Mathematical Formulas: Part 2

Edgar Valdebenito

abstract

Some formulas related with the constant pi:

$$\pi = 3.1415926535\ ...$$

Keywords:número pi,series,productos infinitos.

1. Introducción . Notación

$$\mathbb{N} = \{1,\ 2,\ 3,\ ...\} \tag{1}$$

$$\mathbb{N}_0 = \mathbb{N} \bigcup \{0\} \tag{2}$$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n\,(n-1)\,(n-1)\,...\,(n-k+1)}{k!} \tag{3}$$

$$(a)_n = a\,(a+1)\,(a+2)\,...\,(a+n-1)\ ,\ (a)_0 = 1\ ,\ n \in \mathbb{N} \tag{4}$$

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2.71828\ ... \tag{5}$$

$$\zeta^*(x) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^x}\ ,\ x > 0 \tag{6}$$

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}-\ln n\right) = 0.5772\ ... \tag{7}$$

2. función arcotangente

$$\tan^{-1} x = \sum_{n=0}^{\infty}\frac{(1/m)_n}{n!}\,a(m,\ x,\ n)\ ,\ m \in \mathbb{N}\ ,\ 0 < x < \sqrt{2^{1/m}-1} \tag{8}$$

$$a(m,\ x,\ n) = \int_0^x \left(1-\left(1+u^2\right)^m\right)^n du \tag{9}$$

$$a(m,\ x,\ n) = \sum_{k=0}^{n}\sum_{s=0}^{m\,k}(-1)^k\binom{n}{k}\binom{m\,k}{s}\frac{x^{2\,s+1}}{2\,s+1} \tag{10}$$

casos particulares:

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \ , \ m = 1 \ , \ 0 < x < 1 \tag{11}$$

$$\pi = 4 \sum_{n=0}^{\infty} \frac{(1/m)_n}{n!} \left(a\left(m, \frac{1}{2}, n\right) + a\left(m, \frac{1}{3}, n\right) \right) \ , \ m = 1, 2, 3 \tag{12}$$

$$\pi = 4 \sum_{n=0}^{\infty} \frac{(1/3)_n}{n!} \sum_{k=0}^{n} \sum_{s=0}^{3k} (-1)^k \binom{n}{k} \binom{3k}{s} \frac{(2^{-2s-1} + 3^{-2s-1})}{2s+1} \ , \ m = 3 \tag{13}$$

$$\pi = 2\sqrt{3} \sum_{n=0}^{\infty} \frac{(1/m)_n}{n!} \sum_{k=0}^{n} \sum_{s=0}^{mk} \binom{n}{k} \binom{mk}{s} \frac{(-1)^k 3^{-s}}{2s+1} \ , \ m = 1, 2 \tag{14}$$

$$\pi = 8 \sum_{n=0}^{\infty} \frac{(1/m)_n}{n!} \sum_{k=0}^{n} \sum_{s=0}^{mk} \binom{n}{k} \binom{mk}{s} \frac{(-1)^k \left(\sqrt{2} - 1\right)^{2s+1}}{2s+1} \ , \ m = 1, 2, 3, 4 \tag{15}$$

$$\pi = 4 \sum_{n=0}^{\infty} \frac{(1/m)_n}{n!} \sum_{k=0}^{n} \sum_{s=0}^{mk} (-1)^k \binom{n}{k} \binom{mk}{s} \frac{(2 \cdot 3^{-2s-1} + 7^{-2s-1})}{2s+1} \tag{16}$$

$$m = 1, 2, 3, 4, 5, 6$$

$$\pi = 4 \sum_{n=0}^{\infty} \frac{(1/m)_n}{n!} \sum_{k=0}^{n} \sum_{s=0}^{mk} (-1)^k \binom{n}{k} \binom{mk}{s} \frac{(3 \cdot 2^{-4s-2} + (99/5)^{-2s-1})}{2s+1} \tag{17}$$

$$m = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$$

$$\pi = 4 \sum_{n=0}^{\infty} \frac{(1/m)_n}{n!} \sum_{k=0}^{n} \sum_{s=0}^{mk} (-1)^k \binom{n}{k} \binom{mk}{s} \frac{(4 \cdot 5^{-2s-1} - 239^{-2s-1})}{2s+1} \tag{18}$$

$$m = 1, 2, 3, \ldots, 17$$

3. función arcotangente , particiones

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \, p_m(n) \, q(m, x, n) \ , \ m \in \mathbb{N}, \ 0 < x < 1 \tag{19}$$

$$q(m, x, n) = \int_0^x u^{2n} \, s(m, u) \, du \tag{20}$$

$$s(m, u) = \begin{cases} 1 & m = 1 \\ \prod_{k=2}^{m} \left(1 - (-1)^m u^{2k}\right) & m \in \mathbb{N} - \{1\} \end{cases} \tag{21}$$

$p_m(n)$ es la función de particiones de un entero n , $p_m(n) \equiv 1$, $m > n$

$$p_1(n) = \{1, 1, 1, \ldots\} \tag{22}$$

$$p_2(n) = \{1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, 7, 7, 8, 8, \ldots\} \tag{23}$$

$$p_3(n) = \{1, 1, 1, 3, 4, 5, 7, 8, 10, 12, 14, 16, 19, 21, 24, 27, \ldots\} \tag{24}$$

$$p_4(n) = \{1, 1, 1, 1, 5, 6, 9, 11, 15, 18, 23, 27, 34, 39, 47, 54, \ldots\} \tag{25}$$

cosos particulares de (12):

$$\pi = 4\sum_{n=0}^{\infty}(-1)^n\, p_2(n)\left(\frac{1}{2n+1}-\frac{1}{2n+5}\right) \tag{26}$$

$$\pi = 2\sqrt{3}\sum_{n=0}^{\infty}(-1/3)^n\, p_2(n)\left(\frac{1}{2n+1}-\frac{1/9}{2n+5}\right) \tag{27}$$

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n\, p_2(n)}{(2n+1)(2n+5)}\left(\left(\frac{1}{2}\right)^{2n+5}(30n+79)+\left(\frac{1}{3}\right)^{2n+5}(160n+404)\right) \tag{28}$$

$$\pi = 4\sum_{n=0}^{\infty}(-1)^n\, p_3(n)\left(\frac{1}{2n+1}-\frac{1}{2n+5}+\frac{1}{2n+7}-\frac{1}{2n+11}\right) \tag{29}$$

4. serie para pi

$$\pi = 4\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^n\binom{n}{k}\left(\frac{2}{3}\right)^k\int_0^{\ln\sqrt{3}}(\operatorname{ch}x)^k\,dx \tag{30}$$

$$I_k = \int_0^{\ln\sqrt{3}}(\operatorname{ch}x)^k\,dx\ ,\ k\in\mathbb{N}_0 \tag{31}$$

$$I_{2k} = \frac{2^{2k-2}}{3^k\,k}+\frac{2k-1}{2k}I_{2k-2}\ ,\ k\in\mathbb{N},\ I_0=\frac{\ln 3}{2} \tag{32}$$

$$I_{2k+1} = \frac{2^{2k}\sqrt{3}}{3^{k+1}(2k+1)}+\frac{2k}{2k+1}I_{2k-1}\ ,\ k\in\mathbb{N},\ I_1=\frac{\sqrt{3}}{2} \tag{33}$$

$$J_{2k+1} = \frac{2^{2k}}{3^{k+1}(2k+1)}+\frac{2k}{2k+1}J_{2k-1}\ ,\ k\in\mathbb{N},\ J_1=\frac{1}{3} \tag{34}$$

$$I_{2k+1} = \sqrt{3}\,J_{2k+1}\ ,\ k\in\mathbb{N}_0 \tag{35}$$

$$G_k = \left(\frac{2}{3}\right)^k I_k\ ,\ k\in\mathbb{N}_0 \tag{36}$$

$$\pi = 4\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k\binom{n}{k}G_k \tag{37}$$

$$\pi = 2\ln 3+4\sum_{n=1}^{\infty}\left(\sum_{k=0}^{[n/2]}\binom{n}{2k}G_{2k}-\sum_{k=0}^{[(n-1)/2]}\binom{n}{2k+1}G_{2k+1}\right) \tag{38}$$

$$G_{2k} = \frac{2^{4k-2}}{3^{3k}\,k}+\frac{2k-1}{2k}\left(\frac{4}{9}\right)G_{2k-2}\ ,\ k\in\mathbb{N},\ G_0=\frac{\ln 3}{2} \tag{39}$$

$$G_{2k+1} = \frac{2^{4k+1}\sqrt{3}}{3^{3k+2}(2k+1)}+\frac{2k}{2k+1}\left(\frac{4}{9}\right)G_{2k-1}\ ,\ k\in\mathbb{N},\ G_1=\frac{2\sqrt{3}}{9} \tag{40}$$

$$G_{2k+1} = \sqrt{3}\,H_{2k+1}\ ,\ k\in\mathbb{N}_0 \tag{41}$$

$$H_{2k+1} = \frac{2^{4k+1}}{3^{3k+2}(2k+1)} + \frac{2k}{2k+1}\left(\frac{4}{9}\right)H_{2k-1}\ ,\ k \in \mathbb{N},\ H_1 = \frac{2}{9} \tag{42}$$

$$G_{2k} = \left(\frac{2}{3}\right)^{2k}\left(\frac{\ln 3}{2^{2k+1}}\binom{2k}{k} + \frac{1}{2^{2k}}\sum_{m=0}^{k-1}\binom{2k}{m}\left(\frac{3^{k-m} - 3^{-(k-m)}}{2k-2m}\right)\right),\ k \in \mathbb{N}_0 \tag{43}$$

$$G_{2k+1} = \left(\frac{2}{3}\right)^{2k+1}\frac{1}{\sqrt{3}}\sum_{m=0}^{k}\binom{k}{m}\frac{3^{-m}}{2m+1}\ ,\ k \in \mathbb{N}_0 \tag{44}$$

$$I_k = \int_1^{\sqrt{3}}\left(\frac{x+x^{-1}}{2}\right)^k \frac{1}{x}\,dx = \int_{1/\sqrt{3}}^{1}\left(\frac{x+x^{-1}}{2}\right)^k \frac{1}{x}\,dx\ ,\ k \in \mathbb{N}_0 \tag{45}$$

5. series para pi

$$\pi = 3\sum_{n=0}^{\infty}2^{-n-1}\sum_{k=0}^{n}\binom{n}{k}\binom{2n}{k}\frac{16^{-k}}{2k+1} \tag{46}$$

$$\frac{2\sqrt{3}}{27}\pi + \frac{4}{3} = \sum_{n=0}^{\infty}2^{-n-1}\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}^{-1} \tag{47}$$

$$\frac{2\sqrt{3}}{9}\pi = \sum_{n=0}^{\infty}2^{-n-1}\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}^{-1}\frac{1}{2k+1} \tag{48}$$

$$\pi = 3 + 6\sum_{n=0}^{\infty}2^{-n-1}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{(6k+5)(6k+7)} \tag{49}$$

$$\pi^2 = 10 - \sum_{n=0}^{\infty}2^{-n-1}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{((k+1)(k+2))^3} \tag{50}$$

$$\frac{2\sqrt{2}}{\pi} = \sum_{n=0}^{\infty}\frac{(-1)^n(54n+15)}{2^{3n+3}}\sum_{k=0}^{n}\frac{(1/2)_k^3}{(k!)^3} \tag{51}$$

$$\frac{2\sqrt{2}}{\pi} = \sum_{n=0}^{\infty}\frac{(54n+15)}{2^{3n+3}}\sum_{k=0}^{n}\frac{(-1)^k(1/2)_k^3}{(k!)^3} \tag{52}$$

6. series , sucesiones

Sean $a_n,\ b_n,\ c_n,\ n \in \mathbb{N}$ las sucesiones definidas por :

$$a_{n+2} = -2a_{n+1} + 4b_{n+1} - a_n \tag{53}$$

$$b_{n+2} = 2a_{n+1} - 2b_{n+1} - b_n \tag{54}$$

$$a_1 = -1\,,\ \ a_2 = 5\,,\ \ b_1 = 1\,,\ \ b_2 = -4 \tag{55}$$

$$c_{n+2} = 2\left(\sqrt{2} - 1\right) c_{n+1} - c_n \tag{56}$$

$$c_1 = \sqrt{2} - 1\,,\ \ c_2 = 5 - 4\sqrt{2} \tag{57}$$

$$c_n = a_n + b_n \sqrt{2}\,,\ \ n \in \mathbb{N} \tag{58}$$

$$a_n = \{-1,\ 5,\ -25,\ 113,\ -521,\ 2405,\ \ldots\} \tag{59}$$

$$b_n = \{1,\ -4,\ 17,\ -80,\ 369,\ -1700,\ \ldots\} \tag{60}$$

$$c_n = \left\{-1 + \sqrt{2}\,,\ 5 - 4\sqrt{2}\,,\ -25 + 17\sqrt{2}\,,\ \ldots\right\} \tag{61}$$

se tiene:

$$\frac{\pi}{4} \tan^{-1} \sqrt{\frac{\sqrt{2} - 1}{2}} = \sum_{n=0}^{\infty} \frac{c_{2n+1}}{(2n+1)^2} = \frac{-1 + \sqrt{2}}{1^2} + \frac{-25 + 17\sqrt{2}}{3^2} + \ldots \tag{62}$$

$$\frac{\pi^2}{3} - \left(\tan^{-1} \sqrt{2\left(\sqrt{2} + 1\right)}\right)^2 = \frac{\pi^2}{12} + \pi \tan^{-1} \sqrt{\frac{\sqrt{2} - 1}{2}} - \left(\tan^{-1} \sqrt{\frac{\sqrt{2} - 1}{2}}\right)^2 = 4 \sum_{n=1}^{\infty} \frac{(-1)^{n-1} c_n}{n^2} \tag{63}$$

$$\frac{\pi z}{3\sqrt{3}} = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} c_n}{(3n)^2 - 1} \tag{64}$$

$$z = \frac{1}{2} \sqrt[3]{2\sqrt{2} - 2 + 3\sqrt[3]{2\sqrt{2} - 2 + \ldots}} = \cos\left(\frac{1}{3} \tan^{-1} \sqrt{2\left(\sqrt{2} + 1\right)}\right) \tag{65}$$

$$\frac{\pi}{16} \sqrt{2\left(2 + \sqrt{2\sqrt{2}}\right)} = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} c_n}{(4n)^2 - 1} = \frac{1}{2} + \frac{-1 + \sqrt{2}}{3 \cdot 5} - \frac{5 - 4\sqrt{2}}{7 \cdot 9} + \ldots \tag{66}$$

7. número λ

$$0 < \lambda < 1\,,\ \ \operatorname{sen}(\lambda \pi) = \lambda \Longrightarrow \lambda = 0.736484448 \ldots \tag{67}$$

$$\cos(\lambda \pi) = -\sqrt{1 - \lambda^2} \tag{68}$$

$$\tan(\lambda \pi) = -\frac{\lambda}{\sqrt{1 - \lambda^2}} \tag{69}$$

$$\pi = \frac{\operatorname{sen}^{-1} \lambda}{1 - \lambda} \tag{70}$$

$$\lambda = 1 - \frac{1}{\pi} \operatorname{sen}^{-1}\left(1 - \frac{1}{\pi} \operatorname{sen}^{-1}(1 - \ldots)\right) \tag{71}$$

$$\pi = \Gamma(1 + \lambda)\, \Gamma(1 - \lambda) \tag{72}$$

$$\pi = \lim_{n\to\infty} \frac{n^2 (n!)^2}{(2^2-\lambda^2)(3^2-\lambda^2)\ldots((n+1)^2-\lambda^2)} \tag{73}$$

$$\frac{1}{\pi} = \prod_{n=0}^{\infty}\left(1-\left(\frac{\lambda}{n+1}\right)^2\right) \tag{74}$$

$$\ln\pi = \sum_{n=1}^{\infty}\frac{\lambda^{2n}\zeta(2n)}{n} \tag{75}$$

$$\ln\pi = \ln\left(\frac{2}{1-\sqrt{1-\lambda^2}}\right) - 2\sum_{n=1}^{\infty}\frac{(2^{2n-1}-1)\lambda^{2n}\zeta(2n)}{n\,2^{2n}} \tag{76}$$

$$\prod_{n=2}^{\infty}\left(1-\left(\frac{2\lambda}{2n-1}\right)^2\right) = \frac{\sqrt{1-\lambda^2}}{4\lambda^2-1} \tag{77}$$

$$\pi e^{2\gamma} = \frac{1}{1-\lambda^2}\prod_{n=1}^{\infty} e^{2/n}\left(1+\frac{2}{n}+\frac{1-\lambda^2}{n^2}\right)^{-1} \tag{78}$$

$$\pi = F(\lambda, \lambda; 1+\lambda; 1) \tag{79}$$

$$\pi = \frac{1}{1-\lambda^2}\prod_{n=1}^{\infty}\frac{(n+1)^2}{(n+1)^2-\lambda^2} \tag{80}$$

$$\pi = 1 + 2\lambda^2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2-\lambda^2} \tag{81}$$

$$\pi = \frac{\lambda}{1-\lambda} + 2\lambda(1-\lambda)\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2-(1-\lambda)^2} \tag{82}$$

$$\pi = 4\sqrt{1-\lambda^2}\sum_{n=1}^{\infty}\frac{(-1)^n(2n-1)}{(2n-1)^2-4\lambda^2} \tag{83}$$

$$\pi = 4\sqrt{1-\lambda^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2n-1)}{(2n-1)^2-4(1-\lambda)^2} \tag{84}$$

$$\pi = -\frac{1}{\sqrt{1-\lambda^2}} + \frac{2\lambda^2}{\sqrt{1-\lambda^2}}\sum_{n=1}^{\infty}\frac{1}{n^2-\lambda^2} \tag{85}$$

$$\pi = 8\sqrt{1-\lambda^2}\left(\frac{1}{4\lambda^2-1} - \sum_{n=2}^{\infty}\frac{1}{(2n-1)^2-4\lambda^2}\right) \tag{86}$$

$$\pi^2 = 1 + 2\lambda^2\sum_{n=1}^{\infty}\frac{n^2+\lambda^2}{(n^2-\lambda^2)^2} \tag{87}$$

$$\pi^2 = 4\left(1-\lambda^2\right)\sum_{n=1}^{\infty}\left(\frac{1}{(2n-1-2\lambda)^2} + \frac{1}{(2n-1+2\lambda)^2}\right) \tag{88}$$

$$\frac{1}{\pi} = \sqrt{\frac{1}{2}-\frac{\sqrt{1-\lambda^2}}{2}}\ \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}-\frac{\sqrt{1-\lambda^2}}{2}}}\ \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}-\frac{\sqrt{1-\lambda^2}}{2}}}}\ \ldots \tag{89}$$

8. pi , función zeta alternada

$$\pi = \sigma\, e^{-\zeta^*(1/2)\sqrt{\sigma\, e^{-\zeta^*(1/2)\sqrt{\sigma\, e^{-\cdots}}}}} \tag{90}$$

donde

$$\sigma = 2\exp\left(\int_0^\infty \frac{1 - e^{-x} + \sqrt{x}}{x\,(1 + e^x)}\, dx\right) = \pi \exp\left(\sqrt{\pi}\ \zeta^*(1/2)\right) = 9.17869 \ldots \tag{91}$$

$$\zeta^*(1/2) = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{\sqrt{n}} = 0.60489 \ldots \tag{92}$$

$$\ln\left(\frac{\pi}{2}\right) + \sqrt{\pi}\ \zeta^*(1/2) = \int_0^\infty \frac{1 - e^{-x} + \sqrt{x}}{x\,(1 + e^x)}\, dx \tag{93}$$

$$\ln\left(\frac{\pi}{2}\right) + \sqrt{\pi}\ \zeta^*(1/2) = \int_0^1 \frac{1 - x + \sqrt{-\ln x}}{x\,(1 + x)\,(-\ln x)}\, dx \tag{94}$$

$$\ln\left(\frac{\pi}{2}\right) + \sqrt{\pi}\ \zeta^*(1/2) = \int_1^\infty \frac{x\left(1 + \sqrt{\ln x}\right) - 1}{x\,(1 + x)\ln x}\, dx \tag{95}$$

$$\ln\left(\frac{\pi}{2}\right) + \sqrt{\pi}\ \zeta^*(1/2) = 2\int_0^\infty \frac{1 + e^{-x^2} + x}{x\left(1 + e^{x^2}\right)}\, dx \tag{96}$$

$$\ln\left(\frac{\pi}{2}\right) + \sqrt{\pi}\ \zeta^*(1/2) = \int_{-\infty}^\infty \frac{1 - e^{-e^x} + e^{x/2}}{1 + e^{e^x}}\, dx \tag{97}$$

9. productos infinitos

$$\frac{4\,a_n}{\pi} = \sqrt{2}\ (n+1)\prod_{k=1}^\infty\left(1 - \left(\frac{n+1}{4\,k}\right)^2\right) - n\prod_{k=1}^\infty\left(1 - \left(\frac{n}{4\,k}\right)^2\right),\ \ n \in \mathbb{N} \tag{98}$$

$$a_{n+2} = \sqrt{2}\ a_{n+1} - \frac{1}{2}\, a_n\ ,\ \ a_1 = \frac{1}{\sqrt{2}},\ a_2 = 0 \tag{99}$$

$$a_n = 2^{-1-(n/2)}\left((1 - i)^n + (1 + i)^n\right) \tag{100}$$

$$a_{4n-3} = \frac{(-1)^{n+1}}{\sqrt{2}}\ ,\ \ a_{4n-2} = 0\ ,\ \ a_{4n-1} = \frac{(-1)^n}{\sqrt{2}}\ ,\ \ a_{4n} = (-1)^n \tag{101}$$

$$\frac{3\sqrt{3}\ a_n}{\pi} = 2\,(n+1)\prod_{k=1}^\infty\left(1 - \left(\frac{n+1}{3\,k}\right)^2\right) - n\prod_{k=1}^\infty\left(1 - \left(\frac{n}{3\,k}\right)^2\right),\ \ n \in \mathbb{N} \tag{102}$$

$$a_{n+2} = a_{n+1} - a_n \ , \ a_1 = \frac{1}{2} \, , \ a_2 = -\frac{1}{2} \tag{103}$$

$$a_n = \frac{1}{2}\left(\left(\frac{1 - i\sqrt{3}}{2}\right)^n + \left(\frac{1 + i\sqrt{3}}{2}\right)^n\right) \tag{104}$$

$$a_{3n-2} = \frac{(-1)^{n+1}}{2} \, , \ a_{3n-1} = \frac{(-1)^n}{2} \, , \ a_{3n} = (-1)^n \tag{105}$$

$$\frac{6\,a_n}{\pi} = 2\,(n+1)\prod_{k=1}^{\infty}\left(1 - \left(\frac{n+1}{6\,k}\right)^2\right) - \sqrt{3}\; n \prod_{k=1}^{\infty}\left(1 - \left(\frac{n}{6\,k}\right)^2\right) , \ n \in \mathbb{N} \tag{106}$$

$$a_{n+2} = \sqrt{3}\; a_{n+1} - a_n \ , \ a_1 = \frac{\sqrt{3}}{2} \, , \ a_2 = \frac{1}{2} \tag{107}$$

$$a_n = \frac{1}{2}\left(\left(\frac{\sqrt{3} + i}{2}\right)^n + \left(\frac{\sqrt{3} - i}{2}\right)^n\right) \tag{108}$$

$$a_{6n-5} = \frac{(-1)^{n+1}\sqrt{3}}{2} \, , \ a_{6n-4} = \frac{(-1)^{n+1}}{2} \, , \ a_{6n-3} = 0 \, , \ a_{6n-2} = \frac{(-1)^n}{4} \, , \ a_{6n-1} = \frac{(-1)^n\sqrt{3}}{2} \, , \ a_{6n} = (-1)^n \tag{109}$$

$$\frac{12\left(\sqrt{6} - \sqrt{2}\right)a_n}{\pi} = 4\,(n+1)\prod_{k=1}^{\infty}\left(1 - \left(\frac{n+1}{12\,k}\right)^2\right) - \left(\sqrt{6} + \sqrt{2}\right) n \prod_{k=1}^{\infty}\left(1 - \left(\frac{n}{12\,k}\right)^2\right) , \ n \in \mathbb{N} \tag{110}$$

$$a_{n+2} = \frac{\sqrt{6} + \sqrt{2}}{2}\, a_{n+1} - a_n \ , \ a_1 = \frac{\sqrt{6} + \sqrt{2}}{4} \, , \ a_2 = \frac{\sqrt{3}}{2} \tag{111}$$

$$a_n = \frac{1}{2}\left(\left(\frac{\sqrt{6} + \sqrt{2}}{4} + i\,\frac{\sqrt{6} - \sqrt{2}}{4}\right)^n + \left(\frac{\sqrt{6} + \sqrt{2}}{4} - i\,\frac{\sqrt{6} - \sqrt{2}}{4}\right)^n\right) \tag{112}$$

$$a_{12n-11} = \frac{(-1)^{n+1}\left(\sqrt{6} + \sqrt{2}\right)}{4} \, , \ a_{12n-10} = \frac{(-1)^{n+1}\sqrt{3}}{2} \tag{113}$$

$$a_{12n-9} = \frac{(-1)^{n+1}}{\sqrt{2}} \, , \ a_{12n-8} = \frac{(-1)^{n+1}}{2} \, , \ a_{12n-7} = \frac{(-1)^{n+1}\left(\sqrt{6} - \sqrt{2}\right)}{4} \tag{114}$$

$$a_{12n-6} = 0 \, , \ a_{12n-5} = \frac{(-1)^n\left(\sqrt{6} - \sqrt{2}\right)}{4} \, , \ a_{12n-4} = \frac{(-1)^n}{2} \tag{115}$$

$$a_{12n-3} = \frac{(-1)^n}{\sqrt{2}} \, , \ a_{12n-2} = \frac{(-1)^n\sqrt{3}}{2} \, , \ a_{12n-1} = \frac{(-1)^n\left(\sqrt{6} + \sqrt{2}\right)}{4} \tag{116}$$

$$a_{12n} = (-1)^n \tag{117}$$

$$\frac{2^{m+1}\,s_m\,a_{n,m}}{\pi} = (n+1)\prod_{k=1}^{\infty}\left(1 - \left(\frac{n+1}{2^{m+1}\,k}\right)^2\right) - c_m\,n\prod_{k=1}^{\infty}\left(1 - \left(\frac{n}{2^{m+1}\,k}\right)^2\right) , \ n, \, m \in \mathbb{N} \tag{118}$$

$$a_{n+2,m} = 2\,c_m\,a_{n+1,m} - a_{n,m} \ , \ a_{1,m} = c_m \, , \ a_{2,m} = c_m^2 - s_m^2 = c_{m-1} \tag{119}$$

$$s_m = \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}\ ,\ m\in\mathbb{N} \qquad (120)$$
$$\longleftarrow\ —\ —\ m\text{–radicales}\ —\ —\ \longrightarrow$$

$$c_m = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}\ ,\ m\in\mathbb{N} \qquad (121)$$
$$\longleftarrow\ —\ —\ m\text{–radicales}\ —\ —\ \longrightarrow$$

10. serie trigonométrica

$$\pi = A\sum_{n=1}^{\infty}\frac{B^{2n-1}}{2n-1}\,\mathrm{sen}((4n-2)\,C) \qquad (122)$$

$$A=24\ ,\ B=\frac{1-\sqrt{2\sqrt{3}-3}}{1+\sqrt{2\sqrt{3}-3}}\ ,\ C=\tan^{-1}\left(\sqrt{\frac{2\sqrt{3}-3}{3}}\right) \qquad (123)$$

$$A=24\ ,\ B=\frac{\sqrt[4]{3}-1}{\sqrt[4]{3}+1}\ ,\ C=\tan^{-1}\left(\frac{1}{\sqrt[4]{3}}\right) \qquad (124)$$

$$A=24\ ,\ B=\frac{\sqrt[4]{3}-1}{\sqrt[4]{3}+1}\ ,\ C=\tan^{-1}\left(\sqrt[4]{3}\right) \qquad (125)$$

$$A=12\ ,\ B=\frac{\sqrt{3}-1}{\sqrt{3}+1}\ ,\ C=\pi/4 \qquad (126)$$

$$A=8\ ,\ B=\sqrt{2}-1\ ,\ C=\pi/4 \qquad (127)$$

$$A=8\ ,\ B=\frac{\sqrt{3}-\sqrt{7}+2}{\sqrt{3}+\sqrt{7}-2}\ ,\ C=\pi/3 \qquad (128)$$

$$A=12\ ,\ B=1/3\ ,\ C=\tan^{-1}\left(\frac{3\sqrt{3}-\sqrt{11}}{4}\right) \qquad (129)$$

11. series

$$I_n=\sum_{k=0}^{\infty}\frac{(n)_k}{k!\,(1+a)^{k+n}}\sum_{m=0}^{k}\sum_{s=0}^{m}\binom{k}{m}\binom{m}{s}\frac{(-1)^m\,a^{k-m}}{2m-s+1}\ ,\ n\in\mathbb{N},\ a>1/2 \qquad (130)$$

$$I_{n+1}=\frac{1}{3n}\left(\frac{1}{3^{n-1}}-1\right)+\frac{2(2n-1)}{3n}I_n\ ,\ I_1=\frac{\pi}{3\sqrt{3}} \qquad (131)$$

$$\pi = \frac{4}{\sqrt{\sqrt{2}-1}} \tan^{-1}\left(\sqrt{\sqrt{2}-1}\right) + 4 \sum_{n=1}^{\infty} \left(\sqrt{2}\right)^n \sum_{k=n}^{\infty} \binom{k}{n} \frac{(-1)^k \left(\sqrt{2}-1\right)^k}{2k+1} \tag{132}$$

$$\pi = 4\left(\sqrt{2}+1\right) \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{\sqrt{2}-1}{4}\right)^n \sum_{k=0}^{n-1} (-1)^k \binom{2n}{2k+1} \left(\sqrt{2}+1\right)^{2k} \tag{133}$$

$$\pi = 12 \sum_{n=1}^{\infty} \sum_{k=0}^{n} \frac{(-1)^{k+1}}{n} \binom{n}{k} 2^k \left(\frac{2}{\sqrt{2}+\sqrt{6}}\right)^{2n-k} \operatorname{sen}\left(\frac{(2n-k)\pi}{4}\right) \tag{134}$$

$$\pi = 12 \sum_{n=1}^{\infty} \sum_{k=0}^{n} \frac{(-1)^{k+1}}{n} \binom{n}{k} 2^k \left(\frac{2}{2+\sqrt{6}+\sqrt{14+4\sqrt{6}}}\right)^{2n-k} \operatorname{sen}\left(\frac{(2n-k)\pi}{3}\right) \tag{135}$$

$$\pi = 12 \sum_{n=1}^{\infty} \sum_{k=0}^{n} \frac{(-1)^{n-1}}{n} \binom{n}{k} 2^k \left(\frac{2}{2+3\sqrt{2}+\sqrt{18+12\sqrt{2}}}\right)^{2n-k} \operatorname{sen}\left(\frac{(2n-k)\pi}{3}\right) \tag{136}$$

$$\pi = 12 \sum_{n=1}^{\infty} \sum_{k=0}^{n} \frac{(-1)^{n-1}}{n} \binom{n}{k} 2^k \left(\frac{2}{2+\sqrt{2}+2\sqrt{3}+\sqrt{6}}\right)^{2n-k} \operatorname{sen}\left(\frac{(2n-k)\pi}{4}\right) \tag{137}$$

$$\pi = 2\tan^{-1}\left(\frac{e^x}{e^x-1}\right) + \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \left(-\frac{1}{2}\right)^k \left(\frac{1-e^{-(n+k+1)x}}{n+k+1}\right), \quad x > 0 \tag{138}$$

$$\pi = 4\tan^{-1}\left(\frac{e^x}{e^x-1}\right) - 2 \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \left(-\frac{1}{2}\right)^k \frac{e^{-(n+k+1)x}}{(n+k+1)}, \quad x \ge 0 \tag{139}$$

$$\pi = 2 \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \left(-\frac{1}{2}\right)^k \frac{1}{(n+k+1)} \tag{140}$$

$$\pi = 4\tan^{-1}\left(\frac{e^x}{e^x+1}\right) + 2 \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} (-1)^n \frac{e^{-(n+k+1)x}}{(n+k+1)\,2^k}, \quad x > -\ln\left(\sqrt{3}-1\right) \tag{141}$$

$$\pi = 3\tan^{-1}\left(\sqrt{3}\,\operatorname{th} x\right) + \frac{3\sqrt{3}}{2} \sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{e^{-(6n-4)x}}{3n-2} + \frac{e^{-(6n-2)x}}{3n-1}\right), \quad x \ge 0 \tag{142}$$

$$\pi = 6 \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^{n+k} (\ln 3/2)^{2k+1} (1/3)^{n-k}}{(2k+1)!\,(2n-2k+1)} \tag{143}$$

$$\pi = 6 \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k (-\ln 3/2)^{n-k} (1/3)^k}{(2k+1)\,(n-k)!} \tag{144}$$

$$\pi = 4 \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^{n+k} \left(\ln\left(\sqrt{2}+1\right)\right)^{2k+1}}{(2k+1)!\,(2n-2k+1)} \tag{145}$$

$$\pi = 8 \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^n}{(2k+1)\left(\sqrt{2}+1\right)^{2k} \left(\sqrt{2}\right)^{n-k+1}} \tag{146}$$

$$\pi = 12 \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^n}{(2k+1)\, 3^k \left(\sqrt{3} + \sqrt{2}\right)^{2n-2k+1}} \tag{147}$$

$$\pi = 3 \ln 3 + 12 \sum_{n=1}^{\infty} (-2)^n I_n \tag{148}$$

$$I_n = \frac{2+\sqrt{3}}{2n\left(6+4\sqrt{3}\right)^n} - \frac{2n-1}{2n} I_{n-1} \;,\; n \in \mathbb{N} \;,\; I_0 = \frac{\ln 3}{2} \tag{149}$$

$$\pi = \frac{6}{q} \sqrt{p^2 - q^2} \sum_{n=0}^{\infty} \binom{2n}{n} 2^{-2n} I(p, q, n) \;,\; p > q > 0 \tag{150}$$

$$I(p, q, n) = \left(\frac{p}{q}\right)^{2n} \left(\ln\left(\frac{2p}{2p-q}\right) + \sum_{m=1}^{2n} \binom{2n}{m} \frac{(-p)^{-m}}{m} \left(\left(\frac{2(p^2-q^2)}{2p-q}\right)^m - \left(\frac{p^2-q^2}{p}\right)^m \right) \right) \tag{151}$$

$$\pi = 4 \sum_{k=1}^{n} \frac{P_k(-1)}{k!\, 2^k} + \frac{1}{n!} \int_0^1 \frac{P_{n+1}(-t)}{1+t^2} \left(\frac{t}{1+t^2}\right)^n dt \;,\; n \in \mathbb{N} \tag{152}$$

$$\pi = 12 \sum_{k=1}^{n} \frac{P_k\left(1/\sqrt{3}\right)}{k!} \left(\frac{3-\sqrt{3}}{4}\right)^k + \frac{1}{n!} \int_{1/\sqrt{3}}^{1} \frac{P_{n+1}(t)}{1+t^2} \left(\frac{1-t}{1+t^2}\right)^n dt \;,\; n \in \mathbb{N} \tag{153}$$

$$P_{k+1}(x) = \left(1+x^2\right) \frac{d P_k(x)}{dx} - 2kx\, P_k(x) \;,\; P_1(x) = 1 \tag{154}$$

$$P_k(-1) = \{1, 2, 4, 0, -96, \ldots\} \tag{155}$$

$$P_k\left(1/\sqrt{3}\right) = \left\{1, -\frac{2}{\sqrt{3}}, 0, \frac{16}{\sqrt{3}}, -\frac{128}{3}, \ldots\right\} \tag{156}$$

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