**The real parts of the nontrivial Riemann zeta function zeros**

Igor Turkanov

ABSTRACT

This theorem is based on holomorphy of studied functions and the fact that nearby of a singularity point the imaginary part of some rational function can take random arbitrary preassigned value.

The colored markers are:
- ● - assumption or a fact which is not proven at present;
- ● - the statement which requires additional attention;
- ● - statement which is proved earlier or clearly undestandable.

THEOREM

● The real parts of all the nontrivial Riemann zeta function zeros $\rho$ lie on the line $Re\left(\rho\right)=\frac{1}{2}$.

PROOF:

● According to the functional equality [10, p. 22], [5, p. 8-11]:

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta\left(s\right)=\Gamma\left(\frac{1-s}{2}\right)\pi^{-\frac{1-s}{2}}\zeta\left(1-s\right),\qquad Re\left(s\right)>0 \tag{1}$$

$\zeta\left(s\right)$ - the Riemann zeta function, $\Gamma\left(s\right)$ - the Gamma function.

- From [5, p. 8-11] $\zeta(\bar{s}) = \overline{\zeta(s)}$, it means that $\forall \rho = \sigma + it$: $\zeta(\rho) = 0$ and $0 \leqslant \sigma \leqslant 1$ we have:

$$\zeta(\bar{\rho}) = \zeta(1-\rho) = \zeta(1-\bar{\rho}) = 0 \tag{2}$$

- From [11], [9, p. 128], [10, p. 45] we know that $\zeta(s)$ has no nontrivial zeros on the line $\sigma = 1$ and consequently on the line $\sigma = 0$ also, in accordance with (2) they don't exist.

- Let's denote the set of nontrivial zeros $\zeta(s)$ through $\mathcal{P}$ (multiset with consideration of multiplicitiy):

$$\mathcal{P} \equiv \{\rho : \ \zeta(\rho) = 0, \ \rho = \sigma + it, \ 0 < \sigma < 1\},$$

and

$$\mathcal{P}_1 \equiv \left\{\rho : \ \zeta(\rho) = 0, \ \rho = \sigma + it, \ 0 < \sigma < \frac{1}{2}\right\} \tag{3}$$
$$\mathcal{P}_2 \equiv \left\{\rho : \ \zeta(\rho) = 0, \ \rho = \frac{1}{2} + it\right\}$$
$$\mathcal{P}_3 \equiv \left\{\rho : \ \zeta(\rho) = 0, \ \rho = \sigma + it, \ \frac{1}{2} < \sigma < 1\right\}$$

Then:

$$\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 \text{ and } \mathcal{P}_1 \cap \mathcal{P}_2 = \mathcal{P}_2 \cap \mathcal{P}_3 = \mathcal{P}_1 \cap \mathcal{P}_3 = \varnothing, \ \|\mathcal{P}_1\| = \|\mathcal{P}_3\|$$

- Hadamard's theorem (Weierstrass preparation theorem) on the decomposition of function through the roots gives us the following result [10, p. 30], [5, p. 31], [12]:

$$\zeta(s) = \frac{\pi^{\frac{s}{2}} e^{as}}{s(s-1)\Gamma\left(\frac{s}{2}\right)} \prod_{\rho \in \mathcal{P}} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}, \qquad Re(s) > 0 \tag{4}$$

$$a = \ln 2\sqrt{\pi} - \frac{\gamma}{2} - 1, \ \gamma - \text{Euler's constant and}$$

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{2}\ln\pi + a - \frac{1}{s} + \frac{1}{1-s} - \frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} + \sum_{\rho\in\mathcal{P}}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) \tag{5}$$

- According to the fact that $\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}$ - Digamma function of [10, p. 31], [5, p. 23] we have:

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{1-s} + \sum_{\rho\in\mathcal{P}}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) + \sum_{n=1}^{\infty}\left(\frac{1}{s+2n} - \frac{1}{2n}\right) + C, \tag{6}$$

$$C = const$$

.

- From [4, p. 160], [8, p. 272], [3, p. 81]:

$$\sum_{\rho\in\mathcal{P}}\frac{1}{\rho} = 1 + \frac{\gamma}{2} - \ln 2\sqrt{\pi} = 0,0230957\ldots \tag{7}$$

- Indeed, from (2):

$$\sum_{\rho\in\mathcal{P}}\frac{1}{\rho} = \frac{1}{2}\sum_{\rho\in\mathcal{P}}\left(\frac{1}{1-\rho} + \frac{1}{\rho}\right)$$

- From (5):

$$2\sum_{\rho\in\mathcal{P}}\frac{1}{\rho} = \lim_{s\to 1}\left(\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{1-s} + \frac{1}{s} - a - \frac{1}{2}\ln\pi + \frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\right).$$

- Also it's known, for example, from [10, p. 49], [3, p. 98] that the number of nontrivial zeros of $\rho = \sigma + it$ in strip $0 < \sigma < 1$, the imaginary parts of which $t$ are less than some number $T > 0$ is limited, i.e.

$$\parallel \{\rho :\ \rho\in\mathcal{P},\ \rho = \sigma + it,\ |t| < T\} \parallel < \infty.$$

- Besides, it can be presented that on the contrary the sum of $\sum_{\rho\in\mathcal{P}}\frac{1}{\rho}$ would have been unlimited.

- Thus $\forall T > 0\ \exists\, \delta_x > 0,\ \delta_y > 0$ such that

$$\text{in area } 0 < t \leqslant \delta_y, 0 < \sigma \leqslant \delta_x \text{ there are no zeros } \rho = \sigma + it \in \mathcal{P}. \tag{8}$$

Let's consider random root $q \in \mathcal{P}_1 \cup \mathcal{P}_2$
Let's denote $k(q)$ the multiplicity of the root $q$.
Let's examine the area $Q\,(R) \overset{\text{def}}{=} \{s : \|s - q\| \leqslant R, R > 0\}$.

- From the fact of finiteness of set of nontrivial zeros $\zeta(s)$ in the limited area follows $\exists\ R > 0$, such that $Q(R)$ does not contain any root from $\mathcal{P}$ except $q$.

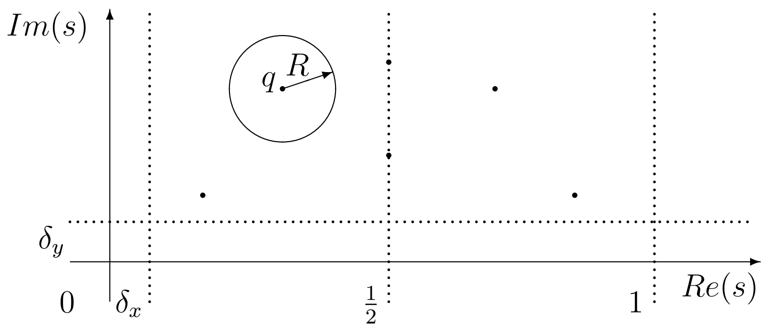


Fig. 1.

- From [1], [10, p. 31], [5, p. 23] we know that the Digamma function $\dfrac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}$

in the area $Q(R)$ has no poles, i.e. $\forall s \in Q(R)$

$$\left\| \frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \right\| < \infty.$$

Let's denote:

$$I_{\mathcal{P}}(s) \overset{\text{def}}{=} -\frac{1}{s} + \frac{1}{1-s} + \sum_{\rho \in \mathcal{P}} \frac{1}{s - \rho}$$

and

$$I_{\mathcal{P} \setminus \{q\}}(s) = -\frac{1}{s} + \frac{1}{1-s} + \sum_{\rho \in \mathcal{P} \setminus \{q\}} \frac{1}{s - \rho}. \tag{9}$$

Hereinafter $\mathcal{P}\setminus\{q\} \overset{\text{def}}{=} \mathcal{P}\setminus\{(q,k(q))\}$ (the difference in the multiset).

Summation - $\sum_{\rho\in\mathcal{P}} \dfrac{1}{s-\rho}$ and $\sum_{\rho\in\mathcal{P}\setminus\{q\}} \dfrac{1}{s-\rho}$ further we shall consider as the sum of pairs $\left(\dfrac{1}{s-\rho}+\dfrac{1}{s-(1-\rho)}\right)$ and $\sum_{\rho\in\mathcal{P}} \dfrac{1}{\rho}$ as the sum of pairs $\left(\dfrac{1}{\rho}+\dfrac{1}{1-\rho}\right)$ as a consequence of division of the sum from (6) $\sum_{\rho\in\mathcal{P}}\left(\dfrac{1}{s-\rho}+\dfrac{1}{\rho}\right)$ into $\sum_{\rho\in\mathcal{P}} \dfrac{1}{s-\rho}+\sum_{\rho\in\mathcal{P}} \dfrac{1}{\rho}$. As specifed in [4], [6], [8], [10].

- Let's note that $I_{\mathcal{P}\setminus\{q\}}(s)$ is complex differentiable function $\forall\ s\in Q(R)$.

Then from (5) we have:

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{2}\ln\pi + a - \frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} + \sum_{\rho\in\mathcal{P}} \frac{1}{\rho} + I_{\mathcal{P}}(s). \tag{10}$$

And in view of (7):

$$Im\frac{\zeta'(s)}{\zeta(s)} = Im\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} + I_{\mathcal{P}}(s)\right). \tag{11}$$

Let's note that from the equality of

$$\sum_{\rho\in\mathcal{P}} \frac{1}{1-s-\rho} = -\sum_{(1-\rho)\in\mathcal{P}} \frac{1}{s-(1-\rho)} = -\sum_{\rho\in\mathcal{P}} \frac{1}{s-\rho} \tag{12}$$

follows that:

$$I_{\mathcal{P}}(1-s) = -I_{\mathcal{P}}(s),\ I_{\mathcal{P}\setminus\{q\}}(1-s) = -I_{\mathcal{P}\setminus\{1-q\}}(s),\ Re\,(s)>0.$$

- Besides

$$I_{\mathcal{P}\setminus\{q\}}(s) = I_{\mathcal{P}}(s) - \frac{k(q)}{s-q}$$

and $I_{\mathcal{P}\setminus\{q\}}(s)$ is limited in the area of $s\in Q(R)$ as a result of absence of its poles in this area as well as its differentiability in each point of this area.

- From (1) follows:

$$\frac{\zeta'(s)}{\zeta(s)}+\frac{\zeta'(1-s)}{\zeta(1-s)}=-\frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}-\frac{1}{2}\frac{\Gamma'\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}+\ln\pi,\ \ Re\,(s)>0. \quad (13)$$

- Let's examine a circle with the center in a point $q$ and radius $r \leqslant R$, laying in the area of $Q(R)$:

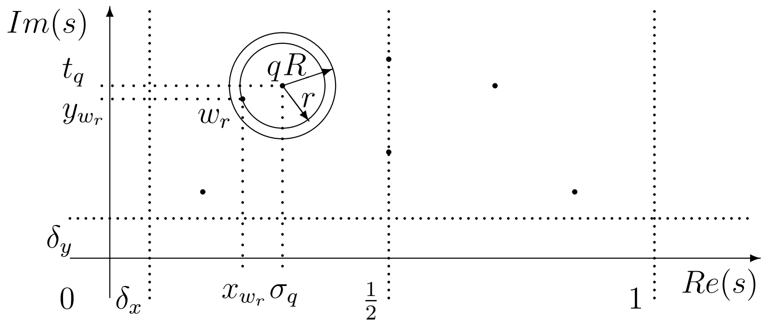


Fig. 2.

- For $s=x+iy,\ q=\sigma_q+it_q$

$$Im\frac{k(q)}{s-q}=Im\frac{k(q)}{x+iy-\sigma_q-it_q}=\frac{k(q)(t_q-y)}{(x-\sigma_q)^2+(y-t_q)^2}=k(q)\frac{t_q-y}{r^2}.$$

- On each of the semicircles: on the left - $\{s:\|s-q\|=r,\ \sigma_q-r\leqslant x\leqslant\sigma_q\}$ and on the right - $\{s:\|s-q\|=r,\ \sigma_q\leqslant x\leqslant\sigma_q+r\}$ function $Im\frac{k(q)}{s-q}$ is continuous and takes values from $-\frac{k(q)}{r}$ to $\frac{k(q)}{r}$, $r>0$ and $k(q)\geqslant 1$ as the multiplicity of the root.

- From (11) function

$$Im\frac{\zeta'(s)}{\zeta(s)}-Im\frac{k(q)}{s-q}=Im\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}+I_{\mathcal{P}\setminus\{q\}}(s)\right)$$

in the area of $Q(R)$ is limited so $\exists H_1(R) > 0,\ H_1(R) \in \mathbb{R}$:

$$\left| Im\frac{\zeta'(s)}{\zeta(s)} - Im\frac{k(q)}{s-q} \right| < H_1(R), \qquad \forall s \in Q(R). \tag{14}$$

- From the characteristics of continuous functions on the interval to take all intermediate values between its extremes, it follows that $\exists R_1 > 0$ :

$$R_1 < R,\ \frac{k(q)}{R_1} > H_1(R)$$

and $\forall r > 0,\ r < R_1$ there is a point on the left semicircle $w_r \stackrel{\text{def}}{=} x_{w_r} + iy_{w_r}$ such that:

$$Im\frac{\zeta'(w_r)}{\zeta(w_r)} - Im\frac{k(q)}{w_r - q} = -Im\frac{k(q)}{w_r - q},$$

i.e.

$$Im\frac{\zeta'(w_r)}{\zeta(w_r)} = 0, \quad \forall r > 0, r < R_1. \tag{15}$$

- And from (7), (10) follows:

$$k(q)\frac{t_q - y_{w_r}}{r^2} = -Im\left( -\frac{1}{2}\frac{\Gamma'\left(\frac{w_r}{2}\right)}{\Gamma\left(\frac{w_r}{2}\right)} + I_{\mathcal{P}\setminus\{q\}}(w_r) \right), \tag{16}$$

for $\forall r > 0,\ r < R_1$.

Let's note the real function:

$$\alpha(s) \stackrel{\text{def}}{=} -\frac{1}{k(q)} Im\left( -\frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} + I_{\mathcal{P}\setminus\{q\}}(s) \right).$$

- Note that the function $\alpha(s)$ and its derivative are limited $\forall s \in Q(R_1)$.

- I.e. at $r \to 0$, since $\|q - w_r\| = r^2$:

$$\begin{cases} t_q - y_{w_r} = & \alpha(w_r) r^2 \\ x_{w_r} - \sigma_q = & r(1 - O(r^2)). \end{cases} \tag{17}$$

For definiteness let's consider, that $w_r$ we set on the left semicircle as on the right semicircle a point with the same - (15) characteristics does exist.

- Suppose that:

$$\alpha(q) = -\frac{1}{k(q)} Im \left( -\frac{1}{2} \frac{\Gamma'\left(\frac{q}{2}\right)}{\Gamma\left(\frac{q}{2}\right)} + I_{\mathcal{P}\setminus\{q\}}(q) \right) \neq 0. \tag{18}$$

- Then, according to the theorem on the preservation of the sign of the continuous real function of two variables $\alpha(x+iy)$ there exists a surrounding of $q$ in which this function does not come to 0 and has the same sign.

I.e. $\exists\ R_2 \in \mathbb{R},\ 0 < R_2 \leqslant R_1$ :

$$\forall\ s \in Q(R_2)\ \ \alpha(s) \neq 0,\ \ sign(\alpha(s)) = const. \tag{19}$$

From this point and from (17) follows, that $\forall\ s \in Q(R_2)$, $\forall r_1 > 0,\ r_2 > 0,$ $r_1 < R_2,\ r_2 < R_2$ :

$$(t_q - y_{w_{r_1}})(t_q - y_{w_{r_2}}) > 0, \tag{20}$$

i.e. points $w_{r_1}$ and $w_{r_2}$ lay at the same time or in the top semicircle, not including the piece of a circle laying on a straight line $y = t_q$, or in bottom. Let for definiteness it be - the bottom semicircle, i.e. $\alpha(q) > 0$.

Given (17) to point $w_r$, $s = x + iy$, consider the partial derivative:

$$\frac{\partial}{\partial x}\left(-\frac{1}{2}Im\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} + ImI_{\mathcal{P}\setminus\{q\}}(s) + \frac{k(q)(t_q - y_{w_r})}{(x-\sigma_q)^2 + (y_{w_r} - t_q)^2}\right)_{s=w_r} =$$

$$= \frac{\partial}{\partial x}\left(-\frac{1}{2}Im\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} + ImI_{\mathcal{P}\setminus\{q\}}(s)\right)_{s=w_r} +$$

$$+\frac{2k(q)(y_{w_r} - t_q)(x_{w_r} - \sigma_q)}{((x_{w_r} - \sigma_q)^2 + ((y_{w_r} - t_q)^2)^2} =$$

$$= O(1)_{r\to 0} + \frac{2k(q)\alpha(w_r)(1 - O(r^2)_{r\to 0})}{r((1 - O(r^2)_{r\to 0})^2 + \alpha(w_r)^2 r^2)^2}.$$

- I.e. $\exists\, R_3 \in \mathbb{R},\ 0 < R_3 \leqslant R_2 :\ \forall\, 0 < r < R_3 :$

$$\frac{\partial}{\partial x}\left(-\frac{1}{2}Im\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} + ImI_{\mathcal{P}\setminus\{q\}}(s) + \frac{k(q)(t_q - y_{w_r})}{(x-\sigma_q)^2 + (y_{w_r} - t_q)^2}\right)_{s=w_r} \neq 0.$$

- This, in turn, by the implicit function theorem, means that for $s = x+iy \in Q(R_3),\ t_q - y = \alpha(w_r)r^2,\ x - \sigma_q = r(1 - O(r^2))$, there exists a continuous function $x = W(y)$:

$$Im\frac{\zeta'(x+iy)}{\zeta(x+iy)} = 0 \ \Leftrightarrow x = W(y). \tag{21}$$

I.e. $\forall\, y : t_q - y < \alpha(w_{R_3})R_3^2,\ \exists\, 0 < r < R_3 :$

$$x_{w_r} = W(y). \tag{22}$$

- Based on the (21) and (22) let's choose $r_1,\ r_2,\ B_\beta > 0$, satisfying the conditions:

$$r_1 = r,\ \ r_2 :\ x_{w_{r_2}} = W(y_{w_{r_1}} + \frac{\alpha(w_{r_1})}{B_\beta}r_1^3). \tag{23}$$

- I.e.

$$r_2 = \sqrt{(W(y_{w_{r_1}} + \frac{\alpha(w_{r_1})}{B_\beta} r_1^3) - \sigma_q)^2 + (t_q - y_{w_{r_1}} - \frac{\alpha(w_{r_1})}{B_\beta} r_1^3)^2}.$$

- Let's join the points $w_{r_1}$ and $w_{r_2}$ by the curve described by a function that has a derivative at each interior point of the segment.
Let's designate: $f_{r_1,r_2}(\tau) \stackrel{\text{def}}{=} f_x(\tau) + i f_y(\tau)$:

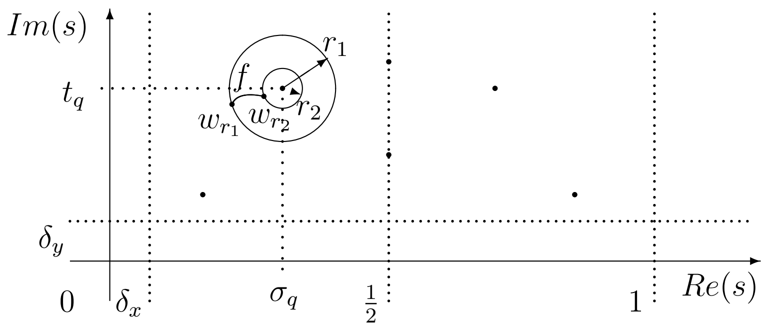

Fig. 3.

According to construction let's designate $\tau_{r_1}$ and $\tau_{r_2}$ such, that:

$$f_{r_1,r_2}(\tau_{r_1}) = w_{r_1},\ f_{r_1,r_2}(\tau_{r_2}) = w_{r_2}.$$

- Function $\dfrac{\zeta'(s)}{\zeta(s)}$ is meromorphic as a division of two holomorphic functions, therefore the function $Im\dfrac{\zeta'(s)}{\zeta(s)}$ is differentiable and so is continuous and differentiable by $\tau$ function $Im\dfrac{\zeta'(f_{r_1,r_2}(\tau))}{\zeta(f_{r_1,r_2}(\tau))}$.

It means that continuous on a segment and differentiated on an internal interval of this segment the real function according to (15) takes identical values on its ends:

$$Im\frac{\zeta'(f_{r_1,r_2}(\tau_{r_1}))}{\zeta(f_{r_1,r_2}(\tau_{r_1}))} = Im\frac{\zeta'(f_{r_1,r_2}(\tau_{r_2}))}{\zeta(f_{r_1,r_2}(\tau_{r_2}))} = 0.$$

- By Rolle's theorem on the extremum of a differentiable function on the interval we have:

$$\exists \tau_1 \in (\tau_{r_1}, \tau_{r_2}): \quad \left( Im \frac{\zeta'\left(f_{r_1,r_2}(\tau)\right)}{\zeta\left(f_{r_1,r_2}(\tau)\right)} \right)'_{\tau=\tau_1} = 0. \tag{24}$$

I.e. on a curve described by function $f_{r_1,r_2}(\tau), \ \ \tau \in (\tau_{r_1}, \tau_{r_2})$, there is a point $\Theta_w = \Theta_w(r_1, r_2) \stackrel{\text{def}}{=} f_{r_1,r_2}(\tau_1)$ for which it is true (24).

- As the curve which connect points $w_{r_1}$ and $w_{r_2}$, let's examine the following variant of the line.

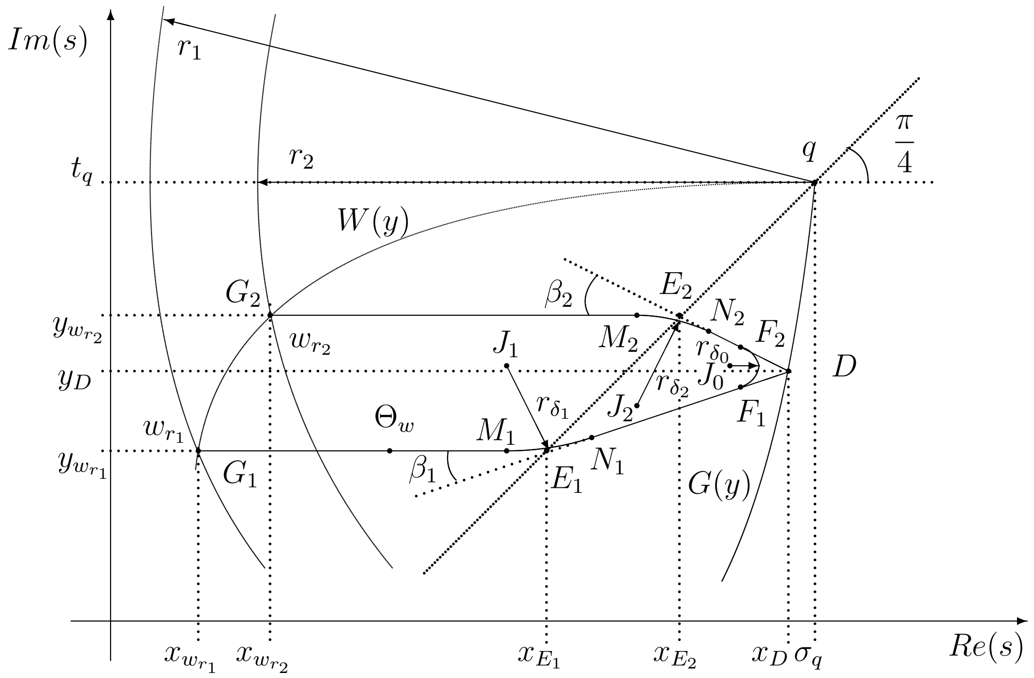


Fig. 4.

The curve connecting $w_{r_1}$ and $w_{r_2}$, consists of seven sections: $G_1M_1$, $M_1N_1$, $N_1F_1$, $F_1F_2$, $F_2N_2$, $N_2M_2$, $M_2G_2$. Where $G_1$ is $w_{r_1}$, $G_2$ - $w_{r_2}$, $G_1M_1$, $G_2M_2$, $N_1F_1$, $N_2F_2$ - the segments laying on straight lines, $M_1N_1$, $M_2N_2$, $F_1F_2$ - arches of the circles, touching the corresponding straight lines in points $M_1$, $N_1$, $F_1$, $F_2$, $N_2$, $M_2$.

Segments $G_1F_1$ and $F_2G_2$ are parallel to the real axis and lay on the straight lines $y = y_{w_{r_1}}$ and $y = y_{w_{r_2}}$.

- Let's choose a point $D \stackrel{\text{def}}{=} (x_D, y_D)$ as follows:

$$\frac{y_{w_{r_2}} - y_D}{y_D - y_{w_{r_1}}} \stackrel{\text{def}}{=} a_D > 0, \quad y_D = \frac{y_{w_{r_2}} + a_D y_{w_{r_1}}}{1 + a_D}, \tag{25}$$

$$G(y) \stackrel{\text{def}}{=} \sigma_q - \frac{(t_q - y)^{\frac{3}{2}}}{\sqrt{\alpha(q)}}, \quad x_D = G(y_D) = \sigma_q - \frac{(t_q - y_D)^{\frac{3}{2}}}{\sqrt{\alpha(q)}}. \tag{26}$$

I.e. let's choose $y_D$ so that this number lays strictly between numbers $y_{w_{r_1}}$ and $y_{w_{r_2}}$, and fix a proportion in which the length $\left|y_{w_{r_2}} - y_{w_{r_1}}\right|$ shares on $\left|y_{w_{r_2}} - y_D\right| > 0$ and $\left|y_D - y_{w_{r_1}}\right| > 0$ for $\forall\ 0 < r_2 < r_1 < R_3$.

The point $D$ is an intersection of a straight lays $y = y_D$ and a curve $x = G(y)$.

The point $D$ for $\forall\ 0 < r_2 < r_1 < R_3$ lays between the straight line $x = \sigma_q$ and the straight line that goes from point $q$ under angle $\frac{\pi}{4}$ to the real axis, in accordance with $(t_q - y_D) < R_3 \leqslant R < 1$.

Let's draw straight lines $DE_1$ and $DE_2$ so that their points of intersection with straight lines $G_1E_1$ and $G_2E_2$ - $E_1$ and $E_2$ accordingly, lay on a straight line that goesfrom point $q$ under angle $\frac{\pi}{4}$ to real axis.

Points $F_1$ and $F_2$ are the points of tangency of the circle of radius $r_{\delta_0}$ centered at $J_0$ and straight lines $E_1D$ and $E_2D$. These lines have an angle of inclination to the real axis $\beta 1$ and $-\beta 2$ respectively.

At such construction of the circle, touching the specified straight lines - $DE_1$ and $G_1E_1$, $DE_2$ and $G_2E_2$ in points $M_1,\ N_1$ and $M_2,\ N_2$, with the centers in points $J_1$ and $J_2$ respectively, can have as small radiuses as posible $r_{\delta_1} > 0$ and $r_{\delta_2} > 0$.

I.e. the points of a curve which are set on arches $M_1N_1$ and $M_2N_2$, can be as much as closely located to a straight line that goes from point $q$ under angle $\frac{\pi}{4}$ to real axis, for $\forall\ 0 < r_2 < r_1 < R_3$.

- Let's suppose that the point $\Theta_w$ lays on the segment of the $N_1F_1$ or $N_2F_2$.

Let's take $Re(s) = x$ as $\tau$, then:

$$f_x(x) = x,$$

for the segment $N_1F_1$, $N_1 \stackrel{\text{def}}{=} (x_{N_1}, y_{N_1})$ :

$$y = f_y(x) = y_{N_1} + \operatorname{tg}(\beta_1)(x - x_{N_1}), \quad y(\tau_1)' = \operatorname{tg}(\beta_1),$$

for the segment $N_2F_2$, $N_2 \stackrel{\text{def}}{=} (x_{N_2}, y_{N_2})$ :

$$y = f_y(x) = y_{N_2} + \operatorname{tg}(-\beta_2)(x - x_{N_2}), \quad y(\tau_1)' = -\operatorname{tg}(\beta_2).$$

From (23), (26), for $r_1 \to 0$:

$$\sigma_q - x_D = \frac{(t_q - y_D)^{\frac{3}{2}}}{\sqrt{\alpha(q)}} = \frac{\left(t_q - \dfrac{y_{w_{r_2}} + a_D y_{w_{r_1}}}{1 + a_D}\right)^{\frac{3}{2}}}{\sqrt{\alpha(q)}} =$$

$$= \frac{\left(\dfrac{t_q - y_{w_{r_2}} + a_D(t_q - y_{w_{r_1}})}{1 + a_D}\right)^{\frac{3}{2}}}{\sqrt{\alpha(q)}} =$$

$$= \frac{\left(\dfrac{\alpha(w_{r_1})(r_1^2 - B_\beta^{-1} r_1^3) + a_D \alpha(w_{r_1}) r_1^2}{1 + a_D}\right)^{\frac{3}{2}}}{\sqrt{\alpha(q)}} =$$

$$= \frac{(\alpha(w_{r_1})(r_1^2 - (1 + a_D)^{-1} B_\beta^{-1} r_1^3))^{\frac{3}{2}}}{\sqrt{\alpha(q)}} =$$

$$= \alpha(w_{r_1}) r_1^3 \sqrt{\frac{\alpha(w_{r_1})}{\alpha(q)}} (1 - (1 + a_D)^{-1} B_\beta^{-1} r_1)^{\frac{3}{2}} =$$

$$= \alpha(w_{r_1}) r_1^3 \left(1 + \left(\sqrt{\frac{\alpha(w_{r_1})}{\alpha(q)}} - 1\right)\right) (1 - (1 + a_D)^{-1} B_\beta^{-1} r_1)^{\frac{3}{2}} =$$

$$= \alpha(w_{r_1}) r_1^3 + o(r_1^3). \quad (27)$$

- In view from the (23), (27), setting point $E_1: \ \sigma_q - x_{E_1} = t_q - y_{w_{r_1}}$, $r_1 \to 0$:

$$\begin{aligned}
\operatorname{tg}(\beta_1) = \frac{y_D - y_{w_{r_1}}}{x_D - x_{E_1}} &= \frac{y_D - y_{w_{r_1}}}{\sigma_q - x_{E_1} + (x_D - \sigma_q)} = \frac{y_D - y_{w_{r_1}}}{t_q - y_{w_{r_1}} - (\sigma_q - x_D)} = \\
&= \frac{\frac{y_{w_{r_2}} - y_{w_{r_1}}}{1 + a_D}}{t_q - y_{w_{r_1}} - (\sigma_q - x_D)} = \frac{\frac{\alpha(w_{r_1}) r_1^3}{B_\beta(1 + a_D)}}{\alpha(w_{r_1}) r_1^2 - \alpha(w_{r_1}) r_1^3 + o(r_1^3)} = \\
&= \frac{1}{B_\beta(1 + a_D)} r_1 + O(r_1^2). \qquad (28)
\end{aligned}$$

- Similarly, for the point of $E_2: \ \sigma_q - x_{E_2} = t_q - y_{w_{r_2}}$, in view of (23) and $r_1 \to 0$:

$$\begin{aligned}
\operatorname{tg}(\beta_2) &= \frac{y_{w_{r_2}} - y_D}{x_D - x_{E_2}} = \frac{y_{w_{r_2}} - y_D}{\sigma_q - x_{E_2} + (x_D - \sigma_q)} = \frac{y_{w_{r_2}} - y_D}{t_q - y_{w_{r_2}} - (\sigma_q - x_D)} = \\
&= \frac{\frac{a_D(y_{w_{r_2}} - y_{w_{r_1}})}{1 + a_D}}{t_q - y_{w_{r_2}} - (\sigma_q - x_D)} = \frac{\frac{a_D \alpha(w_{r_1}) r_1^3}{B_\beta(1 + a_D)}}{\alpha(w_{r_1}) r_1^2 - B_\beta^{-1} \alpha(w_{r_1}) r_1^3 - \alpha(w_{r_1}) r_1^3 + o(r_1^3)} = \\
&= \frac{a_D}{B_\beta(1 + a_D)} r_1 + O(r_1^2). \qquad (29)
\end{aligned}$$

- And for the $s = x + iy$ equation (24) at the point $\Theta_w = (\tau_1, y(\tau_1))$ can be written as follows:

$$\begin{aligned}
y(\tau_1)' \frac{(y(\tau_1) - t_q)^2 - (\tau_1 - \sigma_q)^2}{((\tau_1 - \sigma_q)^2 + (y(\tau_1) - t_q)^2)^2} - \frac{2(\tau_1 - \sigma_q)(t_q - y(\tau_1))}{((\tau_1 - \sigma_q)^2 + (y(\tau_1) - t_q)^2)^2} = \\
= \alpha(x + iy(x))'_{x=\tau_1} + y(\tau_1)' \alpha(x + iy(x))'_{y=y(\tau_1)}. \qquad (30)
\end{aligned}$$

For convenience let's designate:

$$\begin{aligned}
\Delta_{\tau_1} &\stackrel{\text{def}}{=} \sigma_q - \tau_1, \\
g(\tau) &\stackrel{\text{def}}{=} \alpha(x + iy(x))'_{x=\tau} + y(\tau)' \alpha(x + iy(x))'_{y=y(\tau)},
\end{aligned}$$

for the case when the point $\Theta_w$ lays on the segment $N_1F_1$:

$$\beta \stackrel{\text{def}}{=} \beta 1,$$

for $N_2F_2$:

$$\beta \stackrel{\text{def}}{=} -\beta 2.$$

- From inequalities on the segments $N_1F_1$ and $N_2F_2$, in view of (23):

$$0 < \sigma_q - \tau_1 < (t_q - y_{w_{r_1}}) = \alpha(w_{r_1})r_1^2$$

and

$$t_q - y_{\tau_1} < (t_q - y_{w_{r_1}}) = \alpha(w_{r_1})r_1^2, \tag{31}$$

$$t_q - y_{\tau_1} > (t_q - y_{w_{r_2}}) = \alpha(w_{r_1})(r_1^2 - B_\beta^{-1}r_1^3) \tag{32}$$

follows:

$$((\tau_1 - \sigma_q)^2 + (y(\tau_1) - t_q)^2)^2 = \Theta(\alpha(w_{r_1})^4 r_1^8)_{r_1 \to 0}.$$

Where the symbol $\Theta$ means that $\exists\ 0 < R_4 \leqslant R_3,\ k_1 > 0,\ k_2 > 0:$
$\forall\ 0 < r_1 < R_4:$

$$k_1\alpha(w_{r_1})^4 r_1^8 < ((\tau_1 - \sigma_q)^2 + (y(\tau_1) - t_q)^2)^2 < k_2\alpha(w_{r_1})^4 r_1^8.$$

In view of limitedness of partial derivatives of the function $\alpha(s)$ for $\forall\ s \in Q(R)$ and (28), (29) we have:

$$g(\tau_1) = O(1)_{r_1 \to 0} + O(r_1)_{r_1 \to 0} = O(1)_{r_1 \to 0}.$$

Let's rewrite (30) in new designations at $r_1 \to 0$:

$$\begin{aligned} &\mathrm{tg}(\beta)(\Delta_{\tau_1}^2 - (y(\tau_1) - t_q)^2) - 2\Delta_{\tau_1}(t_q - y(\tau_1)) = \\ &= -O(1)\Theta(\alpha(w_{r_1})^4 r_1^8). \end{aligned} \tag{33}$$

- In view of the inequalities (31) and (32) at $r_1 \to 0$:

$$(y(\tau_1) - t_q)^2 = \alpha(w_{r_1})^2 r_1^4 + O(r_1^5).$$

And then (33) can be written:

$$\operatorname{tg}(\beta)\Delta_{\tau_1}^2 - 2(t_q - y(\tau_1))\Delta_{\tau_1} = \operatorname{tg}(\beta)\alpha(w_{r_1})^2 r_1^4 + O(\operatorname{tg}(\beta) r_1^5)$$

- If this quadratic equation with regards the variable $\Delta_{\tau_1}$ has roots $\Delta_{\tau_1,1}$ and $\Delta_{\tau_1,2}$, then they are:

$$\Delta_{\tau_1,1,2} =$$

$$= \frac{2(t_q - y(\tau_1)) \pm \sqrt{4(t_q - y(\tau_1))^2 + 4\operatorname{tg}(\beta)^2\alpha(w_{r_1})^2 r_1^4 + O(\operatorname{tg}(\beta)^2 r_1^5)}}{2\operatorname{tg}(\beta)}.$$

Let's note, that $\exists\ 0 < R_5 \leqslant R_4 :\ \forall\ 0 < r_1 < R_5$ the discriminant of the equation (33) is positive, i.e. at the equation analyzed has two real roots.

For the case when the point $\Theta_w$ allegedly lays on the segment $N_1F_1$ we have: $\beta = \beta_1 > 0$ and $\operatorname{tg}(\beta) > 0$, then the first root of the equation (33) is negative:

$$\Delta_{\tau_1,1} =$$

$$= \frac{2(t_q - y(\tau_1)) - \sqrt{4(t_q - y(\tau_1))^2 + 4\operatorname{tg}(\beta_1)^2\alpha(w_{r_1})^2 r_1^4 + O(\operatorname{tg}(\beta_1)^2 r_1^5)}}{2\operatorname{tg}(\beta_1)}.$$

I.e.

$$x_{\tau_1} > \sigma_q,$$

it's impossible according to construction of the curve where the whole curve segment $N_1F_1$ lays to the left of a straight line $x = \sigma_q$.

The second root at $r_1 \to 0$:

$$\Delta_{\tau_1,2} =$$

$$= \frac{2(t_q - y(\tau_1)) + \sqrt{4(t_q - y(\tau_1))^2 + 4\operatorname{tg}(\beta_1)^2\alpha(w_{r_1})^2 r_1^4 + O(\operatorname{tg}(\beta_1)^2 r_1^5)}}{2\operatorname{tg}(\beta_1)}.$$

- Let's give estimation $\Delta_{\tau_1,2}$ at the bottom at $r_1 \to 0$:

$$\begin{aligned}\Delta_{\tau_1,2} > \frac{4(t_q - y(\tau_1))}{2\,\mathrm{tg}(\beta_1)} > \frac{2(t_q - y_{w_{r_2}})}{\mathrm{tg}(\beta_1)} = \\ = \frac{2\alpha(w_{r_1})(r_1^2 - B_\beta^{-1} r_1^3)}{\dfrac{1}{B_\beta(1+a_D)} r_1 + O(r_1^2)} = 2B_\beta(1+a_D)\alpha(w_{r_1})r_1 + O(r_1^2).\end{aligned}$$

Then at $r_1 \to 0$:

$$\begin{aligned}\Delta_{\tau_1,2} - (\sigma_q - x_{N_1}) > \Delta_{\tau_1,2} - (\sigma_q - x_{w_{r_1}}) > \\ > 2B_\beta(1+a_D)\alpha(w_{r_1})r_1 + O(r_1^2) - \alpha(w_{r_1})r_1^2 = \\ = \alpha(w_{r_1})(2B_\beta(1+a_D) - r_1)r_1 + O(r_1^2).\end{aligned}$$

- Therefore $\exists\, 0 < R_6 \leqslant R_5 : \ \forall\, 0 < r_1 < R_6 :$

$$\Delta_{\tau_1,2} - (\sigma_q - x_{N_1}) = x_{N_1} - x_{\tau_1} > 0.$$

I.e.

$$x_{\tau_1} < x_{N_1},$$

which is impossible because of the assumption that the point $\Theta_w$ lays on the segment $N_1F_1$.

Thus, none of the roots of the equation (33) approach this segment, hence the assumption that the point $\Theta_w$ lays on the segment $N_1F_1$ is false.

- For the case when the point $\Theta_w$ allegedly lays on the segment $N_2F_2$: $\beta = -\beta_2 < 0$ and $\mathrm{tg}(\beta) < 0$, then the first root of the equation (33):

$$\Delta_{\tau_1,1} =$$
$$= \frac{-2(t_q - y(\tau_1)) + \sqrt{4(t_q - y(\tau_1))^2 + 4\,\mathrm{tg}(\beta_2)^2\alpha(w_{r_1})^2 r_1^4 + O(\mathrm{tg}(\beta_2)^2 r_1^5)}}{2\,\mathrm{tg}(\beta_2)} <$$
$$< \frac{4\,\mathrm{tg}(\beta_2)^2\alpha(w_{r_1})^2 r_1^4 + O(\mathrm{tg}(\beta_2)^2 r_1^5)}{2\,\mathrm{tg}(\beta_2)(4(t_q - y(\tau_1)))} = \frac{\mathrm{tg}(\beta_2)\alpha(w_{r_1})^2 r_1^4 + O(\mathrm{tg}(\beta_2) r_1^5)}{2(t_q - y(\tau_1))} <$$
$$< \frac{\mathrm{tg}(\beta_2)\alpha(w_{r_1})^2 r_1^4 + O(\mathrm{tg}(\beta_2) r_1^5)}{2(t_q - y_{w_{r_2}})} = \frac{\frac{a_D}{B_\beta(1+a_D)} r_1 \alpha(w_{r_1})^2 r_1^4 + O(r_1^6)}{2\alpha(w_{r_1})(r_1^2 - B_\beta^{-1} r_1^3)} =$$
$$= \frac{a_D \alpha(w_{r_1})}{2B_\beta(1+a_D)} r_1^3 + O(r_1^4).$$

From (27) at $r_1 \to 0$:

$$\sigma_q - x_D = \alpha(w_{r_1}) r_1^3 + o(r_1^3).$$

- Therefore $\exists\ 0 < R_7 \leqslant R_6 :\ \forall\ 0 < r_1 < R_7,\ B_\beta > 0 :$

$$\Delta_{\tau_1,1} - (\sigma_q - x_D) = (\sigma_q - x_{\tau_1}) - (\sigma_q - x_D) = x_D - x_{\tau_1} <$$
$$< \alpha(w_{r_1}) r_1^3 \left(\frac{a_D}{2B_\beta(1+a_D)} - 1\right) + o(r_1^3) < 0.$$

- The last inequality is satisfied, provided:

$$\frac{a_D}{2B_\beta(1+a_D)} < 1 \ \Leftrightarrow\ \frac{a_D}{2(1+a_D)} < B_\beta. \tag{34}$$

I.e.

$$x_{\tau_1} > x_D,$$

it's impossible to build a curve where the whole curve segment $N_2F_2$ lays to the left of the point $D$.

- Let's examine the second root of the equation (33):

$$\Delta_{\tau_1,2} =$$
$$= \frac{-2(t_q - y(\tau_1)) - \sqrt{4(t_q - y(\tau_1))^2 + 4\,\mathrm{tg}(\beta_2)^2\alpha(w_{r_1})^2 r_1^4 + O(\mathrm{tg}(\beta_2)^2 r_1^5)}}{2\,\mathrm{tg}(\beta_2)}.$$

Obviously $\forall\ 0 < r_1 < R_7$ :

$$\Delta_{\tau_1,2} < 0 \;\Leftrightarrow\; x_{\tau_1} > \sigma_q > x_D > x_{F_2}.$$

It's impossible because of the assumption that the point $\Theta_w$ lays on the segment $N_2F_2$.

Thus, none of the roots of the equation (33) approach this segment, hence the assumption that the point $\Theta_w$ lays on the segment $N_2F_2$ is false.

So, from a certain moment at $r_1 \to 0$, the point $\Theta_w$ can not lay on the segments $N_1F_1$ and $N_2F_2$.
Similar considerations apply to the top semicircle, when $\alpha(q) < 0$. In all four cases, we can find such value of $0 < R_7 \leqslant R :\ \forall\ 0 < r_1 < R_7$ : that roots of the equation (33) will either be negative or be out of ordinates of the corresponding segments.

- Let's assume that the point $\Theta_w$ lays on the circle with center at $J_0 \overset{\mathrm{def}}{=} (x_{J_0}, y_{J_0})$ and the radius $r_{\delta_0} > 0$.

Let's take $Im(s) = y$ as $\tau$, then:

$$f_y(y) = y,\;\; x = f_x(y) = \sigma_q - \sqrt{r_{\delta_0}^2 - (y - y_{J_0})^2}.$$

Then, for $s = x + iy$ the equation (24) in the point $\Theta_w = (x(\tau_1), \tau_1)$ can be written as follows:

$$\frac{(\tau_1 - t_q)^2 - (x(\tau_1) - \sigma_q)^2}{((x(\tau_1) - \sigma_q)^2 + (\tau_1 - t_q)^2)^2} + x(\tau_1)' \frac{2(\sigma_q - x(\tau_1))(t_q - \tau_1)}{((x(\tau_1) - \sigma_q)^2 + (\tau_1 - t_q)^2)^2} = \\ = \alpha(x(y) + iy)'_{y=\tau_1} + x(\tau_1)'\alpha(x(y) + iy)'_{x=x(\tau_1)}. \tag{35}$$

On construction $x(\tau)'$ on an arch $F_1F_2$ can take values:

$$-\frac{1}{\mathrm{tg}(\beta_1)} \leqslant x(\tau)' \leqslant \frac{1}{\mathrm{tg}(\beta_2)}.$$

Partial derivatives of the function $\alpha(s)$ are also limited for $\forall s \in Q(R)$.

At $r_{\delta_0} \to 0$ :

$$x(\tau_1) \to x_D.$$

- Based on (17), (23), (27) the equation (35) can be written as follows:

$$\frac{\alpha(w_{r_1})^2 r_1^4 + O(r_1^5) - \alpha(w_{r_1})^2 r_1^6 + o(r_1^6)}{(\alpha(w_{r_1})^2 r_1^6 + o(r_1^6) + \alpha(w_{r_1})^2 r_1^4 + O(r_1^5))^2} + \\ + x(\tau_1)' \frac{2(\alpha(w_{r_1}) r_1^3 + o(r_1^3))(\alpha(w_{r_1}) r_1^2 + O(r_1^3))}{(\alpha(w_{r_1})^2 r_1^6 + o(r_1^6) + \alpha(w_{r_1})^2 r_1^4 + O(r_1^5))^2} = \\ = \frac{1}{\alpha(w_{r_1})^2 r_1^4} + O\left(\frac{1}{r_1^3}\right) + x(\tau_1)' \left(\frac{2}{\alpha(w_{r_1})^2 r_1^3} + O\left(\frac{1}{r_1^2}\right)\right) = \\ = \frac{1}{\alpha(w_{r_1})^2 r_1^3} \left(\frac{1}{r_1} + 2x(\tau_1)'\right) + O(r_1^{-3}) = \\ = \alpha(x(y) + iy)'_{y=\tau_1} + x(\tau_1)'\alpha(x(y) + iy)'_{x=x(\tau_1)} = O(r_1^{-1}). \tag{36}$$

I.e. at $r_1 \to 0$ :

$$\frac{1}{r_1} + 2x(\tau_1)' = O(1). \tag{37}$$

- The equation (37) can be executed only if for the value of $x(\tau_1)'$ is negative. This derivative takes its maximal on the module negative value on an arch $F_1F_2$ in the point $\tau_1 = F_1$ and it's equal in view of (28):

$$x(\tau_1)' = -\frac{1}{\mathrm{tg}(\beta_1)} = -\frac{1}{\dfrac{1}{B_\beta(1+a_D)}r_1 + O(r_1^2)} = -\frac{B_\beta(1+a_D)}{r_1} + O(r_1).$$

This means that if:

$$2B_\beta(1+a_D) < 1, \tag{38}$$

then the equation (37), from a certain moment, is not executed for any point $\tau_1$ from $F_1F_2$.

- I.e. $\exists\ 0 < R_8 \leqslant R_7 :\ \forall\ 0 < r_1 < R_8 :$

$$\frac{1}{r_1} + 2x(\tau_1)' = \Theta(r_1^{-1}) \neq O(1).$$

- Inequalities (34) and (38) limit the possible values of $B_\beta$:

$$\frac{a_D}{2(1+a_D)} < B_\beta < \frac{1}{2(1+a_D)},$$

therefore:

$$a_D < 1,$$

and for example for $a_D = \dfrac{1}{2}$:

$$\frac{1}{6} < B_\beta < \frac{1}{3},$$

i.e. there are such $a_D$ and $B_\beta$, for example:

$$a_D = \frac{1}{2},\ \ B_\beta = \frac{1}{4},$$

that the point $\Theta_w$ will not lay on the constructed curve between points $N_1$ and $N_2$ for $\forall\ 0 < r_1 < R_8$.

- Let's assume, that the point $\Theta_w$ lays on arches: $M_1N_1$ or $N_2M_2$ of the circles with the centers in points $J_1 \stackrel{\text{def}}{=} (x_{J_1}, y_{J_1})$, $J_2 \stackrel{\text{def}}{=} (x_{J_2}, y_{J_2})$ and radiuses $r_{\delta_1} > 0$ and $r_{\delta_2} > 0$ accordingly.

  Let's take $Re(s) = x$ as $\tau$, then:

$$f_x(x) = x,$$

  for the arc $M_1N_1,\ M_1 \stackrel{\text{def}}{=} (x_{M_1}, y_{M_1}):$

$$y = f_y(x) = y_{M_1} + \sqrt{r_{\delta_1}^2 - (x - y_{J_1})^2},$$

$$\max_{\Theta_w \in M_1N_1} (|y(x)'|) = |\mathrm{tg}(\beta_1)| = O(r_1),$$

  for the arc $N_2M_2,\ M_2 \stackrel{\text{def}}{=} (x_{M_2}, y_{M_2}):$

$$y = f_y(x) = y_{M_2} - \sqrt{r_{\delta_2}^2 - (x - y_{J_2})^2},$$

$$\max_{\Theta_w \in N_2M_2} (|y(x)'|) = |\mathrm{tg}(\beta_2)| = O(r_1).$$

  At $r_{\delta_1} \to 0,\ r_{\delta_2} \to 0$ the point $\Theta_w$ will be approaching a straight line that goes from point $q$ under angle $\dfrac{\pi}{4}$ to real axis.

- And for $s = x + iy$ equation (24) at the point $\Theta_w = (\tau_1, y(\tau_1))$ can be written similarly (30):

$$y(\tau_1)' \frac{(y(\tau_1) - t_q)^2 - (\tau_1 - \sigma_q)^2}{((\tau_1 - \sigma_q)^2 + (y(\tau_1) - t_q)^2)^2} - \frac{2(\tau_1 - \sigma_q)(t_q - y(\tau_1))}{((\tau_1 - \sigma_q)^2 + (y(\tau_1) - t_q)^2)^2} =$$

$$= \alpha(x + iy(x))'_{x=\tau_1} + y(\tau_1)'\alpha(x + iy(x))'_{y=y(\tau_1)}.$$

- Near by the straight line that goes from point $q$ under angle $\frac{\pi}{4}$ to real axis:

$$(y(\tau_1) - t_q)^2 \sim (\tau_1 - \sigma_q)^2,$$

and in view of limitation $y(\tau_1)' = O(r_1)$, the first member of the sum of the left part of last equality can be reduced up to limited value in case when the corresponding radiuses are approaching the zero,
i.e. $\forall\, r_1 > 0,\ \epsilon > 0,\ \exists\, r_{\delta_1} = r_{\delta_1}(r_1) > 0,\ r_{\delta_2} = r_{\delta_2}(r_1) > 0$:

$$\left| y(\tau_1)' \frac{(y(\tau_1) - t_q)^2 - (\tau_1 - \sigma_q)^2}{((\tau_1 - \sigma_q)^2 + (y(\tau_1) - t_q)^2)^2} \right| < \epsilon.$$

And $\exists\ 0 < R_9 \leqslant R_8 :\ \ \forall\ 0 < r_1 < R_9$ the equation (24) in a point $\Theta_w = (\tau_1, y(\tau_1))$ will look as follows:

$$\frac{2(\sigma_q - \tau_1)(t_q - y(\tau_1))}{((\tau_1 - \sigma_q)^2 + (y(\tau_1) - t_q)^2)^2} = O(1). \qquad (39)$$

Or:

$$\frac{\alpha(w_{r_1})^2 r_1^4 + O(r_1^5)}{2(\alpha(w_{r_1})^2 r_1^4 + O(r_1^5))^2} = O(1),$$

which is impossible with the radius $r_1 \to 0$.

Thus, $\exists\, 0 < R_{10} \leqslant R_9 :\ \forall\, 0 < r_1 < R_{10}$ the equation (39) is incorrect and, accordingly, and the assumption that the point $\Theta_w$ lays on the arcs $M_1N_1$ or $N_2M_2$ is not true.

This means that $\forall\ 0 < r_1 < R_{10}$ the point $\Theta_w$ lays on the segments: $G_1M_1$ or $N_2G_2$, or the assumption that $\alpha(q) \neq 0$ is not true.

- Let's assume that the point $\Theta_w$ lays on the segments: $G_1M_1$ or $N_2G_2$.

  Let's take $Re(s) = x$ as $\tau$, then:

$$f_x(x) = x,$$

  for the segment $G_1M_1$

$$y = f_y(x) = y_{w_{r_1}},$$

  for the segment $N_2G_2$

$$y = f_y(x) = y_{w_{r_2}}.$$

  And for $s = x + iy$ the equation (24) in the point $\Theta_w = (\tau_1, y(\tau_1))$ in view of that $y(\tau_1)' = 0$, from (30):

$$\frac{2(\sigma_q - \tau_1)(t_q - y(\tau_1))}{((\tau_1 - \sigma_q)^2 + (y(\tau_1) - t_q)^2)^2} = \alpha(x + iy(x))'_{x=\tau_1}. \tag{40}$$

  Note that for the segment $G_1M_1$:

$$0 < t_q - y(\tau_1) < \sigma_q - \tau_1 < r_1,$$

  for the segment $M_2G_2$:

$$0 < t_q - y(\tau_1) < \sigma_q - \tau_1 < r_2.$$

  It means, that at $r_1 \to 0$:

$$((\tau_1 - \sigma_q)^2 + (y(\tau_1) - t_q)^2)^2 = \Theta((\sigma_q - \tau_1)^4).$$

- And (40) can be written:

$$\frac{2(\sigma_q - \tau_1)(t_q - y(\tau_1))}{\Theta((\sigma_q - \tau_1)^4)} = O(1). \tag{41}$$

I.e. at $r_1 \to 0$ for the segment $G_1M_1$:

$$t_q - y_{w_{r_1}} = t_q - y(\tau_1) = \Theta((\sigma_q - \tau_1)^3) = O(r_1^3)$$

and

$$\alpha(w_{r_1})r_1^2 = t_q - y_{w_{r_1}} = O(r_1^3),$$

for the segment $M_2G_2$:

$$t_q - y_{w_{r_2}} = t_q - y(\tau_1) = \Theta((\sigma_q - \tau_1)^3) = O(r_2^3) = O(r_1^3)$$

and

$$\alpha(w_{r_1})(r_1^2 - B_\beta^{-1}r_1^3) = t_q - y_{w_{r_2}} = O(r_1^3).$$

I.e. при $r_1 \to 0$:

$$\alpha(w_{r_1}) = O(r_1). \tag{42}$$

- And then:

$$\alpha(q) = 0.$$

That contradicts our assumption (18), i.e.:

$$\alpha(q) = -\frac{1}{k(q)} Im \left( -\frac{1}{2} \frac{\Gamma'\left(\frac{q}{2}\right)}{\Gamma\left(\frac{q}{2}\right)} + I_{\mathcal{P}\setminus\{q\}}(q) \right) = 0. \tag{43}$$

The equation (43) is closely connected with random matrices.

Similarly, examine the function $\zeta(1-s)$.

Let's designate:

$$\zeta(s)_- \stackrel{\text{def}}{=} \zeta(1-s),$$

$$\alpha(s)_{-} \stackrel{\text{def}}{=} -\frac{1}{k(q)} Im \left( -\frac{1}{2} \frac{\Gamma'\left(\dfrac{1-s}{2}\right)}{\Gamma\left(\dfrac{1-s}{2}\right)} - I_{\mathcal{P}\setminus\{q\}}(s) \right).$$

Let's examine the same point $q \in \mathcal{P}_1 \cup \mathcal{P}_2$.

- Let's note that the area where the line is considered $G_1 G_2$ localized around a single point, so there is no need to analyze the ambiguity of the imaginary part of the logarithm, and it is not required to build the Riemann surface, especially since the logarithm of the function $\zeta(s)$ itself is not considered, but only its derivative's (equal for all surfaces of multi-valued logarithm) imaginary part is analyzed.

- Let's have the similar reasonig on $\zeta(s)_{-}$ in the area $Q(R)$.

The result of these reasoning is the following equation, similar to (43):

$$\alpha(q)_{-} = -\frac{1}{k(q)} Im \left( -\frac{1}{2} \frac{\Gamma'\left(\dfrac{1-q}{2}\right)}{\Gamma\left(\dfrac{1-q}{2}\right)} - I_{\mathcal{P}\setminus\{q\}}(q) \right) = 0. \tag{44}$$

- Thus, from the (43) and (44) we have $\forall\, q \in \mathcal{P}_1 \cup \mathcal{P}_2$:

$$-k(q)(\alpha(q) + \alpha(q)_{-}) = Im \left( -\frac{1}{2} \frac{\Gamma'\left(\frac{q}{2}\right)}{\Gamma\left(\frac{q}{2}\right)} - \frac{1}{2} \frac{\Gamma'\left(\dfrac{1-q}{2}\right)}{\Gamma\left(\dfrac{1-q}{2}\right)} \right) = 0. \tag{45}$$

- From (6) the equality (45) can be written as follows:

$$\sum_{n=0}^{\infty} \left( \frac{t_q}{(2n+\sigma_q)^2 + t_q^2} - \frac{t_q}{(2n+1-\sigma_q)^2 + t_q^2} \right) = 0.$$

I.e.

$$\sum_{n=0}^{\infty} \frac{t_q((2n+1-\sigma_q)^2-(2n+\sigma_q)^2)}{((2n+\sigma_q)^2+t_q^2)((2n+1-\sigma_q)^2+t_q^2)} =$$

$$= \sum_{n=0}^{\infty} \frac{t_q(1-2\sigma_q)(4n+1)}{((2n+\sigma_q)^2+t_q^2)((2n+1-\sigma_q)^2+t_q^2)} =$$

$$= (1-2\sigma_q) \sum_{n=0}^{\infty} \frac{t_q(4n+1)}{((2n+\sigma_q)^2+t_q^2)((2n+1-\sigma_q)^2+t_q^2)} = 0.$$

Sum

$$\sum_{n=0}^{\infty} \frac{t_q(4n+1)}{((2n+\sigma_q)^2+t_q^2)((2n+1-\sigma_q)^2+t_q^2)}$$

exists and is not equal to 0 when $t_q \neq 0$ so the equality (45) is performed exclusively at

$$\sigma_q = \frac{1}{2}.$$

So, assuming that an arbitrary nontrivial root $q$ of zeta functions belongs to the union $\mathcal{P}_1 \cup \mathcal{P}_2$ we found that it belongs only to $\mathcal{P}_2$, i.e. $\mathcal{P}_1 = \varnothing$.

And according to the fact that $\|\mathcal{P}_3\| = \|\mathcal{P}_1\| = 0$ we have:

$$\mathcal{P}_3 = \mathcal{P}_1 = \varnothing, \;\; \mathcal{P} = \mathcal{P}_2,$$

This proves the basic statement and the assumption which had been made by Bernhard Riemann about the location of the real parts of the nontrivial zeros of zeta function.

# REFERENCES:


[1] Abramowitz, M. and Stegun, I. A. (Eds.). "Polygamma Functions."§6.4 in Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing. New York: Dover, p. 260, 1972.

[2] Allouche, J.-P. "Series and Infinite Products related to Binary Expansions of Integers."1992.

[3] Davenport, H. Multiplicative Number Theory, 2nd ed. New York: Springer-Verlag, 1980.

[4] Edwards, H. M. Riemann's Zeta Function. New York: Dover, 1974.

[5] Karatsuba, A. A. and Voronin, S. M. The Riemann Zeta-Function. Hawthorn, NY: de Gruyter, 1992.

[6] Keiper, J. B. "Power Series Expansions of Riemann's xi Function."Math. Comput. 58, 765-773, 1992.

[7] Knuth, D. E. The Art of Computer Programming, Vol. 1: Fundamental Algorithms, 3rd ed. Reading, MA: Addison-Wesley, 1997.

[8] Lehmer, D. H. "The Sum of Like Powers of the Zeros of the Riemann Zeta Function."Math. Comput. 50, 265-273, 1988.

[9] Smith, D. E. A Source Book in Mathematics. New York: Dover, 1994.

[10] Titchmarsh, E. C. The Theory of the Riemann Zeta Function, 2nd ed. New York: Clarendon Press, 1987.

[11] Valle-Poussin C.J. De la Recherches analytiques sur la theorie des wombres partie I,II,III, Am. Soc. Sci. Bruxelles, Ser. A. - 1896-1897, t. 20-21.

[12] Voros, A. "Spectral Functions, Special Functions and the Selberg Zeta Function."Commun. Math. Phys. 110, 439-465, 1987.