Asymptotic behaviors of normalized maxima for generalized Maxwell distribution under nonlinear normalization*

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Abstract. In this article, the high-order asymptotic expansions of cumulative distribution function and probability density function of extremes for generalized Maxwell distribution are established under nonlinear normalization. As corollaries, the convergence rates of the distribution and density of maximum are obtained under nonlinear normalization.

Keywords. High-order expansion; Extreme; Generalized Maxwell distribution; Nonlinear normalization.

Mathematics Subject Classification(2000): Primary 60F15; Secondary 60G70.

1 Introduction

Let $(X_n,\ n \geq 1)$ be independent and identically distributed (iid) random sequence with cumulative distribution function (cdf). Let $M_n = \max(X_k,\ 1 \leq k \leq n)$ represent the partial maximum of $(X_n,\ n \geq 1)$. A *cdf* F is said to belong to the max domain of attraction of a non-degenerate distribution function H under power normalization (or nonlinear normalization), written as $F \in D_p(H)$, if there exist normalized constants $\alpha_n > 0$, $\beta_n > 0$ and $n > 1$, such that

$$\lim_{n\to\infty} P\big(|M_n/\alpha_n|^{1/\beta_n} \operatorname{sign}(M_n) \leq x \big) = \lim_{n\to\infty} F^n\big(\alpha_n |x|^{\beta_n} \operatorname{sign}(x)\big) = H(x)$$

for all $x \in C(H)$,the set of continuity points of function H, where $\operatorname{sign}(x) = -1, 0$ or 1 according as $x < 0, x = 0$ or $x > 0$. Pancheva (1985) showed that H can be one of only power types of the following six power max stable laws:

$$H_{1,\alpha}(x) = \begin{cases} 0, & \text{if } x \leq 1, \\ \exp\{-(\log x)^{-\alpha}\}, & \text{if } x > 1, \end{cases}$$

$$H_{2,\alpha}(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ \exp\{-(-\log x)^{\alpha}\}, & \text{if } 0 < x < 1, \\ 1, & \text{if } x \geq 1, \end{cases}$$

*Project supported by NSFC (grant No.11171275), the Science and Technology Plan Project of Guizhou Province(No. LH[2015]7001, No. LH[2015]70049, No. LH[2015]70055).
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$$H_{3,\alpha}(x) = \begin{cases} 0, & \text{if } x \le -1, \\ \exp\{-(-\log(-x))^{-\alpha}\}, & \text{if } -1 < x < 0, \\ 1, & \text{if } x \ge 0, \end{cases}$$

$$H_{4,\alpha}(x) = \begin{cases} \exp\{-(\log(-x))^{\alpha}\}, & \text{if } x < -1, \\ 1, & \text{if } x \ge -1, \end{cases}$$

$$H_{5,\alpha}(x) = \Phi_1(x) = \begin{cases} 0, & \text{if } x \le 0, \\ \exp\{-x^{-1}\}, & \text{if } x > 0, \end{cases}$$

$$H_{6,\alpha}(x) = \Psi_1(x) = \begin{cases} \exp\{x\}, & \text{if } x < 0, \\ 1, & \text{if } x \ge 0, \end{cases}$$

where α is a positive parameter. Mohan and Ravi (1993) showed that max stable laws under linear normalization attract less distributions than under power normalization. Necessary and sufficient conditions for F belonging to $D_p(H)$ can be found in Christoph and Falk (1996), Mohan and Subramanya (1991) and Subramanya (1994). Barakat et al. (2010) established the relationship between the convergence rates under linear and power normalization. As one of very important problem in extreme value analysis, the asymptotic property of extreme of special distribution under power normalization has been considered in plenty of literatures in recent years. Chen et al. (2012) obtained the exact uniform convergence rates of limit distribution of maximum from general error distribution under power normalization. Chen and Feng (2014) and Huang et al. (2016) proved a similar result respectively for the short-tailed symmetric and logarithmic normal distribution. Peng et al. (2013) considered the convergence of moments and densities of extremes under power normalization. Yang and Li (2015) investigated the distributional expansions of maximum of logarithmic general error distribution. Jiang et al. (2016) got the high-order expansions of extremes for skew-normal distribution. For limit property of maximum of given distribution under linear normalization, Nair (1981) investigated the asymptotic expansions of the distribution and the moments of extremes for standard normal distribution. Liao et al. (2013, 2014) extended the result to the case of skew-normal distribution. Jia and Li (2014) gained the higher-order expansion for the distribution of normalized maximum from general error distribution. Jia et al. (2015) derived the higher-order expansion for the moment of normalized maximum from general error distribution. Other related works, see Li and Li (2015), Du and Chen (2015, 2016).

In this article, the aim to establish the high-order expansion of cdf and probability density function (pdf) of normalized maxima for generalized Maxwell distribution (GMD for short) under power normalization. Vodă (2009) used a modified Weibull hazard rate as generator to obtain a generalized Maxwell distribution which pdf is given by

$$f_k(x) = \frac{k}{2^{k/2}\sigma^{2+1/k}\Gamma(1+k/2)} x^{2k} \exp\left(-\frac{x^{2k}}{2\sigma^2}\right), \ x > 0, \tag{1.1}$$

where k, σ is positive and $\Gamma(\cdot)$ represents the Gamma function. For $k = 1$, GMD(1) reduces to the classic Maxwell distribution. The tail behavior and the limit distribution of maxima from GMD was studied by Huang and Chen (2015). To our knowledge, there is no works about the high-order expansion of extremes of GMD under nonlinear normalization. This paper is to fill this gap.

In order to gain the high-order expansions of maxima for GMD under nonlinear normalization, we will give the power norming constants. In this sequel, let F_k stand for the cdf of GMD. Huang and Chen (2015) showed that F_k belongs to the domain of attraction of the Gumbel extreme value distribution. It follows Theorem 3.1 of Mohan and Ravi (1993) that we have $F_k \in D_p(\Phi_1)$. Huang and Chen (2015) proved that:

$$\frac{1-F_k(x)}{f_k(x)} \sim \frac{\sigma^2}{k} x^{1-2k}, \tag{1.2}$$

as $x \to \infty$.

According to Huang and Chen (2015), for large x we have

$$1-F_k(x) = c(x)\exp\left(-\int_1^x \frac{g(t)}{f(t)}\,\mathrm{dt}\right) \tag{1.3}$$

where

$$c(x) \to \frac{\exp\big(-1/(2\sigma^2)\big)}{2^{k/2}\sigma^{1/k}\Gamma(1+k/2)},$$

as $x \to \infty$

$$f(x) = k^{-1}\sigma^2 x^{1-2k}, \tag{1.4}$$

and

$$g(x) = 1 - k^{-1}\sigma^2 x^{-2k}.$$

By Theorem 3.1 of Mohan and Ravi (1993) and (1.4), we may choose the suitable power norming constants α_n and β_n in such a way that

$$1 - F_k(\alpha_n) = 1/n, \tag{1.5}$$

$$\beta_n = k^{-1}\sigma^2\alpha_n^{1-2k}, \tag{1.6}$$

such that

$$\lim_{n\to\infty} F_k^n\big(\alpha_n x^{\beta_n}\big) = \exp\big(-x^{-1}\big) = \Phi_1(x), \tag{1.7}$$

for all $x > 0$. By (1.2) and (1.5), for large n we could obtain

$$\alpha_n^{2k} \sim 2\sigma^2 \log n. \tag{1.8}$$

The reminder of this paper is organized as follows: Section 2, we give the main results. Section 3, some auxiliary lemmas and associated proofs are provided. Section 4, the proofs of the results are presented.

2 Main result

In this section, we present the high-order expansions of the cdf and pdf of normalized maxima for GMD with parameter $k > 0$ under nonlinear normalization. First of all, the high-order expansions of the cdf of normalized maximum from GMD with $k > 0$ under nonlinear normalization are stated as follows.

Theorem 2.1. *Let $F_k(x)$ represent the cdf of GMD with $k>0$. For normalizing constants α_n and β_n determined respectively by (1.5) and (1.6), we have*

$$\alpha_n^{2k}\Big(\alpha_n^{2k}\big(F_k^n(\alpha_n x^{\beta_n})-\Phi_1(x)\big)-S_k(x)\Phi_1(x)\Big)\to\Big(T_k(x)+\frac{S_k^2(x)}{2}\Big)\Phi_1(x)$$

as $n\to\infty$, where $S_k(x)$ and $T_k(x)$ are respectively defined by

$$S_k(x)=\sigma^2x^{-1}\big(\log x-k^{-1}\big)\log x$$

and

$$T_k(x)=\sigma^4x^{-1}\Big(-\frac{1}{2}(\log x)^3+\Big(\frac{2}{3}+k^{-1}\Big)(\log x)^2-\frac{1}{2}k^{-2}\log x+2k^{-1}\Big)\log x.$$

Remark 2.1. *For the case of parameter $k=1$, i.e., the classic Maxwell case, the associated result follows.*

Noting that $\alpha_n^{-2k}\sim 1/(2\sigma^2\log n)$, it follows Theorem 2.1 that we can establish the convergence rate of the distribution of maxima to its limit described as follows.

Corollary 2.1. *Let α_n and β_n given respectively by (1.5) and (1.6) and for $x>0$, we have*

$$F_k^n(\alpha_n x^{\beta_n})-\Phi_1(x)\sim\frac{(\log x-k^{-1})\log x}{2x\log n}\Phi_1(x)$$

for $k>0$ and large n.

In the following the high-order expansion of the pdf of normalized maxima is established. In this sequel, set $V_n(x)=(F_k^n(\alpha_n x^{\beta_n}))'-\Phi_1'(x)$, where $(F_k^n(\alpha_n x^{\beta_n}))'$ and $\Phi_1'(x)$ respectively denote the pdf of $(M_n/\alpha_n)^{1/\beta_n}$ and $\Phi_1(x)$ for $x>0$.

Theorem 2.2. *Under the condition of Theorem 2.1, we have*

$$\alpha_n^{2k}\Big(\alpha_n^{2k}V_n(x)-P_k(x)\Phi_1'(x)\Big)\to Q_n(x)\Phi_1'(x)$$

as $n\to\infty$, where $P_k(x)$ and $Q_k(x)$ are respectively defined by

$$P_k(x)=\sigma^2\left((x^{-1}-1)(\log x)^2+\big(-k^{-1}x^{-1}+k^{-1}+2\big)\log x-k^{-1}\right)$$

and

$$\begin{aligned}Q_k(x)=&\sigma^4\Bigg(\frac{1}{2}(x^{-2}-3x^{-1}+1)(\log x)^4+\Big(-k^{-1}x^{-2}+(3k^{-1}+\frac{8}{3})x^{-1}-\frac{8}{3}-k^{-1}\Big)(\log x)^3\\&+\Big(\frac{1}{2}k^{-2}x^{-2}-(\frac{3}{2}k^{-1}+3)k^{-1}x^{-1}+\frac{1}{2}k^{-2}+3k^{-1}+2\Big)(\log x)^2\\&+\big((k^{-1}+2)k^{-1}x^{-1}-(k^{-1}+2)k^{-1}\big)\log x+k^{-1}\Bigg).\end{aligned}$$

Remark 2.2. *As parameter $k=1$, that is the classic Maxwell distribution, the corresponded result cloud be gained.*

Observing that $\alpha_n^{-2k} \sim 1/(2\sigma^2 \log n)$, by utlizing Theorem 2.2, we can derive the convergence rate of the density of maxima tending to its extreme value limit depicted as follows.

Corollary 2.2. *Let the normalizing constants α_n and β_n given respectively by (1.5) and (1.6) and for $x > 0$, we have*

$$(F_k^n(\alpha_n x^{\beta_n}))' - \Phi_1'(x) \sim \frac{\left((x^{-1}-1)(\log x)^2 + (-k^{-1}x^{-1} + k^{-1} + 2)\log x - k^{-1}\right)}{2x^2 \log n}\Phi_1(x)$$

for $k > 0$ and large n.

3 Some Technical lemmas

The following lemmas are been utilized in the process of the proofs of main results.

Lemma 3.1. *Let $F_k(x)$ represent the cdf of GMD with $k > 0$. For large x, we have*

$$\begin{aligned}1 - F_k(x) =& f_k(x)\frac{\sigma^2}{k}x^{1-2k}\left(1 + k^{-1}\sigma^2 x^{-2k} + k^{-2}(1-2k)\sigma^4 x^{-4k} + O(x^{-6k})\right)\\ =& \frac{\exp(-1/(2\sigma^2))}{2^{k/2}\sigma^{1/k}\Gamma(1+k/2)}\big(1 + k^{-1}\sigma^2 x^{-2k} + k^{-2}(1-2k)\sigma^4 x^{-4k}\\ &+ O(x^{-6k})\big)\exp\left(-\int_1^x \frac{g(t)}{f(t)}\,\mathrm{dt}\right) \end{aligned} \tag{3.1}$$

with $f(t)$ and $g(t)$ given by (1.4).

Proof. The proof can be found in Lemma 3.1 of Huang and Liu (2015). □

Lemma 3.2. *For fixed $x > 0$ and $k > 0$, set*

$$A_n(x) = 1 + k^{-1}\sigma^2 x^{-2k} + k^{-2}(1-2k)\sigma^4 x^{-4k} + O(x^{-6k})$$

and

$$C_n(x) = \frac{A_n(\alpha_n x^{\beta_n})}{A_n(\alpha_n)}$$

with the norming constants α_n and β_n defined by (1.5) and (1.6). Then,

$$\alpha_n^{2k}\left(C_n(x) - 1\right) \to 0$$

and

$$\alpha_n^{4k}\left(C_n(x) - 1\right) \to -2k^{-1}\sigma^4 \log x,$$

as $n \to \infty$.

Proof. It is easy to check that $n(1 - F(\alpha_n x^{\beta_n})) \to x^{-1}$ as $n \to \infty$ by (1.7). By some elementary calculations, we have $C_n(x) \to 1$ as $n \to \infty$. By applying (1.5) and (1.6), for large n we have

$$(\alpha_n x^{\beta_n})^{-2k} - \alpha_n^{-2k} = -2\sigma^2\alpha_n^{-4k}\log x + 2\sigma^4\alpha_n^{-6k}(\log x)^2 + O(\alpha_n^{-8k})$$

and

$$(\alpha_n x^{\beta_n})^{-4k} - \alpha_n^{-4k} = -4\sigma^2\alpha_n^{-6k}\log x + 8\sigma^4\alpha_n^{-8k}(\log x)^2 + O(\alpha_n^{-10k}),$$

which implies

$$\begin{aligned}C_n(x)-1=&\Big(k^{-1}\sigma^2\left((\alpha_n x^{\beta_n})^{-2k}-\alpha_n^{-2k}\right)+k^{-2}(1-2k)\sigma^4\left((\alpha_n x^{\beta_n})^{-4k}-\alpha_n^{-4k}\right)+O(\alpha_n^{-8k})\Big)\\&\times(1+o(1))\\=&\Big(-2k^{-1}\sigma^4(\log x)\alpha_n^{-4k}+2k^{-1}\sigma^6(\log x-2k^{-1}+4)(\log x)\alpha_n^{-6k}+O(\alpha_n^{-8k})\Big)\\&\times(1+o(1)).\end{aligned}$$

Then, we have

$$\lim_{n\to\infty}\alpha_n^{2k}(C_n(x)-1)=0$$

and

$$\lim_{n\to\infty}\alpha_n^{4k}(C_n(x)-1)=-2k^{-1}\sigma^4\log x.$$

The proof is complete.

□

Lemma 3.3. *Let $M_n(x)=k\sigma^{-2}\alpha_n^{2k}\beta_n x^{2k\beta_n-1}-\beta_n x^{-1}-x^{-1}$ with norming constants α_n and β_n defined by (1.5) and (1.6). Then,*

$$\alpha_n^{2k}M_n(x)\to\sigma^2x^{-1}(2\log x-k^{-1}) \tag{3.2}$$

and

$$\alpha_n^{4k}\left(M_n(x)-\sigma^2x^{-1}(2\log x-k^{-1})\alpha_n^{-2k}\right)\to 2\sigma^4x^{-1}(\log x)^2 \tag{3.3}$$

as $n\to\infty$.

Proof. Through some fundamental computations, the desired result follows, so the detailed process is omitted.

□

Lemma 3.4. *Let $U_n(x)=n\log F_k(\alpha_n x^{\beta_n})+x^{-1}$ with norming constants α_n and β_n defined by (1.5) and (1.6). Then,*

$$\alpha_n^{2k}\left(\alpha_n^{2k}U_n(x)-S_k(x)\right)\to T_k(x),$$

as $n\to\infty$, where $S_k(x)$ and $T_k(x)$ are determined by Theorem 2.1.

Proof. For $k>0$, by utilizing (1.2) and (1.5), for all positive integers j and $j>1$ we have

$$\lim_{n\to\infty}\frac{(1-F_k(\alpha_n x^{\beta_n}))^i}{n^{-1}\alpha_n^{-jk}}=0. \tag{3.4}$$

By (1.4), (3.1) and Lemma 3.2, 3.3, we have

$$\begin{aligned}&\frac{1-F_k(\alpha_n x^{\beta_n})}{1-F_k(\alpha_n)}x\\=&C_n(x)\exp\left(-\int_{\alpha_n}^{\alpha_n x^{\beta_n}}\left(\frac{k}{\sigma^2}t^{2k-1}-\frac{1}{t}\right)dt+\log x\right)\end{aligned}$$

$$
\begin{aligned}
&= C_n(x)\exp\left(-\int_1^x M_n(t)dt\right)\\
&= C_n(x)\left(1-\int_1^x M_n(t)dt+\frac{1}{2}\left(\int_1^x M_n(t)dt\right)^2(1+o(1))\right).
\end{aligned} \tag{3.5}
$$

By applying (1.5), (3.2), (3.4), (3.5), Lemma 3.2 and Combining with the dominated convergence theorem, we get

$$
\begin{aligned}
&\lim_{n\to\infty}\alpha_n^{2k}U_n(x)\\
=&\lim_{n\to\infty}\alpha_n^{2k}\left(n\log F_k(\alpha_n x^{\beta_n})+x^{-1}\right)\\
=&\lim_{n\to\infty}\alpha_n^{2k}n\left(\log\left[1-(1-F_k(\alpha_n x^{\beta_n}))\right]+x^{-1}n^{-1}\right)\\
=&\lim_{n\to\infty}\alpha_n^{2k}n\left(-(1-F_k(\alpha_n x^{\beta_n}))-\frac{1}{2}(1-F_k(\alpha_n x^{\beta_n}))^2(1+o(1))+(1-F_k(\alpha_n))x^{-1}\right)\\
=&\lim_{n\to\infty}\alpha_n^{2k}n(1-F_k(\alpha_n))x^{-1}\left(-\frac{1-F_k(\alpha_n x^{\beta_n})}{1-F_k(\alpha_n)}x+1\right)\\
=&x^{-1}\lim_{n\to\infty}\alpha_n^{2k}\left((1-C_n(x))+C_n(x)\int_1^x M_n(t)dt(1+o(1))\right)\\
=&x^{-1}\int_1^x\lim_{n\to\infty}\alpha_n^{2k}M_n(t)dt\\
=&\sigma^2x^{-1}\big(\log x-k^{-1}\big)\log x\\
=:&S_k(x)
\end{aligned}
$$

and by using (1.5), (3.3), (3.4), (3.5), Lemma 3.2 and the dominated convergence theorem, we obtain

$$
\begin{aligned}
&\lim_{n\to\infty}\alpha_n^{2k}\left(\alpha_n^{2k}U_n(x)-S_k(x)\right)\\
=&\lim_{n\to\infty}\alpha_n^{4k}n\Bigg(-(1-F_k(\alpha_n x^{\beta_n}))-\frac{1}{2}(1-F_k(\alpha_n x^{\beta_n}))^2(1+o(1))\\
&+(1-F_k(\alpha_n))x^{-1}\left(1-S_k(x)\alpha_n^{-2k}\right)\Bigg)\\
=&\lim_{n\to\infty}\alpha_n^{4k}x^{-1}\left(-\frac{1-F_k(\alpha_n x^{\beta_n})}{1-F_k(\alpha_n)}x+1-S_k(x)x\alpha_n^{-2k}\right)\\
=&x^{-1}\Bigg(\lim_{n\to\infty}\alpha_n^{4k}\int_1^x\left(M_n(t)-\sigma^2t^{-1}(2\log t-k^{-1})\alpha_n^{-2k}\right)dt\\
&-\lim_{n\to\infty}\frac{1}{2}C_n(x)\left(\alpha_n^{2k}\int_1^x M_n(t)dt\right)^2(1+o(1))+\lim_{n\to\infty}\alpha_n^{4k}(1-C_n(x))\Bigg)\\
=&\sigma^4x^{-1}\Bigg(-\frac{1}{2}(\log x)^3+\left(\frac{2}{3}+k^{-1}\right)(\log x)^2-\frac{1}{2}k^{-2}\log x+2k^{-1}\Bigg)\log x\\
=:&T_k(x).
\end{aligned}
$$

The desired result follows.

□

Lemma 3.5. *Let $F_k(x)$ represent the cdf of GMD and norming constants α_n and β_n defined by (1.5) and (1.6). For large n we have*

$$\begin{aligned}F_k^{n-1}(\alpha_n x^{\beta_n}) &= \left[1+\alpha_n^{-2k}S_k(x)+\alpha_n^{-4k}\left(\frac{1}{2}S_k^2(x)+T_k(x)\right)\right](1+o(1))\Phi_1(x)\\ &=: B_n(x)\Phi_1(x)\end{aligned} \tag{3.6}$$

where $S_k(x)$ and $T_k(x)$ are respectively determined by Theorem 2.1.

Proof. By utilizing Lemma 3.4, for large n we have

$$F_k^{n}(\alpha_n x^{\beta_n}) = \left[1+\alpha_n^{-2k}S_k(x)+\alpha_n^{-4k}\left(\frac{1}{2}S_k^2(x)+T_k(x)\right)+O(\alpha_n^{-6k})\right]\Phi_1(x). \tag{3.7}$$

It is easy to check that $1-F_k(\alpha_n x^{\beta_n}) = O(n^{-1})$ by (1.7). By employing (1.8), we have

$$\begin{aligned}F_k^{n-1}(\alpha_n x^{\beta_n}) &= F_k^{n}(\alpha_n x^{\beta_n})[1-(1-F_k(\alpha_n x^{\beta_n}))]^{-1}\\ &= F_k^{n}(\alpha_n x^{\beta_n})\left(1+O(n^{-1})\right).\end{aligned} \tag{3.8}$$

Combining (3.7) with (3.8), we have

$$F_k^{n-1}(\alpha_n x^{\beta_n}) = \left[1+\alpha_n^{-2k}S_k(x)+\alpha_n^{-4k}\left(\frac{1}{2}S_k^2(x)+T_k(x)\right)\right](1+o(1))\Phi_1(x).$$

The conclusion can be deduced.

□

4 Proofs

Proof of Theorem 2.1. By using Lemma 3.4, we have

$$\begin{aligned}&\alpha_n^{2k}\left(\alpha_n^{2k}(F_k^n(\alpha_n x^{\beta_n})-\Phi_1(x))-S_k(x)\Phi_1(x)\right)\\ =&\alpha_n^{2k}\left(\alpha_n^{2k}\left(\exp\left(n\log F_k(\alpha_n x^{\beta_n})+x^{-1}-x^{-1}\right)-\Phi_1(x)\right)-S_k(x)\Phi_1(x)\right)\\ =&\left(\alpha_n^{2k}\left(\alpha_n^{2k}U_n(x)-S_k(x)\right)+\left(\alpha_n^{2k}U_n(x)\right)^2\left(\frac{1}{2}+O(U_n(x))\right)\right)\Phi_1(x)\\ \to&\left(T_k(x)+\frac{1}{2}S_k^2(x)\right)\Phi_1(x),\end{aligned}$$

as $n\to\infty$, here $S_k(x)$ and $T_k(x)$ are defined by Theorem 2.1. The needed result is complete.

□

Proof of Theorem 2.2. It follows Lemma 3.1 that, for large n we have

$$\begin{aligned}f_k(x) =& \left(1-k^{-1}\sigma^2x^{-2k}+k^{-1}\sigma^4x^{-4k}+O(x^{-6k})\right)\frac{k}{\sigma^2}x^{2k-1}(1-F_k(x))\\ =:&G_n(x)(1-F_k(x)).\end{aligned} \tag{4.1}$$

By employing (3.2), (3.5), (3.6), (4.1) and the dominated convergence theorem, we have

$$
\begin{aligned}
&\lim_{n\to\infty} \alpha_n^{2k} V_n(x) \\
=&\lim_{n\to\infty} \alpha_n^{2k} \left(n\alpha_n \beta_n x^{\beta_n-1} F_k^{n-1}(\alpha_n x^{\beta_n}) f_k(\alpha_n x^{\beta_n}) - \Phi_1^{'}(x) \right) \\
=&\lim_{n\to\infty} \alpha_n^{2k} \left(k^{-1}\sigma^2 \alpha_n^{1-2k} x^{\beta_n-1} B_n(x) G_n(\alpha_n x^{\beta_n}) \frac{1-F_k(\alpha_n x^{\beta_n})}{1-F_k(\alpha_n)} \Phi_1(x) - \Phi_1^{'}(x) \right) \\
=&\lim_{n\to\infty} \alpha_n^{2k} \Bigg(k^{-1}\sigma^2 \alpha_n^{1-2k} x^{\beta_n} C_n(x) \left(1 - \int_1^x M_n(t)dt + \frac{1}{2}\left(\int_1^x M_n(t)dt\right)^2 (1+o(1)) \right) \\
&\times B_n(x) G_n(\alpha_n x^{\beta_n}) - 1 \Bigg) \Phi_1^{'}(x) \\
=&\lim_{n\to\infty} \alpha_n^{2k} \Bigg(C_n(x) \left(1 - \int_1^x M_n(t)dt + \frac{1}{2}\left(\int_1^x M_n(t)dt\right)^2 (1+o(1)) \right) \\
&\times \left(1 + \alpha_n^{-2k} S_k(x) + \alpha_n^{-4k}\left(\frac{1}{2}S_k^2(x) + T_k(x)\right)\right)(1+o(1)) \\
&\times \left(1 + 2\sigma^2 \alpha_n^{-2k} \log x + 2\sigma^4 \alpha_n^{-4k} (\log x)^2 + O(\alpha_n^{-6k}) \right) \\
&\times \left(1 - k^{-1}\sigma^2 \alpha_n^{-2k} + k^{-1}\sigma^4 (2\log x + 1)\alpha_n^{-4k} + O(\alpha_n^{-6k}) \right) - 1 \Bigg) \Phi_1^{'}(x) \\
=&\lim_{n\to\infty} \alpha_n^{2k} \Bigg(\left(C_n(x) - C_n(x)\int_1^x M_n(t)dt + \frac{1}{2} C_n(x) \left(\int_1^x M_n(t)dt\right)^2 (1+o(1)) \right) \\
&\times \Bigg(1 + \left(S_k(x) - (k^{-1} - 2\log x)\sigma^2\right)\alpha_n^{-2k} + \Big[(k^{-1} + 2(\log x)^2)\sigma^4 - (k^{-1} - 2\log x)\sigma^2 S_k(x) \\
&+ \frac{1}{2}S_k^2(x) + T_k(x) \Big] \alpha_n^{-4k} + O(\alpha_n^{-6k}) \Bigg) - 1 \Bigg) \Phi_1^{'}(x) \\
=&\left(S_k(x) - (k^{-1} - 2\log x)\sigma^2 - \lim_{n\to\infty} \alpha_n^{2k} \int_1^x M_n(t)dt \right) \Phi_1^{'}(x) \\
=&\sigma^2 \left((x^{-1} - 1)(\log x)^2 + \left(-k^{-1}x^{-1} + k^{-1} + 2\right)\log x - k^{-1} \right) \Phi_1^{'}(x) \\
=:&P_k(x)\Phi_1^{'}(x)
\end{aligned}
$$

and by applying (3.3), (3.5), (3.6), (4.1) and the dominated convergence theorem, we derive

$$
\begin{aligned}
&\lim_{n\to\infty} \alpha_n^{2k} \left(\alpha_n^{2k} V_n(x) - P_k(x)\Phi_1^{'}(x) \right) \\
=&\lim_{n\to\infty} \Bigg(\left[(k^{-1} + 2(\log x)^2)\sigma^4 - (k^{-1} - 2\log x)\sigma^2 S_k(x) + \frac{1}{2}S_k^2(x) + T_k(x) \right] \\
&- \alpha_n^{4k} \int_1^x \left(M_n(t) - \sigma^2 t^{-1}(2\log t - k^{-1})\alpha_n^{-2k} \right) dt \\
&- \left(S_k(x) - \left(k^{-1} - 2\log x\right)\sigma^2 \right) \alpha_n^{2k} \int_1^x M_n(t)dt \\
&+ \frac{1}{2}\left(\alpha_n^{2k} \int_1^x M_n(t)dt \right)^2 (1+o(1)) + \alpha_n^{4k}\left(C_n(x) - 1\right) \Bigg) \Phi_1^{'}(x)
\end{aligned}
$$

$$
\begin{aligned}
=&\sigma^4\Bigg(\frac{1}{2}(x^{-2}-3x^{-1}+1)(\log x)^4+\left(-k^{-1}x^{-2}+(3k^{-1}+\frac{8}{3})x^{-1}-\frac{8}{3}-k^{-1}\right)(\log x)^3\\
&+\left(\frac{1}{2}k^{-2}x^{-2}-(\frac{3}{2}k^{-1}+3)k^{-1}x^{-1}+\frac{1}{2}k^{-2}+3k^{-1}+2\right)(\log x)^2\\
&+\left((k^{-1}+2)k^{-1}x^{-1}-(k^{-1}+2)k^{-1}\right)\log x+k^{-1}\Bigg)\Phi_1'(x)\\
=&:Q_n(x)\Phi_1'(x).
\end{aligned}
$$

The wanted result follows.

□

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