt = \frac{4\pi}{\kappa_4 H^2} (H\dot{\phi} - 2\phi\dot{H} - \ddot{\phi}) dt. \quad (39)$$

On the other hand, using Eqs. (35) and (36) we obtain

$$\hat{T}_h \frac{dS_I}{dt} = (p + \rho) \frac{dV}{dt} + V \frac{d\rho}{dt}, \quad (40)$$

where $V = \frac{4}{3}\pi R_h^3$. Furthermore, considering the continuity Eq. (24) it is calculated that

$$\hat{T}_h \frac{dS_I}{dt} = -\frac{dE}{dt} - \zeta, \quad (41)$$

where

$$\zeta = \frac{4\pi}{\kappa_4 H^2} (H\dot{\phi} - 2\phi\dot{H} - \ddot{\phi}) \left[\frac{\dot{H}}{3H^2} + 2 \right]. \quad (42)$$

Thence, we can write

$$-dE_I = \hat{T}_h dS_I + \zeta dt. \quad (43)$$

We get a time-dependent extra term in the right-hand-side of this relation, so that this additional term cannot be equal to zero during the certain stage of space-time evolution. Hence, we may conclude that

$$-dE_I \neq \hat{T}_h dS_I. \quad (44)$$

Thus, the first law of gravitational thermodynamics does not hold in teleparallelKaluza-Klein theory for the flat Friedmann-Lemaitre-Robertson-Walker type universe on the apparent horizon. It is known well [63,64] that all matter fields feel the same horizon and have the same Hawking temperature, but some degrees of freedom may see a different horizon definition, space-time metric and Hawking temperature. Due to black holes cannot be in equilibrium in such cases, the first law of galactic thermodynamics is not satisfied. In literature, discussing the effect of spontaneous Lorentz invariance breaking on black hole thermodynamics Dubovski and Sibiryakov [63] have found a similar conclusion.

In 2007, Akbar and Cai [27] found another form of the first law of gravitational thermodynamics

$$-dE_I + WdV = \hat{T}_h dS_I \quad (45)$$

at the dynamical apparent horizon. Here, $W = \frac{1}{2}(\rho - P)$ shows the work density. Next, one may write again Eq. (41) as given below

$$\hat{T}_h dS_I + \hat{T}_h d\tilde{S} = -dE_I + WdV, \quad (46)$$

where

$$W = \frac{1}{2\kappa_4} (6\phi H^2 + 5H\dot{\phi} + 2\phi\dot{H} + \ddot{\phi}), \quad (47)$$

and

$$\hat{T}_h d\tilde{S} = \frac{2\pi}{\kappa_4 H^2} \left((\ddot{\phi} - H\dot{\phi} + 2\phi\dot{H}) \left[\frac{\dot{H}}{3H^2} + 4 \right] - \frac{\dot{H}}{H^2} (6\phi H^2 + 5H\dot{\phi} + 2\phi\dot{H} + \ddot{\phi}) \right) dt. \quad (48)$$

The additional term may be interpreted as extra entropy term comes from the non-equilibrium picture in teleparallelKaluza-Klein theory.

4.2 The Generalized Second Law of Thermodynamics

We now examine the derivative of total entropy, i.e. $S_I = S_m + S_e + S_h$ where S_h denotes the horizon entropy, for the second law of galactic thermodynamics which must be valid on the apparent horizon.

From Eqs. (41) with (42), we have

$$\frac{d(S_m + S_c)}{dt} = \frac{2}{H\kappa_4} (\ddot{\phi} - H\dot{\phi} + 2\phi\dot{H}) \left[1 + \frac{\dot{H}}{3H^2} \right]. \quad (49)$$

On the other hand, concerning the cosmological horizon entropy, we can define it as $S_h = \frac{A}{4G_4}$ with the surface area $A = 4\pi R_h^2$ [23, 39, 61]. Thence we obtain

$$\frac{dS_h}{dt} = -\frac{16\pi^2}{\kappa_4} \frac{\dot{H}}{H^3}. \quad (50)$$

Let us now proceed to the discussion of galactic thermodynamics' second law. We finally calculate the total entropy variation. Using Eqs. (49) and (50), we find:

$$\dot{S}_t = \frac{2}{H\kappa_4} \left\{ (\ddot{\phi} - H\dot{\phi} + 2\phi\dot{H}) \left[1 + \frac{\dot{H}}{3H^2} \right] - \frac{8\pi\dot{H}}{H^2} \right\}. \quad (51)$$

This result shows that the validity of second law, i.e. $\hat{T}_h \dot{S}_t \geq 0$, depends on the teleparallelKaluza-Klein model. For instance, in the teleparallel gravity limit of the theory, i.e. $\phi = 1$, Eq. (51) yields

$$\hat{T}_h \dot{S}_t = \frac{2\dot{H}}{\pi\kappa_4} \left\{ 1 + \frac{\dot{H}}{3H^2} - \frac{4\pi}{H^2} \right\}, \quad (52)$$

and assuming the de Sitter scale factor $a(t) \sim e^{Ht}$ with the case $H = \text{constant}$ gives $\hat{T}_h \dot{S}_t = 0$ which represents the reversible adiabatic phase of our universe. In addition to this implication, $\hat{T}_h \dot{S}_t$ may also tend to infinity which may happen when the expansion rate is very high at the very large time ($t \rightarrow \infty$). In this outlandish condition, all the usable form of energy in our universe will be transformed into an unusable form, the nature of the entropy will be very enigmatic and the cosmological entropy will reach its maximum value. In the language of physics, this is the heat death scenario of a system which is known also as one of the suggested fates of our universe. At this phase, all the thermodynamics free energy in the universe will be decreased and the motion of life cannot sustain any more.

Next, by making use of Friedmann Eqs. (22) and (23), we obtain

$$\rho + p = \frac{1}{\kappa_4} (H\dot{\phi} - 2\phi\dot{H} + \ddot{\phi}). \quad (53)$$

Inserting this value into Eq. (51), it follows that

$$\hat{T}_h \dot{S}_t = -\frac{1}{\pi} \left\{ (\rho + p) \left[1 + \frac{\dot{H}}{3H^2} \right] + \frac{\dot{H}}{G_4 H^2} \right\}, \quad (54)$$

and $\dot{S}_t \geq 0$ implies that

$$\rho + p \geq \frac{-3\dot{H}}{G_4(\dot{H} + 3H^2)}. \quad (55)$$

Now, we focus on some interesting cases involving special conditions. In the teleparallelKaluza-Klein reduction, the equation-of-motion of a scalar field is written as [37]

$$T - 2\partial_\mu T^\mu - 2T^\mu \Gamma_{\nu\mu}^\nu = 0, \quad (56)$$

where we have $\Gamma_{\nu\mu}^\nu = h_i^\nu \partial_\mu h_\nu^i$ with the surviving component

$h_k^\beta \partial_0 h_\beta^k = \frac{3\dot{a}}{a}$. Subsequently, Geng et al. [37] considering this result obtained two special cases. Here, it is seen that Wq. (51) is free from the scalar field model and one can easily find the solution of scale factor. Assuming the solution to be proportional to t^m , they found that possible m values are 0 and $\frac{1}{2}$. Ergo, their results lead us to the following relation

$$a(t) = a_s + b\sqrt{t}, \quad (57)$$

where $a_s (> 0)$ and b are two constant parameters. By substituting this solution into the Eq. (51), Geng et al. [37] found that $a_s b = 0$ must hold and introduced two special cases: (i) $a_s \neq 0, b = 0$ and (ii) $a_s = 0, b \neq 0$. For $a_s \neq 0, b = 0$, they obtained $a(t) = a_s > 0$ which corresponds to a static universe, where the parameter a_s is the scale factor of the acceptable energy scale for the teleparallelKaluza-Klein reduction. Due to the first case describes a static universe, we will ignore this condition. Besides, for the second case $a_s = 0, b \neq 0$, Geng et al. [37] found that $a(t) = b\sqrt{t}$, $H = \frac{1}{2t}$ and $\ddot{a} = -\frac{b}{4t^{3/2}}$. Hence, from this result, one can see that the condition $b < 0$ defines an accelerated expansion while $b > 0$ gives decelerated expansion.

Now we turn back Eqs. (54) and (55). By considering the second case $a(t) = b\sqrt{t}$, we found that $\rho + p \geq \frac{6}{G_4}$ which is the null energy condition. Consequently, we get $\dot{S}_t \geq 0$ and can say that the generalized second law of gravitational thermodynamics holds through the history of time. Therefore, by considering the result produced in this special case, one can say that the validity of galactic second law depends on the null energy condition and background geometry.

5. Discussion

The present study has been devoted to discuss the galactic nature of entropy in the Kaluza-Klein reduction of teleparallel cosmology which is not a scalar-tensor-like theory because of an extra non-minimal coupling term. First of all, we reviewed the teleparallelKaluza-Klein cosmology. After that, assuming our universe as a thermodynamic system which is bounded by the dynamical horizon R_h , we have obtained separately the time-derivation of entropy function for each dark components and for the dynamical horizon itself to investigate thermodynamic properties of the selected model. We prove that the thermodynamic laws are independent of the specific interaction term. Additionally, we have used the thermodynamics laws to investigate the galactic behavior of total entropy function. According to this investigation, we have pointed out some

special cases: (i) the effect of spontaneous Lorentz invariance breaking on the first law of thermodynamics, (ii) the reversible adiabatic scenario, (iii) the heat death fate of our universe, (iv) the null energy condition which indicates the validity of second law.

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Nomenclature

The Greek alphabet ($\mu, \nu, \dots = 0, 1, 2, 3$) represents the space-time while the Latin alphabet ($a, b, \dots = 0, 1, 2, 3$) denotes the tangent space. Additionally, ($M, N, \dots = 0, 1, 2, 3, 5$) and ($I, J, \dots = \hat{0}, \hat{1}, \hat{2}, \hat{3}, \hat{5}$).

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