The real parts of the nontrivial Riemann zeta function zeros

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ABSTRACT

This theorem is based on holomorphy of studied functions and the fact that nearby of a singularity point the imaginary part of the specific function can accept zero value.

The colored markers are:
- ● - assumption or a fact which is not proven at present;
- ● - the statement which requires additional attention;
- ● - statement which is proved earlier or clearly undestandable.

THEOREM

● The real parts of all the nontrivial Riemann zeta function zeros ρ are equal $Re\left(\rho\right) = \frac{1}{2}$.

PROOF:

● According to the functional equality [10, p. 22], [5, p. 8-11]:

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta\left(s\right) = \Gamma\left(\frac{1-s}{2}\right)\pi^{-\frac{1-s}{2}}\zeta\left(1-s\right), \qquad Re\left(s\right) > 0 \qquad (1)$$

$\zeta\left(s\right)$ - the Riemann zeta function, $\Gamma\left(s\right)$ - the Gamma function.

- From [5, p. 8-11] $\zeta(\bar{s}) = \overline{\zeta(s)}$, it means that $\forall \rho = \sigma + it$: $\zeta(\rho) = 0$ and $0 \leqslant \sigma \leqslant 1$ we have:

$$\zeta(\bar{\rho}) = \zeta(1-\rho) = \zeta(1-\bar{\rho}) = 0 \tag{2}$$

- From [11], [9, p. 128], [10, p. 45] we know that $\zeta(s)$ has no nontrivial zeros on the line $\sigma = 1$ and consequently on the line $\sigma = 0$ also, in accordance with (2) they don't exist.

- Let's denote the set of nontrivial zeros $\zeta(s)$ through $\mathcal{P}$ (multiset with consideration of multiplicitiy):

$$\mathcal{P} \overset{\text{def}}{=} \{\rho :\ \zeta(\rho) = 0,\ \rho = \sigma + it,\ 0 < \sigma < 1\}.$$

$$\text{And: } \mathcal{P}_1 \overset{\text{def}}{=} \left\{\rho :\ \zeta(\rho) = 0,\ \rho = \sigma + it,\ 0 < \sigma < \frac{1}{2}\right\}, \tag{3}$$

$$\mathcal{P}_2 \overset{\text{def}}{=} \left\{\rho :\ \zeta(\rho) = 0,\ \rho = \frac{1}{2} + it\right\},$$

$$\mathcal{P}_3 \overset{\text{def}}{=} \left\{\rho :\ \zeta(\rho) = 0,\ \rho = \sigma + it,\ \frac{1}{2} < \sigma < 1\right\}.$$

Then:

$$\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 \quad \text{and} \quad \mathcal{P}_1 \cap \mathcal{P}_2 = \mathcal{P}_2 \cap \mathcal{P}_3 = \mathcal{P}_1 \cap \mathcal{P}_3 = \varnothing,$$

$$\mathcal{P}_1 = \varnothing \Leftrightarrow \mathcal{P}_3 = \varnothing.$$

- Hadamard's theorem (Weierstrass preparation theorem) on the decomposition of function through the roots gives us the following result [10, p. 30], [5, p. 31], [12]:

$$\zeta(s) = \frac{\pi^{\frac{s}{2}} e^{as}}{s(s-1)\Gamma\left(\frac{s}{2}\right)} \prod_{\rho \in \mathcal{P}} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}, \qquad Re(s) > 0 \tag{4}$$

$$a = \ln 2\sqrt{\pi} - \frac{\gamma}{2} - 1,\ \gamma - \text{Euler's constant and}$$

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{2}\ln\pi + a - \frac{1}{s} + \frac{1}{1-s} - \frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} + \sum_{\rho\in\mathcal{P}}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) \qquad (5)$$

- According to the fact that $\dfrac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}$ - Digamma function of [10, p. 31], [5, p. 23] we have:

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{1-s} + \sum_{\rho\in\mathcal{P}}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) + \sum_{n=1}^{\infty}\left(\frac{1}{s+2n} - \frac{1}{2n}\right) + C, \qquad (6)$$

$$C = const$$

.

- From [4, p. 160], [8, p. 272], [3, p. 81]:

$$\sum_{\rho\in\mathcal{P}}\frac{1}{\rho} = 1 + \frac{\gamma}{2} - \ln 2\sqrt{\pi} = 0,0230957\ldots \qquad (7)$$

- Indeed, from (2):

$$\sum_{\rho\in\mathcal{P}}\frac{1}{\rho} = \frac{1}{2}\sum_{\rho\in\mathcal{P}}\left(\frac{1}{1-\rho} + \frac{1}{\rho}\right)$$

- From (5):

$$2\sum_{\rho\in\mathcal{P}}\frac{1}{\rho} = \lim_{s\to 1}\left(\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{1-s} + \frac{1}{s} - a - \frac{1}{2}\ln\pi + \frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\right).$$

- Also it's known, for example, from [10, p. 49], [3, p. 98] that the number of nontrivial zeros of $\rho = \sigma + it$ in strip $0 < \sigma < 1$, the imaginary parts of which t are less than some number $T > 0$ is limited, i.e.

$$\|\{\rho:\ \rho\in\mathcal{P},\ \rho = \sigma + it,\ |t| < T\}\| < \infty.$$

- Indeed, it can be presented that on the contrary the sum of $\sum_{\rho\in\mathcal{P}}\dfrac{1}{\rho}$ would have been unlimited.

- Thus $\forall T>0\ \exists\ \delta_x>0,\ \delta_y>0$ such that

$$\text{in area } 0<t\leqslant\delta_y, 0<\sigma\leqslant\delta_x \text{ there are no zeros } \rho=\sigma+it\in\mathcal{P}. \qquad (8)$$

Let's consider random root $q\in\mathcal{P}_1\cup\mathcal{P}_2$
Let's denote $k(q)$ the multiplicity of the root q.
Let's examine the area $Q(R)\stackrel{\text{def}}{=}\{s:\|s-q\|\leqslant R, R>0\}$.

- From the fact of finiteness of set of nontrivial zeros $\zeta(s)$ in the limited area follows $\exists\ R>0$, such that $Q(R)$ does not contain any root from $\mathcal{P}$ except q.

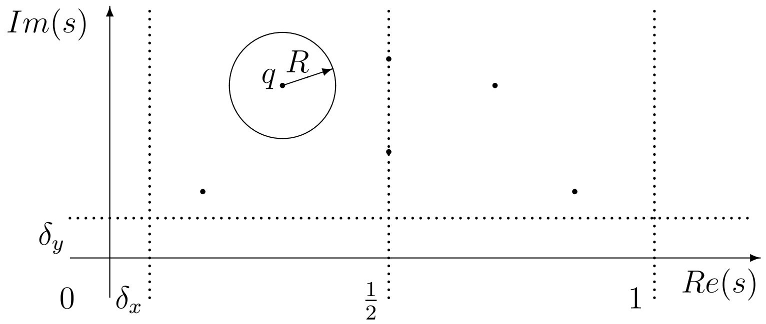

Fig. 1.

- From [1], [10, p. 31], [5, p. 23] we know that the Digamma function $\dfrac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}$ in the area $Q(R)$ has no poles, i.e. $\forall s\in Q(R)$

$$\left\|\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\right\|<\infty.$$

Let's denote:

$$I_{\mathcal{P}}(s)\stackrel{\text{def}}{=}-\frac{1}{s}+\frac{1}{1-s}+\sum_{\rho\in\mathcal{P}}\frac{1}{s-\rho}$$

and

$$I_{\mathcal{P}\setminus\{q\}}(s)=-\frac{1}{s}+\frac{1}{1-s}+\sum_{\rho\in\mathcal{P}\setminus\{q\}}\frac{1}{s-\rho}. \qquad (9)$$

Hereinafter $\mathcal{P}\setminus\{q\} \overset{\text{def}}{=} \mathcal{P}\setminus\{(q,k(q))\}$ (the difference in the multiset).

Summation - $\sum_{\rho\in\mathcal{P}}\dfrac{1}{s-\rho}$ and $\sum_{\rho\in\mathcal{P}\setminus\{q\}}\dfrac{1}{s-\rho}$ further we shall consider as the sum of pairs $\left(\dfrac{1}{s-\rho}+\dfrac{1}{s-(1-\rho)}\right)$ and $\sum_{\rho\in\mathcal{P}}\dfrac{1}{\rho}$ as the sum of pairs $\left(\dfrac{1}{\rho}+\dfrac{1}{1-\rho}\right)$ as a consequence of division of the sum from (6) $\sum_{\rho\in\mathcal{P}}\left(\dfrac{1}{s-\rho}+\dfrac{1}{\rho}\right)$ into $\sum_{\rho\in\mathcal{P}}\dfrac{1}{s-\rho}+\sum_{\rho\in\mathcal{P}}\dfrac{1}{\rho}$. As specifed in [4], [6], [8], [10].

- Let's note that $I_{\mathcal{P}\setminus\{q\}}(s)$ is holomorphic function $\forall\ s\in Q(R)$.

Then from (5) we have:

$$\frac{\zeta'(s)}{\zeta(s)}=\frac{1}{2}\ln\pi+a-\frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}+\sum_{\rho\in\mathcal{P}}\frac{1}{\rho}+I_{\mathcal{P}}(s). \tag{10}$$

And in view of (7):

$$Im\frac{\zeta'(s)}{\zeta(s)}=Im\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}+I_{\mathcal{P}}(s)\right). \tag{11}$$

Let's note that from the equality of

$$\sum_{\rho\in\mathcal{P}}\frac{1}{1-s-\rho}=-\sum_{(1-\rho)\in\mathcal{P}}\frac{1}{s-(1-\rho)}=-\sum_{\rho\in\mathcal{P}}\frac{1}{s-\rho} \tag{12}$$

follows that:

$$I_{\mathcal{P}}(1-s)=-I_{\mathcal{P}}(s),\ I_{\mathcal{P}\setminus\{q\}}(1-s)=-I_{\mathcal{P}\setminus\{1-q\}}(s),\ Re\,(s)>0.$$

- Besides

$$I_{\mathcal{P}\setminus\{q\}}(s)=I_{\mathcal{P}}(s)-\frac{k(q)}{s-q}$$

and $I_{\mathcal{P}\setminus\{q\}}(s)$ is limited in the area of $s\in Q(R)$ as a result of absence of its poles in this area as well as its differentiability in each point of this area.

- If in (5) to replace s with $1-s$ that in view of (7):

$$\frac{\zeta'(s)}{\zeta(s)}+\frac{\zeta'(1-s)}{\zeta(1-s)}=-\frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}-\frac{1}{2}\frac{\Gamma'\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}+\ln\pi,\ Re(s)>0. \quad (13)$$

- Let's examine a circle with the center in a point q and radius $r \leqslant R$, laying in the area of $Q(R)$:

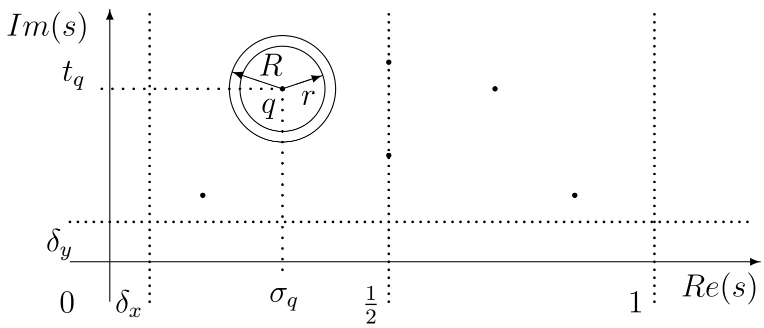

Fig. 2.

- For $s=x+iy,\ q=\sigma_q+it_q$

$$Im\frac{k(q)}{s-q}=Im\frac{k(q)}{x+iy-\sigma_q-it_q}=\frac{k(q)(t_q-y)}{(x-\sigma_q)^2+(y-t_q)^2}=k(q)\frac{t_q-y}{r^2},$$

- Consider also the function $\ln\zeta(s)$ - principal branch function $Ln\zeta(s)$ for which of the (4), in view of (7) $\forall\ s\in Q(R)$ is true:

$$\ln\zeta(s)=\frac{\ln\pi}{2}s-\ln\left(s(1-s)\right)-\ln\Gamma\left(\frac{s}{2}\right)+\sum_{\rho\in\mathcal{P}}\ln\left(1-\frac{s}{\rho}\right). \quad (14)$$

The sum as stipulated earlier, is taken in pairs:

$$\ln\left(1-\frac{s}{\rho}\right)+\ln\left(1-\frac{s}{1-\rho}\right).$$

Let's designate the real function of two variables for $s = x + iy$:

$$\omega(x,y) \stackrel{\text{def}}{=} Im\left(-\ln\left(s(1-s)\right) - \ln\Gamma\left(\frac{s}{2}\right) + \sum_{\rho\in\mathcal{P}\setminus\{q\}} \ln\left(1-\frac{s}{\rho}\right)\right).$$

- Note that the function $\omega(x,y)$ and its partial derivatives on both variables exist and are limited to $\forall\ s = x+iy \in Q(R)$, since

$$\omega(x,y) = Im\ln\zeta(x+iy) - \frac{\ln\pi}{2}y - k(q)Im\ln\left(1-\frac{x+iy}{q}\right).$$

Inside of area $Q(R)$ we take a point $M \stackrel{\text{def}}{=} (x_M, y_M)$, does not coincide which the point q.

- Let's draw a looped curve from the point M so that it doesn't pass through the point q and is described by the function which has a continuous derivative at each point.

Let's designate: $f_M(\tau) \stackrel{\text{def}}{=} f_x(\tau) + if_y(\tau)$:

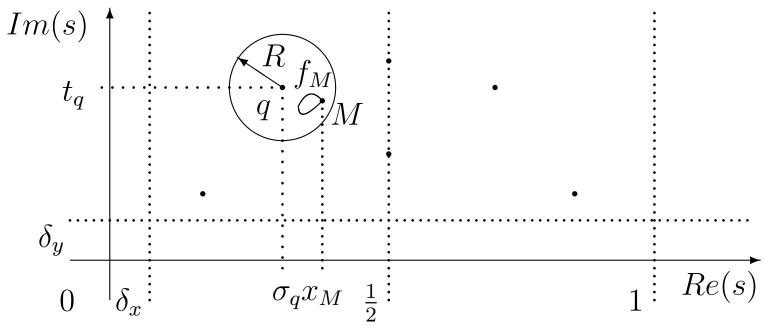

Fig. 3.

In accordance with the construction let's denote $\tau_{M,1}$ and $\tau_{M,2}$ such that:

$$f_M(\tau_{M,1}) = x_M + iy_M, \quad f_M(\tau_{M,2}) = x_M + iy_M.$$

- Function $Im \ln \zeta\,(s) - \dfrac{\ln \pi}{2} Im(s)$ is differentiable and therefore continuous and differentiable in τ function $Im \ln \zeta\,(f_M(\tau)) - \dfrac{\ln \pi}{2} Im(f_M(\tau))$.

  It means that continuous on the segment and differentiable in the domestic range of this segment the real function gets at its ends the same values:

$$Im \ln \zeta\,(f_M(\tau_{M,1})) - \frac{\ln \pi}{2} Im(f_M(\tau_{M,1})) =$$
$$= Im \ln \zeta\,(f_M(\tau_{M,2})) - \frac{\ln \pi}{2} Im(f_M(\tau_{M,2})) =$$
$$= Im \ln \zeta\,(f_M(x_M + iy_M)) - \frac{\ln \pi}{2} Im(f_M(x_M + iy_M)).$$

- By Rolle's theorem on the extremum of a differentiable function on the interval we have:

$$\exists \tau_1 \in (\tau_{M,1}, \tau_{M,2}) : \quad \left(Im \ln \zeta\,(f_M(\tau)) - \frac{\ln \pi}{2} Im(f_M(\tau)) \right)'_{\tau=\tau_1} = 0. \quad (15)$$

  I.e. on a curve described by function $f_M(\tau), \quad \tau \in (\tau_{M,1}, \tau_{M,2})$ there is a point $\Theta_w = \Theta_w(M) \stackrel{\text{def}}{=} f_M(\tau_1)$ for which it is true (15).

- Let's consider the following option line, as a closed curve which passes through a point of M.

  For any $0 < r < R$ construct a circle centered at the point q and the radius r.

  The point of intersection of the left semicircle of the circle and the line $y - t_q = x - \sigma_q$ let's denote as $J \stackrel{\text{def}}{=} (x_J, y_J)$.

  Let's construct a circle with the center in a point J, with radius:

$$0 < r_\delta < \min(r, R - r).$$

As a point of M let's take more distant from q point of intersection of the circle with the line $y - t_q = x - \sigma_q$.

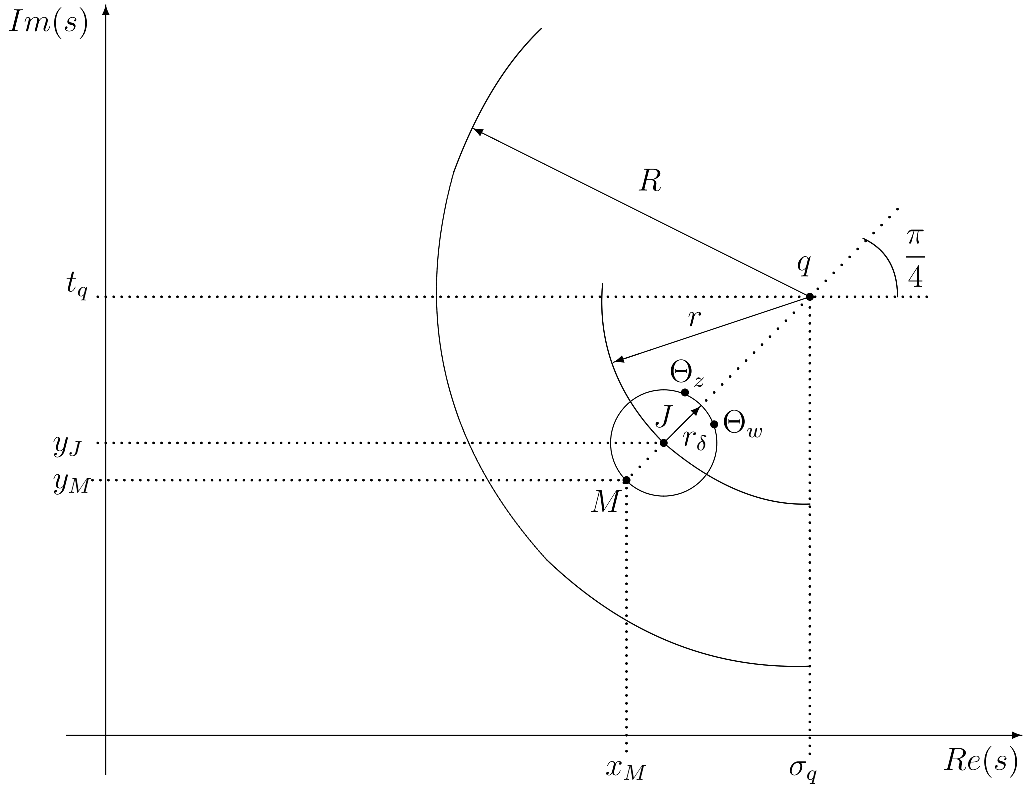

Fig. 4.

- As the desired curve, we consider the circle with center at J and the radius r_δ.

  Let's notice that the received curve satisfies to all declared properties for any as much as small radius since $\min(r, R-r)$.

- Let's assign:

$$0 < r_\delta = O(r^3)_{r \to 0}. \tag{16}$$

Then $\exists\, 0 < R_1 \leqslant R : \; \forall\, 0 < r < R_1$ the condition is satisfied:

$$0 < r_\delta < \min(r, R-r).$$

As τ take $Im(s) = y$, then:

$$f_y(y) = y, \quad x = f_x(y) = x_J \pm \sqrt{r_\delta^2 - (y - y_J)^2}.$$

- According to [1, p. 67, 82]:

$$Im \ln\left(1 - \frac{x + iy}{q}\right) = Im \ln\left(\sigma_q - x + i(t_q - y)\right) - Im \ln\left(q\right) =$$

$$= \arctan\left(\frac{t_q - y}{\sigma_q - x}\right) - \arctan\left(\frac{t_q}{\sigma_q}\right),$$

$$\frac{d}{dx}\arctan(x) = \frac{1}{1 + x^2}.$$

- Then:

$$\frac{d}{d\tau} Im \ln\left(1 - \frac{x(\tau) + i\tau}{q}\right)_{\tau=\tau_1} = \frac{d}{d\tau}\arctan\left(\frac{t_q - \tau}{\sigma_q - x(\tau)}\right)_{\tau=\tau_1} =$$

$$= \frac{1}{1 + \left(\frac{t_q - \tau_1}{\sigma_q - x(\tau_1)}\right)^2}\frac{\partial}{\partial \tau}\left(\frac{t_q - \tau}{\sigma_q - x(\tau_1)}\right)_{\tau=\tau_1} +$$

$$+ \frac{1}{1 + \left(\frac{t_q - \tau_1}{\sigma_q - x(\tau_1)}\right)^2}\frac{\partial}{\partial x}\left(\frac{t_q - \tau_1}{\sigma_q - x}\right)_{x=x(\tau_1)} x'(\tau_1) =$$

$$= -\frac{\sigma_q - x(\tau_1)}{(\sigma_q - x(\tau_1))^2 + (t_q - \tau_1)^2} + \frac{x'(\tau_1)(t_q - \tau_1)}{(\sigma_q - x(\tau_1))^2 + (t_q - \tau_1)^2}. \qquad (17)$$

- And for the $s = x + iy$ equation (15) at the point $\Theta_w = (x(\tau_1), \tau_1)$ can be written as follows:

$$\frac{\sigma_q - x(\tau_1)}{(\sigma_q - x(\tau_1))^2 + (t_q - \tau_1)^2} - \frac{x'(\tau_1)(t_q - \tau_1)}{(\sigma_q - x(\tau_1))^2 + (t_q - \tau_1)^2} = \\ = \omega(x(\tau_1), y)'_{y=\tau_1} + x(\tau_1)'\omega(x, \tau_1)'_{x=x(\tau_1)}. \tag{18}$$

- At $r \to 0$ and (16):

$$(\sigma_q - x(\tau_1))^2 + (t_q - \tau_1)^2 \leqslant (\frac{\sqrt{2}}{2}r + r_\delta)^2 + (\frac{\sqrt{2}}{2}r + r_\delta)^2 = r^2 + O(r^4),$$

$$(\sigma_q - x(\tau_1))^2 + (t_q - \tau_1)^2 \geqslant (\frac{\sqrt{2}}{2}r - r_\delta)^2 + (\frac{\sqrt{2}}{2}r - r_\delta)^2 = r^2 + O(r^4),$$

i.e.

$$(\sigma_q - x(\tau_1))^2 + (t_q - \tau_1)^2 = r^2 + O(r^4).$$

- Hence from the equation (18) at $r \to 0$ $x'(\tau_1)$:

$$x'(\tau_1) = 1 - \omega(x(\tau_1), y)'_{y=\tau_1}\sqrt{2}r + O(r^2). \tag{19}$$

Similar reasonings for $s = x + iy$ in the same area $s \in Q(R)$ points q for function $Im \ln \zeta(1 - s) - \frac{\ln \pi}{2} Im(1 - s)$ we shall come to conclusion that on the circle with the center in J and radius r_δ there should be a point $\Theta_z = \Theta_z(M) \stackrel{\text{def}}{=} f_M(\tau_2)$ for some $\tau_2 \in (\tau_{M,1}, \tau_{M,2})$ for which equality is true:

$$\left(Im \ln \zeta\,(1 - f_M(\tau))) - \frac{\ln \pi}{2} Im(1 - f_M(\tau))\right)'_{\tau=\tau_2} = 0. \tag{20}$$

- Indeed, if we apply the (14) value is $1-s$ instead of s:

$$\ln\zeta(1-s)-\frac{\ln\pi}{2}(1-s)=$$
$$-\ln\left((1-s)s\right)-\ln\Gamma\left(\frac{1-s}{2}\right)+\sum\nolimits_{\rho\in\mathcal{P}}\ln\left(1-\frac{1-s}{\rho}\right)$$

that in view of that $\forall\ \ \rho\in\mathcal{P}$:

$$\left(1-\frac{1-s}{\rho}\right)\left(1-\frac{1-s}{1-\rho}\right)=\left(1-\frac{s}{\rho}\right)\left(1-\frac{s}{1-\rho}\right),$$

we have:

$$\ln\zeta(1-s)-\frac{\ln\pi}{2}(1-s)=-\ln\left(s(1-s)\right)-\ln\Gamma\left(\frac{1-s}{2}\right)+\sum_{\rho\in\mathcal{P}}\ln\left(1-\frac{s}{\rho}\right). \tag{21}$$

Let's designate the real function of two variables for $s=x+iy$:

$$\omega(x,y)_-\stackrel{\text{def}}{=}Im\left(-\ln\left(s(1-s)\right)-\ln\Gamma\left(\frac{1-s}{2}\right)+\sum_{\rho\in\mathcal{P}\setminus\{q\}}\ln\left(1-\frac{s}{\rho}\right)\right).$$

- Note that the function $\omega(x,y)_-$ and its partial derivatives on both variables exist and are limited to $\forall\ s=x+iy\in Q(R)$, since

$$\omega(x,y)_-=Im\ln\zeta(1-x-iy)+\frac{\ln\pi}{2}y-k(q)Im\ln\left(1-\frac{x+iy}{q}\right).$$

And all reasonings for functions $\ln\zeta(1-s)-\dfrac{\ln\pi}{2}(1-s)$ and $\omega(x,y)_-$ are similar to reasonings for appropriate functions $\ln\zeta(s)-\dfrac{\ln\pi}{2}(s)$ and $\omega(x,y)$.

- So, on the circle with the center in J and radius r_δ are two points of Θ_w and Θ_z, for which the equalities (15) and (20) respectively.

And similarly (19) at $r \to 0$:

$$x'(\tau_2) = 1 - \omega_-(x(\tau_2), y)'_{y=\tau_2} r + O(r^2). \tag{22}$$

For $\forall\ s : \|s - (x_J + iy_J)\| < r_\delta$ derivative of function $-\ln\left(s(1-s)\right) + \sum_{\rho \in \mathcal{P}} \ln\left(1 - \dfrac{s}{\rho}\right)$ is continuous and equal to:

$$\frac{d}{ds}\left(-\ln\left(s(1-s)\right) + \sum_{\rho \in \mathcal{P}} \ln\left(1 - \frac{s}{\rho}\right)\right) = I_{\mathcal{P}}(s).$$

Hence for fixed $0 < r < R_1$ from the continuity of the given above function at the point J:
$\forall\ \varepsilon_r > 0, \exists\ \delta_{\varepsilon_r} > 0 : \forall\ s : \|s - (x_J + iy_J)\| < \delta_{\varepsilon_r}$ follows:

$$|ReI_{\mathcal{P}}(s) - ReI_{\mathcal{P}}(x_J + iy_J)| < \frac{\varepsilon_r}{4},$$

$$|ImI_{\mathcal{P}}(s) - ImI_{\mathcal{P}}(x_J + iy_J)| < \frac{\varepsilon_r}{4}.$$

- Let's assign $\varepsilon_r = r$ then $\forall\ 0 < r_\delta < \delta_{\varepsilon_r}$, at $r \to 0$, with the condition (16) similar to (17) where $f_y(\tau) = \tau,\ f_x(\tau) = x(\tau)$ is true:

$$\frac{d}{d\tau} Im\left(-\ln\left(f_M(\tau)(1 - f_M(\tau))\right) + \textstyle\sum_{\rho \in \mathcal{P}} \ln\left(1 - \dfrac{f_M(\tau)}{\rho}\right)\right)_{\tau=\tau_1} -$$

$$-\frac{d}{d\tau} Im\left(-\ln\left(f_M(\tau)(1 - f_M(\tau))\right) + \textstyle\sum_{\rho \in \mathcal{P}} \ln\left(1 - \dfrac{f_M(\tau)}{\rho}\right)\right)_{\tau=\tau_2} =$$

$$= \left(-ReI_{\mathcal{P}}(\Theta_w) + x(\tau_1)' ImI_{\mathcal{P}}(\Theta_w)\right) -$$

$$-\left(-ReI_{\mathcal{P}}(\Theta_z) + x(\tau_2)' ImI_{\mathcal{P}}(\Theta_z)\right) =$$

$$= -ReI_{\mathcal{P}}(\Theta_w) + ReI_{\mathcal{P}}(x_J + iy_J) - ReI_{\mathcal{P}}(x_J + iy_J) + ReI_{\mathcal{P}}(\Theta_z) +$$

$$+x(\tau_1)' ImI_{\mathcal{P}}(\Theta_w) - x(\tau_1)' ImI_{\mathcal{P}}(x_J + iy_J) +$$

$$+x(\tau_2)' ImI_{\mathcal{P}}(x_J + iy_J) - x(\tau_2)' ImI_{\mathcal{P}}(\Theta_z) +$$

$$+x(\tau_1)' ImI_{\mathcal{P}}(x_J + iy_J) - x(\tau_2)' ImI_{\mathcal{P}}(x_J + iy_J) =$$

$$= O\left(\frac{r}{4} + \frac{r}{4} + \frac{r}{4}\left|x(\tau_1)'\right| + \frac{r}{4}\left|x(\tau_2)'\right|\right) +$$

$$+\left(x(\tau_1)' - x(\tau_2)'\right) ImI_{\mathcal{P}}(x_J + iy_J) =$$

$$= \left(x(\tau_1)' - x(\tau_2)'\right) ImI_{\mathcal{P}}(x_J + iy_J) + O\left(r\right) =$$

$$= \left(\omega_-(x(\tau_2), y)'_{y=\tau_2}\sqrt{2}r - \omega(x(\tau_1), y)'_{y=\tau_1}\sqrt{2}r + O(r^2)\right) *$$

$$* \left(ImI_{\mathcal{P}\setminus\{q\}}(x_J + iy_J) + \frac{t_q - y_J}{(\sigma_q - x_J)^2 + (t_q - y_J)^2}\right) + O(r) =$$

$$= \omega_-(x(\tau_2), y)'_{y=\tau_2} - \omega(x(\tau_1), y)'_{y=\tau_1} + O(r) =$$

$$= -\frac{\partial}{\partial y} Im \ln\Gamma\left(\frac{1 - x(\tau_2) - iy}{2}\right)_{y=\tau_2} + \frac{\partial}{\partial y} Im \ln\Gamma\left(\frac{x(\tau_1) + iy}{2}\right)_{y=\tau_1} +$$

$$+ReI_{\mathcal{P}\setminus\{q\}}(\Theta_w) - ReI_{\mathcal{P}\setminus\{q\}}(\Theta_z) + O(r) =$$

$$= -\frac{\partial}{\partial y} Im \ln\Gamma\left(\frac{1 - x(\tau_2) - iy}{2}\right)_{y=\tau_2} + \frac{\partial}{\partial y} Im \ln\Gamma\left(\frac{x(\tau_1) + iy}{2}\right)_{y=\tau_1} +$$

$$+o(1). \tag{23}$$

- Let's consider a difference of the equations (15) and (20) which on construction is equal 0 for $\forall\ 0 < r < R_1$:

$$\lim_{r\to 0} 0 = \lim_{r\to 0}\left(\left(Im \ln\zeta\left(f_M(\tau)\right) - \frac{\ln\pi}{2} Im(f_M(\tau))\right)'_{\tau=\tau_1} - \right.$$
$$\left. - \left(Im \ln\zeta\left(1-f_M(\tau)\right) - \frac{\ln\pi}{2} Im(1-f_M(\tau))\right)'_{\tau=\tau_2}\right) =$$
$$= \lim_{r\to 0}\left(-\left(Im\ln\Gamma\left(\frac{f_M(\tau)}{2}\right)\right)'_{\tau=\tau_1} + \left(Im\ln\Gamma\left(\frac{1-f_M(\tau)}{2}\right)\right)'_{\tau=\tau_2} + \right.$$
$$+\frac{d}{d\tau} Im\left(-\ln\left(f_M(\tau)(1-f_M(\tau))\right) + \sum_{\rho\in\mathcal{P}}\ln\left(1-\frac{f_M(\tau)}{\rho}\right)\right)_{\tau=\tau_1} -$$
$$\left. -\left(\frac{d}{d\tau} Im\left(-\ln\left(f_M(\tau)(1-f_M(\tau))\right) + \sum_{\rho\in\mathcal{P}}\ln\left(1-\frac{f_M(\tau)}{\rho}\right)\right)_{\tau=\tau_2}\right)\right) =$$

And then from (23) follows:

$$= \lim_{r\to 0}\left(-x(\tau_1)'\frac{\partial}{\partial x}\left(Im\ln\Gamma\left(\frac{x+i\tau_1}{2}\right)\right)_{x=x(\tau_1)} - \right.$$
$$-\frac{\partial}{\partial y}\left(Im\ln\Gamma\left(\frac{x(\tau_1)+iy}{2}\right)\right)_{y=\tau_1} +$$
$$+x(\tau_2)'\frac{\partial}{\partial x}\left(Im\ln\Gamma\left(\frac{1-x-i\tau_2}{2}\right)\right)_{x=x(\tau_2)} +$$
$$+\frac{\partial}{\partial y}\left(Im\ln\Gamma\left(\frac{1-x(\tau_2)-iy}{2}\right)\right)_{y=\tau_2} -$$
$$-\frac{\partial}{\partial y}\left(Im\ln\Gamma\left(\frac{1-x(\tau_2)-iy}{2}\right)\right)_{y=\tau_2} +$$
$$\left. +\frac{\partial}{\partial y}\left(Im\ln\Gamma\left(\frac{x(\tau_1)+iy}{2}\right)\right)_{y=\tau_1} + o(1)\right) =$$

$$= \lim_{r\to 0}\left(-\frac{\partial}{\partial x}\left(Im\ln\Gamma\left(\frac{x+iy(\tau_1)}{2}\right)\right)_{x=\tau_1}+\right.$$
$$\left.+\frac{\partial}{\partial x}\left(Im\ln\Gamma\left(\frac{1-x-iy(\tau_2)}{2}\right)\right)_{x=\tau_2}\right)=$$
$$= Im\left(-\frac{d}{ds}\ln\Gamma\left(\frac{s}{2}\right)_{s=q}+\frac{d}{ds}\ln\Gamma\left(\frac{1-s}{2}\right)_{s=q}\right)=$$
$$= Im\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{q}{2}\right)}{\Gamma\left(\frac{q}{2}\right)}-\frac{1}{2}\frac{\Gamma'\left(\frac{1-q}{2}\right)}{\Gamma\left(\frac{1-q}{2}\right)}\right)=0.$$

- Thus for the selected root q is:

$$Im\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{q}{2}\right)}{\Gamma\left(\frac{q}{2}\right)}-\frac{1}{2}\frac{\Gamma'\left(\frac{1-q}{2}\right)}{\Gamma\left(\frac{1-q}{2}\right)}\right)=0,\quad \forall q\in\mathcal{P}_1\cup\mathcal{P}_2. \tag{24}$$

- From (6) equality (24) can be rewritten as follows:

$$\sum_{n=0}^{\infty}\left(\frac{t_q}{(2n+\sigma_q)^2+t_q^2}-\frac{t_q}{(2n+1-\sigma_q)^2+t_q^2}\right)=0.$$

I.e.

$$\textstyle\sum_{n=0}^{\infty}\frac{t_q((2n+1-\sigma_q)^2-(2n+\sigma_q)^2)}{((2n+\sigma_q)^2+t_q^2)((2n+1-\sigma_q)^2+t_q^2)}=$$
$$=\textstyle\sum_{n=0}^{\infty}\frac{t_q(1-2\sigma_q)(4n+1)}{((2n+\sigma_q)^2+t_q^2)((2n+1-\sigma_q)^2+t_q^2)}=$$
$$=(1-2\sigma_q)\textstyle\sum_{n=0}^{\infty}\frac{t_q(4n+1)}{((2n+\sigma_q)^2+t_q^2)((2n+1-\sigma_q)^2+t_q^2)}=0.$$

Sum

$$\sum_{n=0}^{\infty} \frac{t_q(4n+1)}{((2n+\sigma_q)^2+t_q^2)((2n+1-\sigma_q)^2+t_q^2)}$$

exists and is not equal to 0 when $t_q \neq 0$ so the equality (24) is performed exclusively at

$$\sigma_q = \frac{1}{2}.$$

So, assuming that an arbitrary nontrivial root q of zeta functions belongs to the union $\mathcal{P}_1 \cup \mathcal{P}_2$ we found that it belongs only to $\mathcal{P}_2$, i.e. $\mathcal{P}_1 = \varnothing$.

And according to the fact that $\mathcal{P}_1 = \varnothing \Leftrightarrow \mathcal{P}_3 = \varnothing$ we have:

$$\mathcal{P}_3 = \mathcal{P}_1 = \varnothing, \;\; \mathcal{P} = \mathcal{P}_2.$$

This proves the basic statement and the assumption which had been made by Bernhard Riemann about of the real parts of the nontrivial zeros of zeta function.

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