The real parts of the nontrivial Riemann zeta function zeros

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ABSTRACT

This theorem is based on holomorphy of studied functions and the fact that nearby of a singularity point the real part of some rational function can take random arbitrary preassigned value.

The colored markers are:
- 🔴 - assumption or a fact which is not proven at present;
- 🟡 - the statement which requires additional attention;
- 🟢 - statement which is proved earlier or clearly undestandable.

THEOREM

🔴 The real parts of all the nontrivial Riemann zeta function zeros ρ are equal $Re\left(\rho\right) = \frac{1}{2}$.

PROOF:

🟢 According to the functional equality [10, p. 22], [6, p. 8-11]:

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta\left(s\right) = \Gamma\left(\frac{1-s}{2}\right)\pi^{-\frac{1-s}{2}}\zeta\left(1-s\right), \qquad Re\left(s\right) > 0 \qquad (1)$$

$\zeta\left(s\right)$ - the Riemann zeta function, $\Gamma\left(s\right)$ - the Gamma function.

- From [6, p. 8-11] $\zeta(\bar{s}) = \overline{\zeta(s)}$, it means that $\forall \rho = \sigma + it$: $\zeta(\rho) = 0$ and $0 \leqslant \sigma \leqslant 1$ we have:

$$\zeta(\bar{\rho}) = \zeta(1-\rho) = \zeta(1-\bar{\rho}) = 0 \tag{2}$$

- From [11], [9, p. 128], [10, p. 45] we know that $\zeta(s)$ has no nontrivial zeros on the line $\sigma = 1$ and consequently on the line $\sigma = 0$ also, in accordance with (2) they don't exist.

- Let's denote the set of nontrivial zeros $\zeta(s)$ through $\mathcal{P}$ (multiset with consideration of multiplicitiy):

$$\mathcal{P} \stackrel{\text{def}}{=} \{\rho : \ \zeta(\rho) = 0, \ \rho = \sigma + it, \ 0 < \sigma < 1\}.$$

$$\text{And: } \mathcal{P}_1 \stackrel{\text{def}}{=} \left\{\rho : \ \zeta(\rho) = 0, \ \rho = \sigma + it, \ 0 < \sigma < \frac{1}{2}\right\}, \tag{3}$$

$$\mathcal{P}_2 \stackrel{\text{def}}{=} \left\{\rho : \ \zeta(\rho) = 0, \ \rho = \frac{1}{2} + it\right\},$$

$$\mathcal{P}_3 \stackrel{\text{def}}{=} \left\{\rho : \ \zeta(\rho) = 0, \ \rho = \sigma + it, \ \frac{1}{2} < \sigma < 1\right\}.$$

Then:

$$\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 \quad \text{and} \quad \mathcal{P}_1 \cap \mathcal{P}_2 = \mathcal{P}_2 \cap \mathcal{P}_3 = \mathcal{P}_1 \cap \mathcal{P}_3 = \varnothing,$$

$$\mathcal{P}_1 = \varnothing \Leftrightarrow \mathcal{P}_3 = \varnothing.$$

- Hadamard's theorem (Weierstrass preparation theorem) on the decomposition of function through the roots gives us the following result [10, p. 30], [6, p. 31], [12]:

$$\zeta(s) = \frac{\pi^{\frac{s}{2}} e^{as}}{s(s-1)\Gamma\left(\frac{s}{2}\right)} \prod_{\rho \in \mathcal{P}} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}, \qquad Re(s) > 0 \tag{4}$$

$$a = \ln 2\sqrt{\pi} - \frac{\gamma}{2} - 1, \ \gamma - \text{Euler's constant and}$$

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{2}\ln\pi + a - \frac{1}{s} + \frac{1}{1-s} - \frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} + \sum_{\rho\in\mathcal{P}}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) \quad (5)$$

- According to the fact that $\dfrac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}$ - Digamma function of [10, p. 31], [6, p. 23] we have:

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{1-s} + \sum_{\rho\in\mathcal{P}}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) + \sum_{n=1}^{\infty}\left(\frac{1}{s+2n} - \frac{1}{2n}\right) + C, \quad (6)$$

$$C = const$$

.

- From [5, p. 160], [8, p. 272], [4, p. 81]:

$$\sum_{\rho\in\mathcal{P}}\frac{1}{\rho} = 1 + \frac{\gamma}{2} - \ln 2\sqrt{\pi} = 0,0230957\ldots \quad (7)$$

- Indeed, from (2):

$$\sum_{\rho\in\mathcal{P}}\frac{1}{\rho} = \frac{1}{2}\sum_{\rho\in\mathcal{P}}\left(\frac{1}{1-\rho} + \frac{1}{\rho}\right)$$

- From (5):

$$2\sum_{\rho\in\mathcal{P}}\frac{1}{\rho} = \lim_{s\to 1}\left(\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{1-s} + \frac{1}{s} - a - \frac{1}{2}\ln\pi + \frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\right).$$

- Also it's known, for example, from [10, p. 49], [4, p. 98] that the number of nontrivial zeros of $\rho = \sigma + it$ in strip $0 < \sigma < 1$, the imaginary parts of which t are less than some number $T > 0$ is limited, i.e.

$$\|\{\rho:\ \rho\in\mathcal{P},\ \rho = \sigma + it,\ |t| < T\}\| < \infty.$$

- Indeed, it can be presented that on the contrary the sum of $\sum_{\rho\in\mathcal{P}}\frac{1}{\rho}$ would have been unlimited.

- Thus $\forall T > 0 \; \exists \; \delta_x > 0, \; \delta_y > 0$ such that

$$\text{in area } 0 < t \leqslant \delta_y, 0 < \sigma \leqslant \delta_x \text{ there are no zeros } \rho = \sigma + it \in \mathcal{P}. \qquad (8)$$

Let's consider random root $q \in \mathcal{P}_1 \cup \mathcal{P}_2$
Let's denote $k(q)$ the multiplicity of the root q.
Let's examine the area $Q(R) \stackrel{\text{def}}{=} \{s : \|s - q\| \leqslant R, R > 0\}$.

- From the fact of finiteness of set of nontrivial zeros $\zeta(s)$ in the limited area follows $\exists \; R > 0$, such that $Q(R)$ does not contain any root from $\mathcal{P}$ except q.

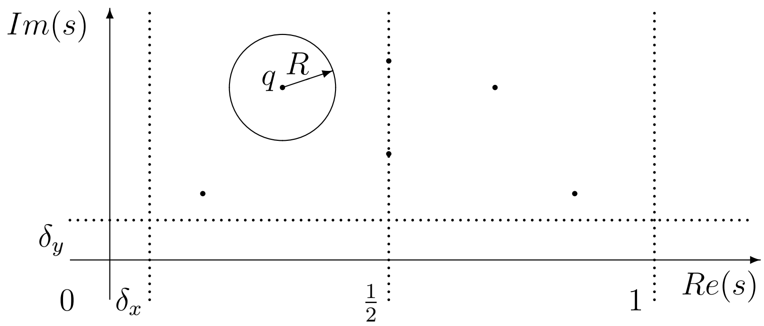

Fig. 1.

- From [1], [10, p. 31], [6, p. 23] we know that the Digamma function $\dfrac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}$ in the area $Q(R)$ has no poles, i.e. $\forall s \in Q(R)$

$$\left\| \frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \right\| < \infty.$$

Let's denote:

$$I_{\mathcal{P}}(s) \stackrel{\text{def}}{=} -\frac{1}{s} + \frac{1}{1-s} + \sum_{\rho \in \mathcal{P}} \frac{1}{s - \rho}$$

and

$$I_{\mathcal{P}\setminus\{q\}}(s) = -\frac{1}{s} + \frac{1}{1-s} + \sum_{\rho \in \mathcal{P}\setminus\{q\}} \frac{1}{s - \rho}. \qquad (9)$$

Hereinafter $\mathcal{P} \setminus \{q\} \overset{\text{def}}{=} \mathcal{P} \setminus \{(q, k(q))\}$ (the difference in the multiset).

Summation - $\sum_{\rho\in\mathcal{P}} \dfrac{1}{s-\rho}$ and $\sum_{\rho\in\mathcal{P}\setminus\{q\}} \dfrac{1}{s-\rho}$ further we shall consider as the sum of pairs $\left(\dfrac{1}{s-\rho} + \dfrac{1}{s-(1-\rho)}\right)$ and $\sum_{\rho\in\mathcal{P}} \dfrac{1}{\rho}$ as the sum of pairs $\left(\dfrac{1}{\rho} + \dfrac{1}{1-\rho}\right)$ as a consequence of division of the sum from (6) $\sum_{\rho\in\mathcal{P}} \left(\dfrac{1}{s-\rho} + \dfrac{1}{\rho}\right)$ into $\sum_{\rho\in\mathcal{P}} \dfrac{1}{s-\rho} + \sum_{\rho\in\mathcal{P}} \dfrac{1}{\rho}$. As specifed in [5], [7], [8], [10].

- Let's note that $I_{\mathcal{P}\setminus\{q\}}(s)$ is holomorphic function $\forall\ s \in Q(R)$.

Then from (5) we have:

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{2}\ln\pi + a - \frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} + \sum_{\rho\in\mathcal{P}} \frac{1}{\rho} + I_{\mathcal{P}}(s). \tag{10}$$

And in view of (7):

$$Re\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{2}\ln\pi + Re\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} + I_{\mathcal{P}}(s)\right). \tag{11}$$

Let's note that from the equality of

$$\sum_{\rho\in\mathcal{P}} \frac{1}{1-s-\rho} = -\sum_{(1-\rho)\in\mathcal{P}} \frac{1}{s-(1-\rho)} = -\sum_{\rho\in\mathcal{P}} \frac{1}{s-\rho} \tag{12}$$

follows that:

$$I_{\mathcal{P}}(1-s) = -I_{\mathcal{P}}(s),\ I_{\mathcal{P}\setminus\{q\}}(1-s) = -I_{\mathcal{P}\setminus\{1-q\}}(s),\ Re\,(s) > 0.$$

- Besides

$$I_{\mathcal{P}\setminus\{q\}}(s) = I_{\mathcal{P}}(s) - \frac{k(q)}{s-q}$$

and $I_{\mathcal{P}\setminus\{q\}}(s)$ is limited in the area of $s \in Q(R)$ as a result of absence of its poles in this area as well as its differentiability in each point of this area.

- If in (5) to replace s with $1-s$ that in view of (7), in a similar way if to take derivative of the basic logarithm (1):

$$\frac{\zeta'(s)}{\zeta(s)}+\frac{\zeta'(1-s)}{\zeta(1-s)}=-\frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}-\frac{1}{2}\frac{\Gamma'\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}+\ln\pi,\ Re\,(s)>0. \quad (13)$$

- Let's examine a circle with the center in a point q and radius $r \leqslant R$, laying in the area of $Q(R)$:

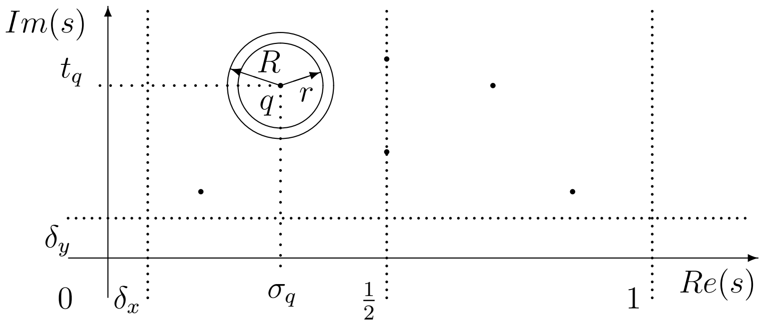

Fig. 2.

- For $s=x+iy,\ q=\sigma_q+it_q$

$$Re\frac{k(q)}{s-q}=Re\frac{k(q)}{x+iy-\sigma_q-it_q}=\frac{k(q)(x-\sigma_q)}{(x-\sigma_q)^2+(y-t_q)^2}=k(q)\frac{x-\sigma_q}{r^2}.$$

- Let us prove the following Lemma:

LEMMA 1

$$\forall\, q\in\mathcal{P}$$

$$\exists\, 0<R_q\leqslant R:\ \ \forall\, 0<r\leqslant R_q\ \ \exists\, m_r:\|m_r-q\|=r,\ Im(m_r)\leqslant Im(q),$$

$$Re\frac{\zeta'(m_r)}{\zeta(m_r)}-Re\frac{\zeta'(1-m_r)}{\zeta(1-m_r)}=0 \quad (14)$$

And for the angle β_{m_r} between the ordinate axis and the straight line passing through the points q and m_r, the following equality holds:

$$\operatorname{tg}\beta_{m_r}=O(r)_{r\to 0}. \quad (15)$$

PROOF:

For $s \in Q(R)$ we consider the function:

$$Re\frac{\zeta'(s)}{\zeta(s)} - Re\frac{\zeta'(1-s)}{\zeta(1-s)} - 2Re\frac{k(q)}{s-q}$$

From (11) and (12), it is equal to:

$$Re\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} + \frac{1}{2}\frac{\Gamma'\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} + 2I_{\mathcal{P}\setminus\{q\}}(s)\right).$$

Since all components of the brace are limited in the area of $s \in Q(R)$, then $\exists\ H_1(R) > 0 : H_1(R) \in \mathbb{R}$:

$$\left|Re\frac{\zeta'(s)}{\zeta(s)} - Re\frac{\zeta'(1-s)}{\zeta(1-s)} - Re\frac{2k(q)}{s-q}\right| < H_1(R), \qquad \forall s \in Q(R).$$

- On each of the semicircles: the bottom -

$\{s : \|s-q\| = r,\ t_q - r \leqslant y \leqslant t_q\}$ and the upper -

$\{s : \|s-q\| = r,\ t_q \leqslant y \leqslant t_q + r\}$ the function $Re\frac{k(q)}{s-q}$ is continuous and takes values from $-\frac{k(q)}{r}$ to $\frac{k(q)}{r}$, $r > 0$.

Consequently $\forall\ 0 < r < \frac{2k(q)}{H_1(R)}$, $\exists\ m_{min,r},\ m_{max,r}$:

$\|m_{min,r} - q\| = r,\ \|m_{max,r} - q\| = r$:

$$Re\frac{2k(q)}{m_{min,r} - q} < -H_1(R),\ Re\frac{2k(q)}{m_{max,r} - q} > H_1(R)$$

and the sum of two functions:

$$Re\frac{\zeta'(s)}{\zeta(s)} - Re\frac{\zeta'(1-s)}{\zeta(1-s)} - Re\frac{2k(q)}{s-q} \qquad \text{and} \qquad Re\frac{2k(q)}{s-q}$$

in points $m_{min,r}$ and $m_{max,r}$ will have values with a different signs.

From the property of a function continuous on a segment to take all the intermediate values between its extrema, it follows that $\exists\ R_q \in \mathbb{R}$,
$R_q > 0$:

$$R_q < R,\ \frac{2k(q)}{R_q} > H_1(R)$$

and then $\forall\ 0 < r \leqslant R_q$
exists on the lower semicircle point $m_r \overset{\text{def}}{=} x_{m_r} + iy_{m_r}$ such that:

$$Re\frac{\zeta'(m_r)}{\zeta(m_r)} - Re\frac{\zeta'(1-m_r)}{\zeta(1-m_r)} = 0.$$

- From (13) and (14) it follows that $\forall\ 0 < r \leqslant R_q$:

$$Re\frac{\zeta'(m_r)}{\zeta(m_r)} = Re\frac{\zeta'(1-m_r)}{\zeta(1-m_r)} =$$
$$= \frac{1}{2}\ln\pi + \frac{1}{2}Re\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{m_r}{2}\right)}{\Gamma\left(\frac{m_r}{2}\right)} - \frac{1}{2}\frac{\Gamma'\left(\frac{1-m_r}{2}\right)}{\Gamma\left(\frac{1-m_r}{2}\right)}\right). \tag{16}$$

- I.e. taking into account the absence of singular points for $\Gamma(s)$, $\forall\ s \in Q(R)$ for $r \to 0$:

$$Re\frac{\zeta'(m_r)}{\zeta(m_r)} = Re\frac{\zeta'(1-m_r)}{\zeta(1-m_r)} = O(1). \tag{17}$$

Point m_r:

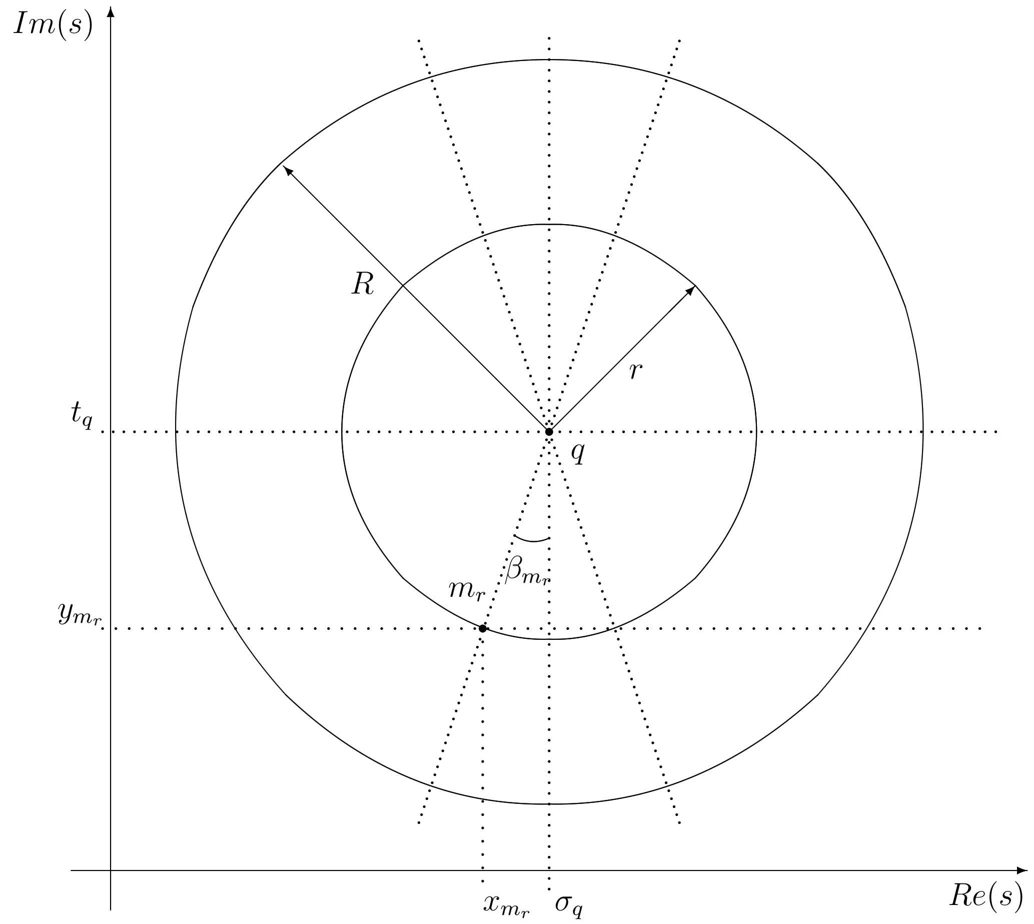

Fig. 3.

- In the case if $y_{m_r} \neq t_q$ the tangent modulus of the angle β_{m_r} is equal to:

$$|\mathrm{tg}\,\beta_{m_r}| = \frac{|\sigma_q - x_{m_r}|}{t_q - y_{m_r}}.$$

From (11) it follows that:

$$k(q)\frac{x_{m_r} - \sigma_q}{r^2} = Re\frac{\zeta'(m_r)}{\zeta(m_r)} - \frac{1}{2}\ln\pi - Re\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{m_r}{2}\right)}{\Gamma\left(\frac{m_r}{2}\right)} + I_{\mathcal{P}\setminus\{q\}}(m_r)\right).$$

In view of (17) at $r \to 0$:

$$\frac{x_{m_r} - \sigma_q}{r^2} = O(1).$$

Then from equality:

$$(\sigma_q - x_{m_r})^2 + (t_q - y_{m_r})^2 = r^2$$

it follows that when $r \to 0$:

$$(t_q - y_{m_r})^2 = r^2 - O(r^4).$$

- I.e. $\exists\, 0 < R_1 \leqslant R_q : \quad \forall\, 0 < r < R_1$

$$t_q - y_{m_r} \neq 0$$

and therefore $r \to 0$:

$$\operatorname{tg} \beta_{m_r} = \frac{O(r^2)}{\theta(r)} = O(r).$$

□

- Let's prove the second lemma:

LEMMA 2

$$\forall\, q \in \mathcal{P}$$

-

$$Re \left(\frac{\Gamma'\left(\frac{q}{2}\right)}{\Gamma\left(\frac{q}{2}\right)} \right)' = Re \left(\frac{\Gamma'\left(\frac{1-q}{2}\right)}{\Gamma\left(\frac{1-q}{2}\right)} \right)'. \tag{18}$$

PROOF:

- From the first Lemma $\forall \quad 0 < r \leqslant R_q$, for $s = x + iy : \quad \|s - q\| = r$ consider the function:

$$g(x,y) \stackrel{\text{def}}{=} Re\frac{\zeta'(s)}{\zeta(s)} Re\frac{\zeta'(1-s)}{\zeta(1-s)}.$$

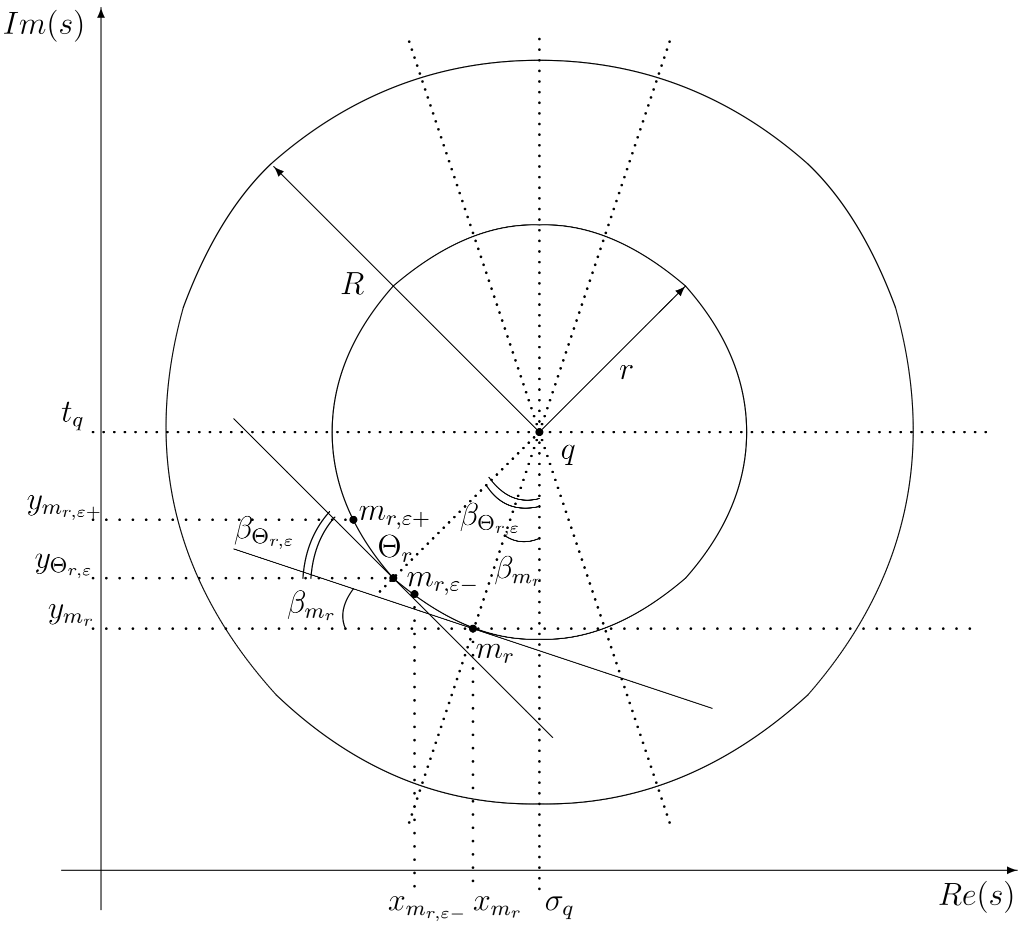

Fig. 4.

For any $\varepsilon > 0$, in view of that function $Re\dfrac{k(q)}{s-q}$ is continuous and accepts values from $-\dfrac{k(q)}{r}$ up to $\dfrac{k(q)}{r}$, there should be a radius $0 < R_2 \leqslant R_q$: $\forall\, 0 < r \leqslant R_2 : \;\; \exists\, m_{r,\varepsilon+},\; m_{r,\varepsilon-}$:

$$Re\frac{\zeta'\left(m_{r,\varepsilon+}\right)}{\zeta\left(m_{r,\varepsilon+}\right)} = Re\frac{\zeta'\left(m_r\right)}{\zeta\left(m_r\right)} + \varepsilon, \quad Re\frac{\zeta'\left(m_{r,\varepsilon-}\right)}{\zeta\left(m_{r,\varepsilon-}\right)} = Re\frac{\zeta'\left(m_r\right)}{\zeta\left(m_r\right)} - \varepsilon. \qquad (19)$$

From (16) follows:

$$Re\frac{\zeta'\left(m_r\right)}{\zeta\left(m_r\right)} = Re\frac{\zeta'\left(1-m_r\right)}{\zeta\left(1-m_r\right)} = \frac{1}{2}\left(Re\frac{\zeta'\left(m_r\right)}{\zeta\left(m_r\right)} + Re\frac{\zeta'\left(1-m_r\right)}{\zeta\left(1-m_r\right)}\right),$$

that means in view of (13):

$$Re\frac{\zeta'\left(1-m_{r,\varepsilon+}\right)}{\zeta\left(1-m_{r,\varepsilon+}\right)} = Re\frac{\zeta'\left(m_r\right)}{\zeta\left(m_r\right)} - \varepsilon, \quad Re\frac{\zeta'\left(1-m_{r,\varepsilon-}\right)}{\zeta\left(1-m_{r,\varepsilon-}\right)} = Re\frac{\zeta'\left(m_r\right)}{\zeta\left(m_r\right)} + \varepsilon. \qquad (20)$$

Let's designate:

$$x_{m_{r,\varepsilon+}} + iy_{m_{r,\varepsilon+}} \stackrel{\text{def}}{=} m_{r,\varepsilon+}, \quad x_{m_{r,\varepsilon-}} + iy_{m_{r,\varepsilon-}} \stackrel{\text{def}}{=} m_{r,\varepsilon-}.$$

From (19) taking into account the unique calculation of the function $Re\dfrac{\zeta'(s)}{\zeta(s)}$ the points $m_{r,\varepsilon+}$ and $m_{r,\varepsilon-}$ are different for $\forall\ \varepsilon > 0$. Without loss of generality, we assume that $x_{m_{r,\varepsilon+}} < x_{m_{r,\varepsilon-}}$.

- In these various points according to (19) and (20) function $g(x, y)$ accepts identical values:

$$g(x_{m_{r,\varepsilon+}}, y_{m_{r,\varepsilon+}}) = g(x_{m_{r,\varepsilon-}}, y_{m_{r,\varepsilon-}}) = \left(Re\frac{\zeta'(m_r)}{\zeta(m_r)}\right)^2 - \varepsilon^2.$$

- The points $m_{r,\varepsilon+}$ and $m_{r,\varepsilon-}$ lie on a circle with center at the point q and radius r, i.e. all the points $s = x+iy$ of the smallest of the arcs that connects them satisfy the equation:

$$y = f_r(x) \stackrel{\text{def}}{=} t_q - \sqrt{r^2 - (\sigma_q - x)^2}.$$

And:

$$f_r(x_{m_{r,\varepsilon+}}) = y_{m_{r,\varepsilon+}}, \ f_r(x_{m_{r,\varepsilon-}}) = y_{m_{r,\varepsilon-}}.$$

- The function $g(x, y)$ is differentiable, which means that $g(x, f_r(x))$ is continuous and differentiable with respect to x.

Thus, a real function that is continuous and differentiable on the inner interval takes on the same values at its ends:

$$g(x_{m_{r,\varepsilon+}}, f_r(x_{m_{r,\varepsilon+}})) = g(x_{m_{r,\varepsilon-}}, f_r(x_{m_{r,\varepsilon-}})).$$

- By Rolle's theorem on the extremum of a differentiable function on an interval, we have:

$$\exists\ x_{\Theta_{r,\varepsilon}} \in (x_{m_{r,\varepsilon+}}, x_{m_{r,\varepsilon-}}) : \quad (g(x, f_r(x)))'_{x=x_{\Theta_{r,\varepsilon}}} = 0. \qquad (21)$$

I.e. $\forall\ \varepsilon > 0,\ \forall\ 0 < r \leqslant R_2$ on the arc described by the function $f_r(x), \quad x \in (x_{m_{r,\varepsilon+}}, x_{m_{r,\varepsilon-}})$ there is a point $\Theta_{r,\varepsilon} \stackrel{\text{def}}{=} (x_{\Theta_{r,\varepsilon}}, f_r(x_{\Theta_{r,\varepsilon}}))$ for which the equality (21) is true.

Let $\beta_{\Theta_{r,\varepsilon}}$ be the angle between the ordinate axis and the straight line passing through the points q and $\Theta_{r,\varepsilon}$.

- By the continuity of the function $Re\dfrac{\zeta'(x+if_r(x))}{\zeta(x+if_r(x))}$ for $\forall\ x \in (x_{m_{r,\varepsilon+}}, x_{m_{r,\varepsilon-}})$ and from (11) and (19) at $r \to 0$:

$$k(q)\frac{x_{m_r} - m_{r,\varepsilon-}}{r^2} = \varepsilon + O(x_{m_r} - m_{r,\varepsilon-}),\ k(q)\frac{x_{m_r} - m_{r,\varepsilon+}}{r^2} = -\varepsilon + O(x_{m_r} - m_{r,\varepsilon+})$$

follows that $\exists\ 0 < R_3 \leqslant R_2 :\ \forall\ 0 < r \leqslant R_3$ we have:

$$\lim_{\varepsilon\to 0} m_{r,\varepsilon+} = m_r,\quad \lim_{\varepsilon\to 0} m_{r,\varepsilon-} = m_r,$$

it means that:

$$\lim_{\varepsilon\to 0} \Theta_{r,\varepsilon} = m_r,\quad \lim_{\varepsilon\to 0} \beta_{\Theta_{r,\varepsilon}} = \beta_{m_r}$$

and taking into account the infinite differentiability of the function $Re\dfrac{\zeta'(x+if_r(x))}{\zeta(x+if_r(x))}$ for $\forall\ x \in (x_{m_{r,\varepsilon+}}, x_{m_{r,\varepsilon-}})$, i.e. to appropriating continuity of derivative function $g(x, f_r(x))$ from equality (21) follows:

$$(g(x, f_r(x)))'_{x=x_{m_r}} = 0. \tag{22}$$

- This equality, taking into account the fact that the angle β_{m_r} between the axis of ordinates and the line passing through the points q and m_r coincides with the angle of inclination of the tangent passing through the point m_r, can be written as follows:

$$(g(x, f_r(x)))'_{x=x_{m_r}} =$$

$$= \frac{d}{dx}\left(Re\frac{\zeta'(x+if_r(x))}{\zeta(x+if_r(x))} Re\frac{\zeta'(1-x-if_r(x))}{\zeta(1-x-if_r(x))}\right)_{x=x_{m_r}} =$$

$$= Re\frac{\zeta'(m_r)}{\zeta(m_r)}\left(Re\frac{\zeta'(m_r)}{\zeta(m_r)}\right)'_x - Re\frac{\zeta'(1-m_r)}{\zeta(1-m_r)}\left(Re\frac{\zeta'(1-m_r)}{\zeta(1-m_r)}\right)'_x +$$
$$+\operatorname{tg}\beta_{m_r}\left(Re\frac{\zeta'(m_r)}{\zeta(m_r)}\left(Re\frac{\zeta'(m_r)}{\zeta(m_r)}\right)'_y -\right.$$
$$\left.-Re\frac{\zeta'(1-m_r)}{\zeta(1-m_r)}\left(Re\frac{\zeta'(1-m_r)}{\zeta(1-m_r)}\right)'_y\right) = 0. \qquad (23)$$

- And taking into account (14), (13):

$$Re\frac{\zeta'(m_r)}{\zeta(m_r)} = Re\frac{\zeta'(1-m_r)}{\zeta(1-m_r)},$$

$$\left(Re\frac{\zeta'(m_r)}{\zeta(m_r)}\right)'_x - \left(Re\frac{\zeta'(1-m_r)}{\zeta(1-m_r)}\right)'_x =$$
$$= \left(Re\frac{\zeta'(m_r)}{\zeta(m_r)} + Re\frac{\zeta'(1-m_r)}{\zeta(1-m_r)}\right)'_x =$$
$$= \frac{\partial}{\partial x}\left(Re\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} - \frac{1}{2}\frac{\Gamma'\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} + \ln\pi\right)\right)_{s=m_r} =$$
$$= Re\frac{d}{ds}\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{m_r}{2}\right)}{\Gamma\left(\frac{m_r}{2}\right)} - \frac{1}{2}\frac{\Gamma'\left(\frac{1-m_r}{2}\right)}{\Gamma\left(\frac{1-m_r}{2}\right)}\right),$$

$$\left(Re\frac{\zeta'(m_r)}{\zeta(m_r)}\right)'_y - \left(Re\frac{\zeta'(1-m_r)}{\zeta(1-m_r)}\right)'_y =$$
$$= \left(Re\frac{\zeta'(m_r)}{\zeta(m_r)} + Re\frac{\zeta'(1-m_r)}{\zeta(1-m_r)}\right)'_y =$$

$$= \frac{\partial}{\partial y}\left(Re\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} - \frac{1}{2}\frac{\Gamma'\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} + \ln\pi \right)\right)_{s=m_r} =$$

$$= Im\frac{d}{ds}\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{m_r}{2}\right)}{\Gamma\left(\frac{m_r}{2}\right)} - \frac{1}{2}\frac{\Gamma'\left(\frac{1-m_r}{2}\right)}{\Gamma\left(\frac{1-m_r}{2}\right)} \right), \qquad (24)$$

- Thus, the equality (23) can be written as follows:

$$(g(x, f_r(x)))'_{x=x_{m_r}} =$$
$$= Re\frac{\zeta'(m_r)}{\zeta(m_r)}\left(Re\frac{d}{ds}\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{m_r}{2}\right)}{\Gamma\left(\frac{m_r}{2}\right)} - \frac{1}{2}\frac{\Gamma'\left(\frac{1-m_r}{2}\right)}{\Gamma\left(\frac{1-m_r}{2}\right)} \right) + \right.$$
$$\left. + \operatorname{tg}\beta_{m_r} Im\frac{d}{ds}\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{m_r}{2}\right)}{\Gamma\left(\frac{m_r}{2}\right)} - \frac{1}{2}\frac{\Gamma'\left(\frac{1-m_r}{2}\right)}{\Gamma\left(\frac{1-m_r}{2}\right)} \right)\right) = 0. \qquad (25)$$

- And taking into account (15), (16) as well as the presence of the last equality of finite limits for all the terms at $r \to 0$ we get:

$$0 = \lim_{r\to 0}(g(x, f_r(x)))'_{x=x_{m_r}} =$$
$$= \left(\frac{1}{2}\ln\pi + \frac{1}{2}Re\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{q}{2}\right)}{\Gamma\left(\frac{q}{2}\right)} - \frac{1}{2}\frac{\Gamma'\left(\frac{1-q}{2}\right)}{\Gamma\left(\frac{1-q}{2}\right)} \right)\right) *$$
$$* \left(Re\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{q}{2}\right)}{\Gamma\left(\frac{q}{2}\right)} - \frac{1}{2}\frac{\Gamma'\left(\frac{1-q}{2}\right)}{\Gamma\left(\frac{1-q}{2}\right)} \right)'\right). \qquad (26)$$

Taking into account (5), (6) and the formula of the Digamma function from [1, p.259 §6.3.16] we estimate the first factor:

$$\frac{1}{2}\ln\pi+\frac{1}{2}Re\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{q}{2}\right)}{\Gamma\left(\frac{q}{2}\right)}-\frac{1}{2}\frac{\Gamma'\left(\frac{1-q}{2}\right)}{\Gamma\left(\frac{1-q}{2}\right)}\right)=$$

$$=\frac{1}{2}Re\left(\ln\pi+\frac{\gamma}{2}+\frac{1}{q}+\sum_{n=1}^{\infty}\left(\frac{1}{q+2n}-\frac{1}{2n}\right)+\right.$$

$$\left.+\frac{\gamma}{2}+\frac{1}{1-q}+\sum_{n=1}^{\infty}\left(\frac{1}{1-q+2n}-\frac{1}{2n}\right)\right)=$$

$$=\frac{1}{2}\left(\ln\pi+\gamma+\frac{\sigma_q}{\sigma_q^2+t_q^2}+\sum_{n=1}^{\infty}\left(\frac{2n+\sigma_q}{(2n+\sigma_q)^2+t_q^2}-\frac{1}{2n}\right)+\right.$$

$$\left.+\frac{1-\sigma_q}{(1-\sigma_q)^2+t_q^2}+\sum_{n=1}^{\infty}\left(\frac{2n+1-\sigma_q}{(2n+1-\sigma_q)^2+t_q^2}-\frac{1}{2n}\right)\right). \qquad (27)$$

- Note that the derivative of the function:

$$\frac{1}{2}\ln\pi+\frac{1}{2}Re\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{x+iy}{2}\right)}{\Gamma\left(\frac{x+iy}{2}\right)}-\frac{1}{2}\frac{\Gamma'\left(\frac{1-x-iy}{2}\right)}{\Gamma\left(\frac{1-x-iy}{2}\right)}\right)$$

along the ordinate axis for any fixed $0<x\leqslant\frac{1}{2}$ and $y>0$ is negative:

$$\frac{\partial}{\partial y}\left(\frac{1}{2}\ln\pi+\frac{1}{2}Re\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{x+iy}{2}\right)}{\Gamma\left(\frac{x+iy}{2}\right)}-\frac{1}{2}\frac{\Gamma'\left(\frac{1-x-iy}{2}\right)}{\Gamma\left(\frac{1-x-iy}{2}\right)}\right)\right)=$$

$$=\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{\partial}{\partial y}\left(\frac{2n+x}{(2n+x)^2+y^2}\right)+\frac{\partial}{\partial y}\left(\frac{2n+1-x}{(2n+1-x)^2+y^2}\right)\right)=$$

$$=-\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{2(2n+x)y}{((2n+x)^2+y^2)^2}+\frac{2(2n+1-x)y}{((2n+1-x)^2+y^2)^2}\right)<0.$$

- Therefore, if the left-hand side of the equality (27) is negative for numbers of the form $q_0 = \sigma_0 + it_0$, where $t_0 > 0$ is fixed and $0 < \sigma_0 \leqslant \dfrac{1}{2}$ is arbitrarily chosen, then it will be negative for any $q = \sigma_q + it_q: \ t_q \geqslant t_0, \ \ 0 < \sigma_q \leqslant \dfrac{1}{2}$.

Consider $q_0 = \sigma_0 + 8i, \ \ 0 < \sigma_0 \leqslant \dfrac{1}{2}$, then from (27) will follow:

$$\frac{1}{2}\ln\pi + \frac{1}{2}Re\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{q_0}{2}\right)}{\Gamma\left(\frac{q_0}{2}\right)} - \frac{1}{2}\frac{\Gamma'\left(\frac{1-q_0}{2}\right)}{\Gamma\left(\frac{1-q_0}{2}\right)}\right) =$$

$$= \frac{1}{2}\left(\ln\pi + \gamma + \frac{1-\sigma_0}{(1-\sigma_0)^2+8^2} + \frac{\sigma_0}{\sigma_0^2+8^2} + \right.$$

$$+ \sum_{n=1}^{\infty}\left(\frac{2n+\sigma_0}{(2n+\sigma_0)^2+8^2} - \frac{1}{2n}\right) +$$

$$\left. + \sum_{n=1}^{\infty}\left(\frac{2n+1-\sigma_0}{(2n+1-\sigma_0)^2+8^2} - \frac{1}{2n}\right)\right) <$$

$$< \frac{1}{2}\left(\ln\pi + \gamma + \frac{1}{8^2} + \sum_{n=1}^{\infty}\left(\frac{2n+\frac{1}{2}}{(2n)^2+8^2} - \frac{1}{2n}\right) + \right.$$

$$\left. + \sum_{n=1}^{\infty}\left(\frac{2n+1}{(2n)^2+8^2} - \frac{1}{2n}\right)\right) =$$

$$= \frac{1}{2}\left(\ln\pi + \gamma + \frac{1}{8^2} + \sum_{n=1}^{\infty}\left(\frac{n-8^2}{2n((2n)^2+8^2)} + \frac{2n-8^2}{2n((2n)^2+8^2)}\right)\right) =$$

$$= \frac{1}{2}\left(\ln\pi + \gamma + \frac{1}{64} + \sum_{n=1}^{\infty}\left(\frac{3}{8(n^2+16)} - \frac{16}{n(n^2+16)}\right)\right). \quad (28)$$

From [1, p.259], [2, § 6.495] :

$$y\sum_{n=1}^{\infty}\frac{1}{n^2+y^2}=-\frac{1}{2y}+\frac{\pi}{2}\coth\pi y.$$

Consequently:

$$\sum_{n=1}^{\infty}\frac{1}{n^2+16}=-\frac{1}{32}+\frac{\pi}{8}\coth 4\pi=0,3614490... \qquad (29)$$

- The remaining amount in the (28) is estimated for the first nine terms:

$$\sum_{n=1}^{\infty}\frac{1}{n(n^2+16)}>\sum_{n=1}^{9}\frac{1}{n(n^2+16)}>1,8873330. \qquad (30)$$

Thus, taking into account (29) and (30) the inequality (28) can be continued:

$$\frac{1}{2}\left(\ln\pi+\gamma+\frac{1}{64}+\sum_{n=1}^{\infty}\left(\frac{3}{8(n^2+16)}-\frac{16}{n(n^2+16)}\right)\right)<$$

$$<\frac{1}{2}\left(1,1447299+0,5772157+0,015625+\frac{3}{8}0,3614491-1,8873330\right)<$$

$$<\frac{1}{2}(1,8731141-1,8873330)<0.$$

I.e. for $\forall\ q=\sigma_q+it_q:\ t_q\geqslant 8,\ \ 0<\sigma_q\leqslant\frac{1}{2}$ the first multiplier of work from (26) is not equal 0.

And taking into account the symmetry of the values of this factor relative to the line $\sigma_q=\frac{1}{2}$ it is not equal to 0 for $\forall\ q=\sigma_q+it_q:\ t_q\geqslant 8,\ \ 0<\sigma_q<1$.

Let's estimate the minimal value $t_q>0$.

For $\forall\ \rho=\sigma+it$:

$$\frac{1}{\rho}+\frac{1}{\bar\rho}+\frac{1}{1-\rho}+\frac{1}{1-\bar\rho}=$$

$$=\frac{\sigma}{\sigma^2+t^2}+\frac{\sigma}{\sigma^2+t^2}+\frac{1-\sigma}{(1-\sigma)^2+t^2}+\frac{1-\sigma}{(1-\sigma)^2+t^2}=$$

$$=\frac{2\sigma}{\sigma^2+t^2}+\frac{2(1-\sigma)}{(1-\sigma)^2+t^2}>\frac{2\sigma}{1+t^2}+\frac{2(1-\sigma)}{1+t^2}=\frac{2}{1+t^2}.$$

Let's designate through $t_1 \stackrel{\text{def}}{=} \min_{\rho\in\mathcal{P}} |Im(\rho)|$ then in view of (7):

$$\frac{2}{1+t_1^2} < \sum_{\rho\in\mathcal{P}} \frac{1}{\rho} < 0,0230958,$$

i.e.

$$t_1 > 9,2518015. \tag{31}$$

Thus $\forall\ q \in \mathcal{P}$ multiplier:

$$\frac{1}{2}\ln\pi + \frac{1}{2}Re\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{q}{2}\right)}{\Gamma\left(\frac{q}{2}\right)} - \frac{1}{2}\frac{\Gamma'\left(\frac{1-q}{2}\right)}{\Gamma\left(\frac{1-q}{2}\right)}\right) \neq 0.$$

Hence the second factor of (26) must be equal to 0, which is equivalent to:

$$Re\left(\frac{\Gamma'\left(\frac{q}{2}\right)}{\Gamma\left(\frac{q}{2}\right)}\right)' = Re\left(\frac{\Gamma'\left(\frac{1-q}{2}\right)}{\Gamma\left(\frac{1-q}{2}\right)}\right)'.$$

□

- Let's prove the third lemma:

LEMMA 3

$$\forall\ s = x + iy,\ 0 < x \leqslant \frac{1}{2},\ y \geqslant 4 :$$

-

$$Re\left(\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\right)' = Re\left(\frac{\Gamma'\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}\right)' \Leftrightarrow \tag{32}$$

$$\Leftrightarrow x = \frac{1}{2}.$$

PROOF:

- From (27), the equality (32) can be written as follows:

$$\sum_{n=0}^{\infty}\left(\frac{(2n+x)^2-y^2}{((2n+x)^2+y^2)^2}-\frac{(2n+1-x)^2-y^2}{((2n+1-x)^2+y^2)^2}\right)=0 \tag{33}$$

In its turn:

$$\sum_{n=0}^{\infty}\left(\frac{(2n+x)^2-y^2}{((2n+x)^2+y^2)^2}-\frac{(2n+1-x)^2-y^2}{((2n+1-x)^2+y^2)^2}\right)=$$
$$=\sum_{n=0}^{\infty}\left(\frac{1}{(2n+x)^2+y^2}-\frac{1}{(2n+1-x)^2+y^2}\right)-$$
$$-2y^2\sum_{n=0}^{\infty}\left(\frac{1}{((2n+x)^2+y^2)^2}-\frac{1}{((2n+1-x)^2+y^2)^2}\right)=$$

$$= \sum_{n=0}^{\infty} \frac{(1-2x)(4n+1)}{((2n+x)^2+y^2)((2n+1-x)^2+y^2)} -$$
$$-2y^2 \sum_{n=0}^{\infty} \frac{(1-2x)(4n+1)((2n+x)^2+(2n+1-x)^2+2y^2)}{((2n+x)^2+y^2)^2((2n+1-x)^2+y^2)^2} =$$
$$= (1-2x) \left(\sum_{n=0}^{\infty} \frac{4n+1}{((2n+x)^2+y^2)((2n+1-x)^2+y^2)} - \right.$$
$$\left. -2y^2 \sum_{n=0}^{\infty} \frac{(4n+1)((2n+x)^2+(2n+1-x)^2+2y^2)}{((2n+x)^2+y^2)^2((2n+1-x)^2+y^2)^2} \right). \quad (34)$$

Let's estimate the sum of the general brackets of equality (34):

$$\sum_{n=0}^{\infty} \frac{4n+1}{((2n+x)^2+y^2)((2n+1-x)^2+y^2)} -$$
$$-2y^2 \sum_{n=0}^{\infty} \frac{(4n+1)((2n+x)^2+(2n+1-x)^2+2y^2)}{((2n+x)^2+y^2)^2((2n+1-x)^2+y^2)^2}.$$

From [1, p.259], [2, § 6.495] :

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2+y^2} = \frac{\pi}{4y} \tanh \frac{\pi y}{2},$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n)^2+y^2} = -\frac{1}{2y^2} + \frac{\pi}{4y} \coth \frac{\pi y}{2}.$$

And then the first composed in the considered sum:

$$\sum_{n=0}^{\infty} \frac{4n+1}{((2n+x)^2+y^2)((2n+1-x)^2+y^2)} <$$
$$< \frac{1}{(x^2+y^2)((1-x)^2+y^2)} + \sum_{n=1}^{\infty} \frac{4n+1}{((2n-1)^2+y^2)((2n)^2+y^2)} <$$
$$< \frac{1}{y^4} + \sum_{n=1}^{\infty} \left(\frac{1}{(2n-1)^2+y^2} - \frac{1}{(2n)^2+y^2} \right) +$$
$$+ \sum_{n=1}^{\infty} \frac{2}{((2n-1)^2+y^2)^2}.$$

Here:

$$\sum_{n=1}^{\infty}\left(\frac{1}{(2n-1)^2+y^2}-\frac{1}{(2n)^2+y^2}\right)=\frac{\pi}{4y}\tanh\frac{\pi y}{2}-\frac{\pi}{4y}\coth\frac{\pi y}{2}+\frac{1}{2y^2},$$

$$\textstyle\sum_{n=1}^{\infty}\frac{2}{((2n-1)^2+y^2)^2}=-\frac{1}{y}\frac{d}{dy}\left(\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2+y^2}\right)=$$
$$=-\frac{1}{y}\frac{d}{dy}\left(\frac{\pi}{4y}\tanh\frac{\pi y}{2}\right)=\frac{\pi}{4y^3}\tanh\frac{\pi y}{2}-\frac{\pi^2}{8y^2}\frac{1}{\cosh^2\frac{\pi y}{2}}$$

I.e.

$$\textstyle\sum_{n=0}^{\infty}\frac{4n+1}{((2n+x)^2+y^2)((2n+1-x)^2+y^2)}<$$
$$<\frac{1}{2y^2}+\frac{\pi}{4y^3}\tanh\frac{\pi y}{2}+\frac{1}{y^4}-$$
$$-\frac{\pi}{4y}\left(\coth\frac{\pi y}{2}-\tanh\frac{\pi y}{2}\right)-\frac{\pi^2}{8y^2}\frac{1}{\cosh^2\frac{\pi y}{2}}.$$

The second composed:

$$\textstyle\sum_{n=0}^{\infty}\frac{(4n+1)((2n+x)^2+(2n+1-x)^2+2y^2)}{((2n+x)^2+y^2)^2((2n+1-x)^2+y^2)^2}=$$
$$=\textstyle\sum_{n=1}^{\infty}\frac{4n-3}{((2n-2+x)^2+y^2)((2n-1-x)^2+y^2)}\,*$$
$$*\left(\frac{1}{(2n-2+x)^2+y^2}+\frac{1}{(2n-1-x)^2+y^2}\right)>$$
$$>\textstyle\sum_{n=1}^{\infty}\frac{4n-1}{((2n-1)^2+y^2)((2n)^2+y^2)}\left(\frac{1}{(2n-1)^2+y^2}+\frac{1}{(2n)^2+y^2}\right)-$$
$$-\textstyle\sum_{n=1}^{\infty}\frac{2}{((2n-1)^2+y^2)((2n)^2+y^2)}\left(\frac{1}{(2n-1)^2+y^2}+\frac{1}{(2n)^2+y^2}\right)>$$
$$>\textstyle\sum_{n=1}^{\infty}\left(\frac{1}{((2n-1)^2+y^2)^2}-\frac{1}{((2n)^2+y^2)^2}\right)-$$
$$-\textstyle\sum_{n=1}^{\infty}\frac{4}{((2n-1)^2+y^2)^3}.$$

Here:

$$\sum_{n=1}^{\infty}\left(\frac{1}{((2n-1)^2+y^2)^2}-\frac{1}{((2n)^2+y^2)^2}\right)=$$

$$=-\frac{1}{2y}\frac{d}{dy}\left(\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2+y^2}-\frac{1}{(2n)^2+y^2}\right)=$$

$$=-\frac{1}{2y}\frac{d}{dy}\left(\frac{\pi}{4y}\tanh\frac{\pi y}{2}+\frac{1}{2y^2}-\frac{\pi}{4y}\coth\frac{\pi y}{2}\right)=$$

$$=\frac{\pi}{8y^3}\tanh\frac{\pi y}{2}-\frac{\pi^2}{16y^2}\frac{1}{\cosh^2\frac{\pi y}{2}}+\frac{1}{2y^4}-\frac{\pi}{8y^3}\coth\frac{\pi y}{2}-\frac{\pi^2}{16y^2}\frac{1}{\sinh^2\frac{\pi y}{2}}.$$

And

$$\sum_{n=1}^{\infty}\frac{4}{((2n-1)^2+y^2)^3}=$$

$$=-\frac{1}{2y}\frac{d}{dy}\left(-\frac{1}{y}\frac{d}{dy}\left(\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2+y^2}\right)\right)=$$

$$=-\frac{1}{2y}\frac{d}{dy}\left(-\frac{1}{y}\frac{d}{dy}\left(\frac{\pi}{4y}\tanh\frac{\pi y}{2}\right)\right)=$$

$$=-\frac{1}{2y}\frac{d}{dy}\left(\frac{\pi}{4y^3}\tanh\frac{\pi y}{2}-\frac{\pi^2}{8y^2}\frac{1}{\cosh^2\frac{\pi y}{2}}\right)=$$

$$=\frac{3\pi}{8y^5}\tanh\frac{\pi y}{2}-\frac{\pi^2}{16y^4}\frac{1}{\cosh^2\frac{\pi y}{2}}-\frac{\pi^2}{8y^4}\frac{1}{\cosh^2\frac{\pi y}{2}}-\frac{\pi^3}{16y^3}\frac{\tanh\frac{\pi y}{2}}{\cosh^2\frac{\pi y}{2}}=$$

$$=-\frac{\pi^2}{8y^3\cosh^2\frac{\pi y}{2}}\left(\frac{3}{2y}+\frac{\pi}{2}\tanh\frac{\pi y}{2}\right)+\frac{3\pi}{8y^5}\tanh\frac{\pi y}{2}.$$

- Hence:

$$\sum_{n=0}^{\infty} \frac{4n+1}{((2n+x)^2+y^2)((2n+1-x)^2+y^2)} -$$
$$-2y^2 \sum_{n=0}^{\infty} \frac{(4n+1)((2n+x)^2+(2n+1-x)^2+2y^2)}{((2n+x)^2+y^2)^2((2n+1-x)^2+y^2)^2} <$$
$$< \frac{1}{2y^2} + \frac{\pi}{4y^3}\tanh\frac{\pi y}{2} + \frac{1}{y^4} -$$
$$-\frac{\pi}{4y}\left(\coth\frac{\pi y}{2} - \tanh\frac{\pi y}{2}\right) - \frac{\pi^2}{8y^2}\frac{1}{\cosh^2\frac{\pi y}{2}} -$$
$$-2y^2\left(\frac{\pi}{8y^3}\tanh\frac{\pi y}{2} - \frac{\pi^2}{16y^2}\frac{1}{\cosh^2\frac{\pi y}{2}} + \frac{1}{2y^4} -\right.$$
$$\left.-\frac{\pi}{8y^3}\coth\frac{\pi y}{2} - \frac{\pi^2}{16y^2}\frac{1}{\sinh^2\frac{\pi y}{2}}\right) -$$
$$-\left(-\frac{\pi^2}{8y^3\cosh^2\frac{\pi y}{2}}\left(\frac{3}{2y} + \frac{\pi}{2}\tanh\frac{\pi y}{2}\right) + \frac{3\pi}{8y^5}\tanh\frac{\pi y}{2}\right) =$$

$$= \frac{1}{y^2}\left(-\frac{1}{2} - \frac{\pi^2}{8}\frac{1}{\cosh^2\frac{\pi y}{2}} - \frac{3\pi}{8y^3}\tanh\frac{\pi y}{2} + \right.$$
$$+\frac{\pi^2}{8}\frac{y^2}{\cosh^2\frac{\pi y}{2}} + \frac{\pi^2}{8}\frac{y^2}{\sinh^2\frac{\pi y}{2}} +$$
$$\left.+\frac{\pi}{4y}\tanh\frac{\pi y}{2} + \frac{1}{y^2} + \frac{\pi^2}{8y\cosh^2\frac{\pi y}{2}}\left(\frac{3}{2y} + \frac{\pi}{2}\tanh\frac{\pi y}{2}\right)\right). \qquad (35)$$

Let's consider positive composed inside of the general bracket of the right part of an inequality (35) at $y \geqslant 4$:

Derivative:

$$\left(\frac{y^2}{\cosh^2\dfrac{\pi y}{2}}\right)' = y\frac{2\cosh\dfrac{\pi y}{2} - \pi y\sinh\dfrac{\pi y}{2}}{\cosh^3\dfrac{\pi y}{2}} < 0,$$

since for $\forall\, y \geqslant 4$

$$\frac{2}{\pi}\coth\frac{\pi y}{2} < y.$$

Similarly, the derivative:

$$\left(\frac{y^2}{\sinh^2\dfrac{\pi y}{2}}\right)' = y\frac{2\sinh\dfrac{\pi y}{2} - \pi y\cosh\dfrac{\pi y}{2}}{\sinh^3\dfrac{\pi y}{2}} < 0,$$

since for $\forall\, y \geqslant 4$

$$\frac{2}{\pi}\tanh\frac{\pi y}{2} < y.$$

Hence $\forall\, y \geqslant 4$:

$$\frac{\pi^2}{8}\frac{y^2}{\cosh^2\dfrac{\pi y}{2}} \leqslant \frac{\pi^2}{8}\frac{16}{\cosh^2 2\pi} < 0,0002754,$$

$$\frac{\pi^2}{8}\frac{y^2}{\sinh^2\dfrac{\pi y}{2}} \leqslant \frac{\pi^2}{8}\frac{16}{\sinh^2 2\pi} < 0,0002754.$$

Further $\forall\, y \geqslant 4$:

$$\frac{\pi}{4y}\tanh\frac{\pi y}{2} < \frac{\pi}{16} < 0,1963496,$$

$$\frac{1}{y^2} \leqslant 0,0625,$$

$$\frac{\pi^2}{8y\cosh^2\dfrac{\pi y}{2}}\left(\frac{3}{2y} + \frac{\pi}{2}\tanh\frac{\pi y}{2}\right) < \frac{\pi^2}{32\cosh^2 2\pi}\left(\frac{3}{8} + \frac{\pi}{2}\right) < 0,0000084.$$

- Hence $\forall\ y \geqslant 4$ the total sum of positive composed in the general bracket does not exceed $\dfrac{1}{2}$:

$$\frac{\pi^2}{8}\frac{y^2}{\cosh^2\dfrac{\pi y}{2}}+\frac{\pi^2}{8}\frac{y^2}{\sinh^2\dfrac{\pi y}{2}}+$$
$$+\frac{\pi}{4y}\tanh\frac{\pi y}{2}+\frac{1}{y^2}+\frac{\pi^2}{8y\cosh^2\dfrac{\pi y}{2}}\left(\frac{3}{2y}+\frac{\pi}{2}\tanh\frac{\pi y}{2}\right)<0,2594088.$$

This means that for $\forall\ y \geqslant 4,\ \ 0 < x \leqslant \dfrac{1}{2}$ the second factor of the right side of the equality (34) does not turn into 0,
hence from (33) and (34):

$$x=\frac{1}{2}.$$

- In a underside the validity of the statement of the Lemma 3 is obvious.

□

So, assuming that an arbitrary nontrivial root q of zeta functions belongs to the union $\mathcal{P}_1 \cup \mathcal{P}_2$ we found that it belongs only to $\mathcal{P}_2$, i.e. $\mathcal{P}_1 = \varnothing$.

And according to the fact that $\mathcal{P}_1 = \varnothing \Leftrightarrow \mathcal{P}_3 = \varnothing$ we have:

$$\mathcal{P}_3=\mathcal{P}_1=\varnothing,\ \ \mathcal{P}=\mathcal{P}_2.$$

This proves the basic statement and the assumption which had been made by Bernhard Riemann about of the real parts of the nontrivial zeros of zeta function.

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