The second order local formalism for time evolution of dynamical systems

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Abstract

The second-order approach to the entropy gradient maximization, for systems with many degrees of freedom provides the dynamic equations of first order and light-like second order without additional ergodicity conditions like conservation laws.

The first order dynamics lead to the definition of the conserved kinetic energy and potential energy. In terms of *proper* degrees of freedom the total energy conservation reproduces the Einstein's mass-energy relation. The newtonian interpretation of the second order dynamic equations suggests the definition for general inertial mass and for the interaction potential.

1 Introduction

The entropy gradient maximization was proposed as an alternative formalism generating the dynamic equations [6] for a closed system with arbitrary degrees of freedom .

It was originally introduced in order to avoid the reversibility problem of statistical dynamics which is based on the conventional classical mechanics.

The formalism has been demonstrated to be a successful and promising approach also for number of other applications.

It is based on the fundamental principle of the *entropy maximization of closed system* (2nd law of thermodynamics). In advance to the historically

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entrenched formalisms, it demands no further artificial pre-assumptions relieving both the conventional lagrangian or hamiltonian formalism and surprisingly also the special relativity.

The ultimate philosophy of this approach is thought to be able to generate all known physical relations, like a newtonian dynamic equations, conservation laws and interaction phenomena, assuming a minimal number of primary statements - ideally the only one, the second law of thermodynamics, and the only one basic object operated thereby - the entropy or statistical weight.

The first-order formalism firstly proposed in [6] provides the first-order dynamics by the conditional maximizing the first order entropy variation with one or more additional "ergodicity" conditions. These conditions can be interpreted as conservation laws [6, 9]. At the same time they produce an interaction between degrees of freedom in the framework of the formalism. The "ergodicity" condition was firstly introduced on an example of the energy conservation and represented a conserved energy as a given scalar function of degree of freedom 's q: $h(q) = \mathcal{E} = \text{const}$

Thus the energy was pre-assumed as an *a priori* given expression, additionally to the given entropy function.

The present extension of the formalism to the next-high order of entropy variation - here the second order - is conditionless. It has been shown to produce the first order dynamic equation without demanding the energy conservation, is therefore *non-ergodic*.

This *second-order approach* was readily demonstrated to generate a collective first-order dynamics, as applied for a bosonic quantum system in terms of occupation numbers. In this problem both the second order entropy variation and the bosonic number conservation as the "ergodicity" have been assumed. The latter has been utilized as an issue of interaction between bosonic degrees of freedom

In the generalized formulation outlined below, the second order extension of the entropy variation is shown to be able producing the dynamic equations without additional conditions. Moreover, the existence of the conserved scalar quantity -kinetic energy- follows contrariwise from the dynamic equation and is therefore provided by the entropy itself. As long as the formalism is restricted on the second order only the potential energy remains constant, since the generalized interaction force between degrees of freedom disappears identically. It means in the second order the formalism describes a system of non-interacting degrees of freedom only

Nevertheless, the dynamic equations are recognizable to recover the iner-

tial mass and the relativistic mass-energy relation

The potential gradient and the inertial mass entering in the resulting second order dynamics both are related back to elements of the matrix are responsible for inertial and interaction properties. Since the only product appears there the identification of each factor is generically not unique. Especially, there is no criteria for recognizing the constant inertial mass. It enables to recover the relativistic velocity-dependence for the inertial factor and to reconstruct the corresponding entropy function.

For a generalized closed system described by arbitrary chosen degrees of freedom , this interpretation suggests an existence of *proper degrees of freedom* defined straightforwardly from the diagonalized matrix of second partial derivatives of entropy.

Furthermore, testing more different constructions of entropy derivatives offers to discover more *conservation laws* or similar *relaxation laws* which appear like conservation ones in a long-time, e.g. quantum scale.

These suggestions cannot be strongly confirmed in a framework of the second order formalism, since the approach should be treated consequently: partial entropy derivatives of orders higher than two in dynamic equations should be omitted, since they disappear in the original entropy variation. Thus a restriction of the formalism on the second order results only in the light-like dynamics with a constant generalized velocity. The subsequent second order dynamics (acceleration) trivially disappear. It means a permanent conservation of generalized momenta, but does not mean a zero inertial mass. In this context the term "massless" is unsubstantial, since the inertial measure is undefined. The linear momentum principle of the mechanics and the interaction-caused momentum dynamics appear first in a higher-order formalism, up from the third order. For this reason a straightforward derivation of the inertial mass as well as potential energy and further conserved quantities is retained for the third order extension.

The content of the article is ordered conventionally as follows. The Section 2 contains the generalized formulation of the second order approach, providing the first order dynamics and causality restrictions

The problem of energy definition and energy conservation is discussed in the following Section 3, the issue of the potential energy and interaction is performed.

The further discussion has rather a speculative status under retention of claims outlined here to be proven in the framework on higher order expansion of the approach.

The second order dynamics is naturally introduced deriving the first oder equations , the Section 4. The mentioned suggestions on the inertance and interaction are outlined here and illustrated by examples of different systems.

In the Section 5 the discussion comes back to the second order restriction. In this context, the renewed revision of conservation laws and suggestion on relaxation laws instead are performed.

The final Section 6 summarizes and discusses the gained results.

2 Formulation

The formalism starts with the scalar field $S(q_I)$ on the space of all degrees of freedom q_I (DoF's) - the *entropy function* of the closed system $\{q_I\}$ One of the DoF's q_I is chosen as the "time reference" τ, $\{q_I\} = \{q_i, \tau\}$, in that sense that in each state of the total system the state of the subsystem $\{q_i\}$ is compared with the one-valued state $\{\tau\}$ [6].

Consider the entropy variation δS with derivatives up to second order:

$$\delta S = S_{,i} dq_i + S_{,\tau} d\tau + \frac{1}{2} \left(S_{,ij} dq_i dq_j + S_{,i\tau} dq_i d\tau \right) + S_{,\tau\tau} d\tau^2 \tag{1}$$

(the conventional tensor notations

$$S_{,i} := \frac{\partial S}{\partial q_i}, S_{,\tau} := \frac{\partial S}{\partial \tau}$$

are used) No further additional *(ergodicity)* conditions are demanded. This is the main difference between the present formulation and the readily outlined *conditional* ones - the first order formalism with global ergodicity [6] and the second-order formalism with local ergodicity [8, 9].

2.1 First-order dynamics

The dynamic equation for a DoF q_i is provided by the first-order requirements [6]:

$$\frac{\partial}{\partial\ dq_i} \delta S = 0; \quad \frac{\partial}{\partial\ d\tau} \delta S = 0,$$

leading to

$$\begin{aligned} & S_{\tau\tau} d\tau = -S_\tau \\ & S_{,i} + S_{,ij} dq_j + S_{,i\tau} d\tau = 0 \end{aligned} \tag{2}$$

For the time scale we introduce the notation

$$\Omega := -\frac{S_\tau}{S_{\tau\tau}}$$

and suppose the time reference to be statistically discoupled [7]

$$S_{,i\tau} = 0.$$

The resulting system of the first-order dynamic equations is

$$\dot{q}_i := \frac{dq_i}{d\tau} = -\frac{1}{\Omega}\left[S_{,ij}\right]^{-1} S_{,j} \tag{3}$$

where $[S_{,ij}]^{-1}$ is the inverse of the matrix $S_{,ij}$ of second partial derivatives of entropy.

For further discussions we use the simplified notations, where the lower index on S means the partial derivative, comma is omitted

$$S_i := S_{,i} \quad S_{ik} \equiv S_{ki} := S_{,ik},$$

and the tilde means the inverse of the matrix of second derivatives.

$$\tilde{S}_{ij} := [S_{,ij}]^{-1}$$

With these notations the dynamic equations have the form

$$\dot{q}_i := -\frac{1}{\Omega}\tilde{S}_{ik}S_k \tag{4}$$

2.2 Causality

As the original formalism claims [6] the time reference τ must obey the time eligibility

$$S_\tau > 0.$$

The further requirement of causality for the matrix of second partial derivatives reads:

$$\begin{bmatrix} S_{ik} & 0 \\ 0 & S_{\tau\tau} \end{bmatrix} - \text{non-positive definite}$$

Here $S_{ik} = S_{,q_iq_k}$ means the matrix of second partial derivatives with respect to all DoF's except the τ, the off-diagonal zeros indicate the discoupled time-reference.

As it follows from the Sylvester-Jacobi criterion, for all admissible trajectories on $\{q_i, \tau\}$ the entropy $S(q_i, \tau)$ obeys:

1.

$$S_{\tau\tau} < 0, \textbf{ concavity with respect to the time-reference } \tau$$

2.

$$S_{ik} \text{ positive-definite: } \textbf{convexity with respect to DoF's } q_i$$

Therefore,

$$\Omega > 0$$

for realized evolution trajectories $q_i(\tau)$.

3 Second order dynamics and energy conservation

We proceed with the discussion taking only the first order dynamics (4) for the basis and "forget" for the present that the generating local functional (1) does not encountered third order derivatives of entropy. This discussion is postponed until the Section 5.

For the total derivative with respect to the time reference τ we use the conventional notation "dot" as the "time derivative". The second time derivative of a DoF q_i following from the dynamic equation (4) then reads

$$\ddot{q}_i := \frac{d^2 q_i}{d\tau^2} = \frac{d}{d\tau}\left[-\frac{1}{\Omega}\tilde{S}_{ij}S_j\right] = \frac{\dot{\Omega}}{\Omega^2}\tilde{S}_{ij}S_j + \frac{1}{\Omega^2}\left(\tilde{S}_{ij,k}S_j\tilde{S}_{kl}S_l + \tilde{S}_{ij}S_j\right)$$

$$= \frac{\dot{\Omega}+1}{\Omega^2}\tilde{S}_{ij}S_j + \frac{1}{\Omega^2}\tilde{S}_{ij,k}S_j\tilde{S}_{kl}S_l \tag{5}$$

On the other hand, we can construct the scalar product like a total "kinetic energy" of the closed system q, τ

$$T := \frac{1}{2}\dot{q}_i\dot{q}_i = \frac{1}{\Omega^2}\tilde{S}_{ij}\tilde{S}_{jk}S_iS_k \tag{6}$$

and check its variation along the dynamic trajectories.

The derivative with respect to the time reference DoF τ is

$$\frac{d}{d\tau}\frac{\dot{q}_i\dot{q}_i}{2} = \dot{q}_i\ddot{q}_i. \tag{7}$$

On the other hand, if we define $\ddot{q}_i$ to be a gradient of some scalar function $-U$,

$$-U_{,i} := \ddot{q}_i \tag{8}$$

where $U = U(\bar{q})$ is a scalar field on the space $\{\bar{q}\}$, the relationship (7) takes the form

$$\frac{d}{d\tau}\frac{\dot{q}_i\dot{q}_i}{2} + U(\bar{q})_{,i}\dot{q}_i = \frac{d}{d\tau}\left[T + U\right] =: \frac{d}{d\tau}\mathcal{E} = 0 \tag{9}$$

It means, the "energy conservation" entirely appears to be a consequence of the definition of $\mathcal{E} = T + U$, by means of (9) where the U is defined in turn by the (13) - the "second Newton's law of mechanics"

Suppose, we aimed to reconstruct the function $U(q)$ - "potential"- in terms of entropy derivatives using the second order dynamic equation (5)

Before proceeding we remark that the symmetry property of mixed partial derivatives for a smooth function $S(q)$ (full commutativity)

$$S_{,ijk} := S_{ijk} = S_{ikj} = S_{kij}$$

is in a general case not assured for the inverse matrix $\tilde{S}_{lj} := [S_{lj}]^{-1}$, since

$$\tilde{S}_{lj,k} \neq \tilde{S}_{lk,j},$$

even though

$$S_{lj} = S_{jl} \Rightarrow \tilde{S}_{lj} = \tilde{S}_{jl}.$$

In order to avoid this inconvenience, we consider a special case of the entropy function S, which possesses the property

$$\tilde{S}_{lj,k} = \tilde{S}_{lk,j},$$

for instance if S_{ik} is a diagonal matrix, $S_{ik} = 0, i \neq k$. In this case the identity (10)

$$\frac{1}{2}\left(\tilde{S}_{lj}S_j\tilde{S}_{lk}S_k\right)_{,i} \equiv \tilde{S}_{lj,i}S_j\tilde{S}_{lk}S_k + \tilde{S}_{ik}S_k \tag{10}$$

may be rewritten as

$$\frac{1}{2}\left(\tilde{S}_{lj}S_j\tilde{S}_{lk}S_k\right)_{,i} = \tilde{S}_{li,j}S_j\tilde{S}_{lk}S_k + \tilde{S}_{ik}S_k.$$

For a normal (non-exotic) time reference we expect (see Sec.5)

$$\dot{\Omega} + 1 = -\left[\frac{S_\tau}{S_{\tau\tau}}\right]_{,\tau} + 1 \sim O(S_{\tau\tau\tau\tau})$$

sufficiently smaller than 1 and then we can recover,

$$-U_{,i} = \left[\frac{1}{\Omega^2}\tilde{S}_{ij,k}S_j\tilde{S}_{kl}S_l\right]_{,i}$$

Thus, the total potential energy of the closed system

$$U = -\frac{1}{\Omega^2}\tilde{S}_{ij}S_j\tilde{S}_{ik}S_k$$

is equal to the total kinetic energy with opposite sign

$$T = \frac{\dot{q}_i\dot{q}_i}{2}$$

and we take the sum of absolute contributions for the total energy of the system

$$\mathcal{E} = |T| + |U| = \dot{q}_i\dot{q}_i$$

It follows further from $\frac{d}{d\tau}\mathcal{E} = 0$, that this case corresponds to the inertial-less *(massless)* or *light-like* dynamics

$$\ddot{q}_i = 0 \quad \Rightarrow \dot{q}_i = c = \text{const}$$

On the other hand, since $\ddot{q}_i = 0$, each $\dot{q}_i = c$ represents an extremal change rate of the DoF q - "velocity" $u_i := \dot{q}_i$. In this view the relation

$$\mathcal{E} = {\dot{q}_i}^2 = c^2$$

with $c^2 = 1$ is a generalization of the *mass-energy relation* $E = mc^2$ for a conventional energy E.

By the way it should be mentioned, that there exists a possibility of the special representation $Q_k = Q_k[q_i]$, where the matrix of second derivatives $\frac{\partial^2}{\partial Q^2}S$ is diagonal.

For this reason Q_k can be defined as *proper degrees of freedom* and $S^p := S[Q_k]$ as the *proper representation of entropy.* Further properties of this representation are amongst others

$$\tilde{S}^p_{kk} = 1/S^p_{kk}$$

$$\mathcal{E} = S^p_i\tilde{S}^p_{ik}\tilde{S}^p_{kj}S^p_j = \sum_k\left[\frac{S^p_k}{S^p_{kk}}\right]^2$$

Now we can see that it follows from the the energy conservation

$$\mathcal{E} = T + U = C = \text{const}$$

and from the second-order dynamics:

$$U_i = -\frac{1}{\Omega^2}\left(\tilde{S}_{ij,k}S_j\tilde{S}_{kl}S_l + \tilde{S}_{ij}S_j\right),$$

where the constants C, C' do not depend on DoF's and consequently on a representation. But since the potential U in proper representation becomes

$$U^p = C' - \tilde{S}_{ij}S_j\tilde{S}_{ik}S_k = C' - T,$$

the rest mass m_0 in the sense of the rest energy E/c^2 can be identified as

$$m_0 = 1/(\Omega^2 S_i^p \tilde{S}_{ik}^p \tilde{S}_{kj}^p S_j^p) = \frac{1}{\Omega^2}\left(\sum_k \left[\frac{S_k}{S_{kk}}\right]^2\right)^{-1}$$

with the entropy S^p as a function of *proper degrees of freedom* Q_k.

4 Suggestions on mechanics of a single degree of freedom

4.1 Basic principles

Consider the single degree of freedom q_i. Its dynamics $q_i(\tau)$ obeys the equation (4), which can be interpreted as the equation for the*generalized velocity* u^i

$$-\Omega\dot{q}_i := u^i = \tilde{S}_{ik}S_k. \tag{11}$$

Since the expression in the right-hand side does not depend explicitly on τ, we can fit the form of the equation by redefining the time parameter (reparametrization of time [7]):

$$\frac{d}{dt} = -\Omega\frac{d}{d\tau}$$

so that

$$\frac{d}{dt}q_i = u^i$$

Then the second order equation corresponds to the *generalized acceleration of* q_i, $w_i = \Omega^2 \ddot{q}_i$

$$w_i = \frac{d}{dt} u^i = u^i_{,k} u^k \tag{12}$$

The form of (12) does not reproduce the newtonian second law of mechanics

$$m \frac{d}{dt} \vec{v} = -\nabla U \tag{13}$$

In terms of u^i it looks rather similar to the Euler's equation of hydrodynamics

$$\frac{d}{dt} \vec{v} = (\vec{v}\nabla)\vec{v}$$

For relatively small velocities, the equation can be understood as an expansion around some state q_i^0 having zero velocity

$$u^i(q_i^0) = 0$$

the $i - stationary$ state.

In order to be interpreted as a newtonian (13), the equation (12) should be representable in a factorized form like

$$a_i = -m^{-1} U_{,i},$$

where the factor m^{-1} is positive, nearly independent of all degrees of freedom q_i, e.g. $[m^{-1}]_{,i}$ should be small compared to m^{-1}.

Obviously, this factorization is not unique, since the resulting equation is strongly dependent on the construction of the generating entropy function $S(q_i)$

The expressions for velocity and acceleration contain only combinations of partial derivatives of entropy S, which can be interpreted relating to the chosen q_i as follows:

The derivatives S_{ii}, S_i describe only the q_i-dependence of the entropy, while the mixed derivatives S_{ik} - the cross-dependence of S on the q_i and q_k together. In this sense it is a measure of the statistical correlation of q_i and q_k, which should be understood as an "interaction" of the DoF q_i with other DoF's $q_k, k \neq i$

The elements $S_{lk},\ l \neq i, k \neq i$ are responsible for "background" correlations between DoF's except the q_i, (although the derivatives

$$S_{lk} := \frac{\partial^2}{\partial q_k \partial q_l} S(\bar{q})$$

can be explicitly dependent on q_i)
The (12) takes than the detailed form:

$$w_i = \left[\tilde{S}_{ik}S_k\right]_{,l}\tilde{S}_{lk}S_k = \left[\frac{1}{|S_{mn}|}\left(\bar{S}_{ii}S_i + \sum_{k\neq i}\bar{S}_{ik}S_k\right)\right]_{,l}\frac{1}{|S_{mn}|}\sum_k \bar{S}_{lk}S_k, \text{ no summation over } i$$

here $\bar{S}_{ik}$ are algebraic complements of elements S_{ik} in the matrix S_{ik} and

$$|S_{mn}| = \bar{S}_{ii}S_{ii} + \sum_{k\neq i}\bar{S}_{ik}S_{ik} -$$

its determinant, here it should be noted that the subdeterminant $\bar{S}_{ii}$ contains only second derivatives responsible for background correlations.

Remarking that the expression for u^i can be rewritten as

$$u^i = \frac{\bar{S}_{ii}S_i + \sum_{k\neq i}\bar{S}_{ik}S_k}{\bar{S}_{ii}S_{ii} + \sum_{k\neq i}\bar{S}_{ik}S_{ik}} = \frac{1}{S_{ii}/S_i}\cdot\frac{1 + \frac{1}{\bar{S}_{ii}S_i}\sum_{k\neq i}\bar{S}_{ik}S_k}{1 + \frac{S_{ii}}{\bar{S}_{ii}}\sum_{k\neq i}\bar{S}_{ik}S_{ik}}$$

we convince us that if the factor $\frac{S_{ii}}{S_i}$ is constant (or nearly constant), its square

$$m_i \sim \frac{S_{ii}^2}{S_i^2}$$

plays the role of the *generalized inertial mass* of the DoF q_i in the closed system $\{\bar{q}, \tau\}$. This hypothesis is also in accordance with the interpretation of the sum

$$m_i = 1/\sum_k \frac{S_k^2}{S_{kk}^2}$$

over all degrees of freedom as the *rest mass-energy* of the system.

Thus S_{ii} and S_i appear as a measure for statistical inertial properties of q_i, which cannot be irelated directly to the *mass*

4.2 Examples

4.2.1 Example I
Bosonic entropy in terms of occupation numbers as DoF's

The entropy as a function of discrete bosonic occupation numbers n_i [9], [4] is given by

$$S[n_i] = \ln\left(\sum_{i=-N}^{N} n_i\right)! - \sum_{i=-N}^{N} \ln n_i! \tag{14}$$

In the continuum limit at very large numbers n_i factorials are replaced by Gamma-functions:

$$S[n(i)] = \ln\ \Gamma\left[\sum_{i=-N}^{N} n(i)\right] - \sum_{i=-N}^{N} \ln\ \Gamma[n(i)] \tag{15}$$

For the both cases (14 - 15) one obtains for partial derivatives [2, 3]

$$S_{n_i} = \sum_i n_i - n_i;\ S_{n_i n_k} = 1; S_{n_i n_i} = 0$$

or in the tensor form

$$S_{ik} = 1 - \delta_{ik},$$

with the inverse matrix

$$\tilde{S}_{ik} = \frac{1}{N-1} - \delta_{ik},$$

where: 1 and $\frac{1}{N-1}$ denote a matrix with all elements equal 1 and $\frac{1}{N-1}$ respectively, N is the number of equivalent (non distinguished) bosonic degrees of freedom - "bosonic dimension" of the system.

For large N we have

$$\lim_{N\to\infty} \tilde{S}_{ik} = -\delta_{ik}.$$

With

$$S_i = \sum_k q_k - q_i$$

the resulting dynamic equations are:

$$u^i = \left[\frac{1}{N-1} - \delta_{ik}\right]\left(\sum_m q_m - q_k\right) = q_i$$

and the newtonian equation is

$$w^i = u^i_{,k} u^k = \delta_{ik} q_k = q_i$$

as well.

Since $q_i = 0$ is the only stationary point, the dynamics describes a DoF with the mass unity in a square potential.

This non-physical behavior of this model occurs due to the the fact, that there are no interactions between degrees of freedom (as it will be discussed below in Sec.6), since there are no derivatives of order higher than second.

In the corresponding "ergodic" model [9] the collective interaction is implied directly by the additional ergodicity condition - here the conservation of the total bosonic number.

In a contrast, in the above example the bosonic degrees of freedom "do not see" each other, hence the unrestricted increase of each bosonic degree of freedom n_i independently is the expected dynamics, providing the locally increasing entropy variation.

By the way, the fermionic entropy with

$$S_{ik} = \tilde{S}_{ik} = \delta_{ik}, S_i = \sum_m q_m$$

provides the same dynamics for discrete DoF's only valued by $0, 1$.

4.2.2 Example II
Quantum *von Neumann* entropy

For the entropy function of *von Neumann type* [1] like

$$S(q) = -\sum_i q_i \ln q_i \tag{16}$$

with

$$S_k := S_{,k} = -\left(\ln q_k + 1\right), \quad S_{kl} := S_{,kl} = -\frac{1}{q_k}\delta_{kl}, \text{no summation over} l$$

Consequently

$$\tilde{S}_{km} = -\mathbf{diag}\ \{q_k, k = m\}$$

and the dynamic equation for this system

$$u^i = \tilde{S}_{ik} S_k = \mathbf{diag}\ \{q_k, i=k\}(\ln q_k + 1) = \begin{bmatrix} . & & & & & & \\ & . & & & & & \\ & & q_{k-1} & & & & \\ & & & q_k & & & \\ & & & & q_{k+1} & & \\ & & & & & . & \\ & & & & & & . \end{bmatrix} \begin{bmatrix} . \\ . \\ \ln q_{k-1} + 1 \\ \ln q_k + 1 \\ \ln q_{k+1} + 1 \\ . \\ . \end{bmatrix}$$

in the tensor form reads finally

$$u^i = q_i \ln q_i + q_i. \tag{17}$$

For the acceleration we obtain further

$$w_i = \sum_k (q_k \ln q_k + q_k)(q_i \ln q_i + q_i)_{,k} = q_i \ln^2 q_i + 3 q_i \ln q_i + 2 q_i$$

From the "zero velocity" condition

$$u^i = q_i \ln q_i + q_i = 0$$

we find two stationary points for each degree of freedom q_i of this system

$$q_i^0 = 0 \text{ and } q_i^0 = 1/e.$$

For a chosen degree of freedom $q_i \equiv q$ the newtonian dynamics around the second stationary state results from the expansion:

$$w(q) = w\left(\frac{1}{e}\right) + w_q\left(\frac{1}{e}\right)\left(q - \frac{1}{e}\right) + \frac{1}{2} w_{qq}\left(q - \frac{1}{e}\right)^2 + ...$$

Since

$$w\left(\frac{1}{e}\right) = 0, \quad w_q\left(\frac{1}{e}\right) = \ln^2 q + 5(\ln q + 1)\Big|_{q=1/e} = 1,$$

the expansion begins at first order

$$w(q) = \ddot{q} = \left(q - \frac{1}{e}\right) = \frac{1}{2}\frac{\partial}{\partial q}\frac{\left(q - \frac{1}{e}\right)^2}{2}$$

which corresponds to the harmonic oscillation around $q = 1/e$ with the mass unity.

The divergent derivative u_q is remarkable for the first stationary point $q_0 = 0$, which suggests an interpretation either as an infinite inertial mass, or as an infinitely strong attraction, preventing an escape from this point - a kind of a *quantum black hole*.

4.2.3 Example III
Lorentzian relativistic dynamics and recovering the entropy

The ambiguity of factorization of second-order dynamics (12) mentioned there by the comment to this equation allows various interpretation of a non-constant factor responsible for the inertial mass, for example as a relativistic velocity-dependent mass.

Obviously, if the acceleration is considered in a state with a *non-zero velocity* rather than *stationary* point, the solution q^0

$$u^i(q) = \tilde{S}_{ik}(q)S_k(q) = v = \text{ const}$$

provides another newtonian equation (12) for acceleration w, where the coefficient m^{-1} contains a dependence on v.

Especially, if we need to recover the Lorentzian dependence

$$m(v) = \frac{m_0}{\sqrt{1-v^2}}$$

for the inertial mass under acceleration by a constant external force $f = \text{ const}$ in the state q^0 obeying

$$u(q^v) = v, \tag{18}$$

we have to solve the reverse problem starting with the acceleration

$$w(q) = u'(q)u(q) = \frac{f}{m_0}\sqrt{1-u^2}, \tag{19}$$

that means in general the system of partial differential equations:

$$\begin{aligned} w^i &= u^k\frac{du^i}{dq_k} = \frac{f^i}{m_0}\sqrt{1-u^k u^k} \\ u^i(q) &= \tilde{S}_{ik}(q)S_k(q), \text{ summation over } k \end{aligned} \tag{20}$$

The backward recovery of the entropy is therefore not unique. It may be any scalar function $S(q)$ obeying (20).

The scheme for this procedure can be sketched for a system with one degree of freedom q. We need to search for the entropy function $S(q)$ producing the second-order dynamics

$$w(q) = u'(q)u(q) = \frac{f}{m_0}\sqrt{1-u^2}, \tag{21}$$

which initially obeys

$$w(q^v) = u'(q^v)u(q^v) = u'\left(q^v(v)\right)v = \frac{f}{m_0}\sqrt{1-v^2},$$

where $q^v(v)$ is now the solution of (18)

$$q^v(v) = u^{-1}(v).$$

By integration of the inverse mapping $q(u)$ in (19)

$$u_q = \frac{du}{dq} = \frac{f}{m_0}\frac{\sqrt{1-u^2}}{u}$$

we obtain

$$\frac{f}{m_0}(q-q_0) = -\sqrt{1-u^2},$$

keeping in mind we consider only positive values for degree of freedom q, which are physically and statistically meaningful, like e.g. quantum occupation numbers, Example I. The displacement from the stationary point $(q_v - q_0)$ is rather positive, because it seems unlikely for a single statistical degree of freedom, that the entropy would increase for decreasing q.

The starting point for acceleration is given by

$$q^v(v) = q_0 - \frac{m_0}{f}\sqrt{1-v^2}.$$

and restoring the function u explicitly

$$u(q) = \pm\sqrt{1-\left[\frac{f}{m_0}(q-q_0)\right]^2}.$$

we conclude that entropy S(q) obeys

$$\tilde{S}_{qq}S_q = \frac{S_q}{S_{qq}} = \sqrt{1 - \left[\frac{f}{m_0}(q - q_0)\right]^2} \tag{22}$$

with the first and the second integral

$$S_q = C^{qq} \exp\left\{-\frac{m_0}{f}\sqrt{1 - \left[\frac{f}{m_0}(q - q_0)\right]^2}\right\};$$

$$S = C^{qq}\int \exp\left\{-\frac{m_0}{f}\sqrt{1 - \left[\frac{f}{m_0}(q - q_0)\right]^2}\right\} dq + C^q$$

respectively, C^{qq}, C^q are constants; It follows from (22), that the rest mass (rest energy) for this system is according to (**??**)

$$m_r = \left[\frac{S_q(q_0)}{S_{qq}(q_0)}\right]^2 = 1$$

and does not coincides necessary with the stationary inertial mass m_0. In this concern the following remark is in order:

The m_0 is introduced in (20) artificially as a predefined inertial factor for influence by the external force f, as well as the f itself.

However *no external forces* are allowed in the framework of this formalism. Since the system $\{q, \tau\}$ is subject to any external action, it cannot be considered as closed anymore and thus the formalism can not be applied. In order to keep the formalism consistent, any external force f acting on the DoF q should arise inside the system through the interaction (in this context - the statistic correlation, see the discussion above) between its own DoF's.

5 Restriction on the second order variation

At this point we remember the original formulation of the approach (Sec.2).

The generating local functional is the entropy variation (1), which contains only contributions of entropy derivatives up to second order. All derivatives of order three ($S_{ijk}, S_{\tau\tau\tau}$) and higher are omitted and should be consequently disregarded in all further relationships derived from this functional.

The first order dynamics remains unchanged under this restriction:

$$\dot{q}_i = \frac{d}{d\tau} q_i = -\frac{1}{\Omega(\tau)} \tilde{S}_{ik} S_k$$

Here, the variation of time reference

$$d\tau = \Omega(\tau) := -\frac{S_\tau}{S_{\tau\tau}}$$

contains the second order derivative. It follows that

$$\dot{\Omega} := \frac{d}{d\tau}\Omega(\tau) = -1 + O\left(S_{,\tau\tau\tau}\right).$$

Even if the variation δS contain the τ derivatives of S up to third order it results for the time reference:

$$d\tau = \Omega(\tau) := \frac{-S_{\tau\tau} \pm \sqrt{S_{\tau\tau}^2 - 2S_{\tau\tau\tau}S_\tau}}{S_{\tau\tau\tau}};$$

$$\dot{\Omega} := \frac{d}{d\tau}\Omega(\tau) = -1 + O\left(S_{,\tau\tau\tau\tau}\right)$$

respect the derivatives S_i, S_{ik} (only up to second order, i.e. omitting S_{ikj} and higher)

$$\dot{u}_i = \frac{d}{d\tau}\dot{q}_i = \frac{\dot{\Omega}+1}{\Omega^2}\tilde{S}_{ik}S_k + \frac{1}{\Omega^2}O\left(S_{,ijk}\right)$$

Concerning interaction forces between DoF's $f_i := -U_{,i}$, they should identically disappear in the second order restriction, since the potential is a bilinear combination of second derivatives of entropy, as follows from (**??**). The *acceleration* $\ddot{q}_i$ and the causing *accelerating force* $-U_{,i}$ of a single degree of freedom q_i appears firstly if the third oder contributions are encountered in the entropy variation δS

Consequently, this non-ergodic formalism provides in the second order the only dynamics of non-interacting degrees of freedom (trivially interacting with a constant potential U), defined above as a light-like dynamics

The subsequent second order dynamics trivially disappear. It means a permanent conservation of generalized momenta, but does not mean that inertial mass m_0 is zero. In this context the term "massless" is unsubstantial, since the inertial mass is undefined.

5.1 Conservation laws

Among this trivial conservation of momentum and consequently also the kinetic energy,the restriction of the formalism on the second order implies some local conservation laws additionally.

Since $\tilde{S}_{ik}$ is the inverse of S_{ik}, regarding the second order restriction we arrive at

$$\left(\tilde{S}_{ik}S_k\right)_{,j} = \delta_{ij},$$

$$\frac{d}{d\tau}\tilde{\mathbf{S}}|S>:= \frac{d}{d\tau}\tilde{S}_{ik}S_k = -\frac{1}{\Omega}\tilde{S}_{ik}S_k =: -\frac{1}{\Omega}\tilde{\mathbf{S}}|S>= \dot{q}_i, \tag{23}$$

where the natural notations are introduced for the matrix

$$\tilde{\mathbf{S}} := \tilde{S}_{ik}$$

and the vector

$$|S>:= S_k$$

respectively.

The restriction on the second order enforces further for traces of second-derivative matrix:

$$\frac{d}{d\tau}|\mathbf{S}| := \frac{d}{d\tau}S_{ii} = 0; \quad \frac{d}{d\tau}|\tilde{\mathbf{S}}| := \frac{d}{d\tau}\tilde{S}_{ii} = 0 \tag{24}$$

and

$$\frac{d}{d\tau}|\mathbf{S}|^2 = \frac{d}{d\tau}|\tilde{\mathbf{S}}|^2 = \frac{d}{d\tau}S_{ik}S_{ki} = \frac{d}{d\tau}\tilde{S}_{ik}\tilde{S}_{ki} = 0 \tag{25}$$

as well as for all traces of all powers S_{ik}^n, $\tilde{S}_{ik}^n$. Each of these traces can represent a scalar conserved quantity - "charge" or "number".

It is also worth remarking that

$$\mathbf{div}\ \tilde{\mathbf{S}}|S>:= \mathbf{div}\ \tilde{S}_{ik}S_k = -\Omega\dot{q_{i,i}} = \dim\{\bar{q}\} = \ \text{const}$$

in the second order restriction. It is the fundamental invariant of the space $\{\bar{q}\}$ of DoF's - its dimension.

5.2 Relaxation laws

The first-order dymanics (4) leads to the evolution equation for the entropy $S(\bar{q})$

$$\frac{d}{d\tau}S = -\frac{1}{\Omega(\tau)}S_i\tilde{S}_{ik}S_k =: -\frac{1}{\Omega(\tau)} < S|\tilde{\mathbf{S}}|S >$$

It describes the total *entropy production rate* of the closed system along the evolution trajectories.

$$\dot{S} := \frac{d}{dt} := -\Omega\frac{d}{d\tau}S =< S|\tilde{\mathbf{S}}|S >$$

The entropy production rate obeys in turn the first order dynamics:

$$\frac{d}{d\tau} < S|\tilde{\mathbf{S}}|S >= -\frac{2}{\Omega(\tau)} < S|\tilde{\mathbf{S}}|S > \qquad \Omega > 0 \tag{26}$$

that provides for the entropy itself using the (5.2):

$$\frac{d^2}{d\tau^2}S = \frac{1}{\Omega^2} < S|\tilde{\mathbf{S}}|S >$$

We established that the scalar $< S|\tilde{\mathbf{S}}|S >$ for a closed system $\{\bar{q}, \tau\}$ is subject to exponential extinction *(relaxation)* as

$$< S|\tilde{\mathbf{S}}|S > [\tau] =< S|\tilde{\mathbf{S}}|S > [0] = \exp\left[-\frac{1}{2}\int\limits_0^\tau \frac{dt}{\Omega(t)}\right].$$

As long as the formalism remains in the framework of the second order, the same holds also for

$$\frac{d}{d\tau}S_i\tilde{S}_{ij}\tilde{S}_{jk}S_k = \frac{d}{d\tau} < S|\tilde{\mathbf{S}}^{\mathbf{2}}|S >= -\frac{2}{\Omega(\tau)} < S|\tilde{\mathbf{S}}^{\mathbf{2}}|S > \qquad \Omega > 0 \tag{27}$$

as well as for an arbitrary power $\tilde{\mathbf{S}}^n$ of the matrix $\tilde{S}$. It provides the series of *relaxing* scalar quantities of the system. The first of this series possessing such behavior is the scalar trace ($n = 0$)

$$\frac{d}{d\tau}(S_iS_i)^{\pm 1} = \mp\frac{2}{\Omega(\tau)}S_iS_i. \tag{28}$$

In a long-time scale $\Omega << 1$, for example in a quantum system observed in a classic time, like the postulation of quantum mechanics, these *fast relaxing quantities* are considered as *nearly constant*.

The corresponding scalar quantities inverse to $< S|\tilde{\mathbf{S}}^n|S >,\ n = 1, 2, ...$

$$\frac{d}{d\tau}\frac{1}{< S|\tilde{\mathbf{S}}|S >} = \frac{2}{\Omega}\frac{1}{< S|\tilde{\mathbf{S}}|S >} \tag{29}$$

$$\frac{d}{d\tau}\frac{1}{< S|\tilde{\mathbf{S}}^{\mathbf{2}}|S >} = \frac{2}{\Omega}\frac{1}{< S|\tilde{\mathbf{S}}^{\mathbf{2}}|S >}, ... \tag{30}$$

are exponentially increasing or *nearly singular* in a long-time scale.

The entropy itself follows the law

$$S(\tau) = \int_0^\tau \exp\left[-\frac{1}{2}\int_0^t \frac{d\theta}{\Omega(\theta)}\right] dt + S(0)$$

for the entropy change between states (instances) $\tau = 0$ and τ and becomes nearly constant in a long-time scale. For this reason the long-time scale quantities which are fast relaxing and nearly conserved, can be also determined as *adiabatically conserved* quantities or adiabatic invariants (adiabatic charges). The other obeying the inverse laws are the *adiabatic singularities* respectively.

The vector quantities with similar properties are the vector S_i

$$\frac{d}{d\tau}S_i := \frac{d}{d\tau}|S >= -\frac{2}{\Omega(\tau)}|S > \tag{31}$$

as well as all products of kind

$$\tilde{\mathbf{S}}^{\mathbf{n}}|S >:= \left[\tilde{S}^n\right]_{ik} S_k \tag{32}$$

which can be related to *adiabatic momenta* of order n.

6 Discussion and conclusion

The extension of the local formalism of the entropy gradient maximization to the second order entropy variation reveals the ability to reproduce the first order dynamics without demanding any additional conditions.

It has been proposed as alternative to the first order formalism recently developed. This primary approach provides the first order dynamic equations based on the maximization of the first order entropy variation and using additional *ergodicity conditions*. As such additional requirements, several balance equations can be utilized, e.g. conventional conservation laws, like the energy conservation. Thus, the conservation laws had to be pre-assumed in order to apply the formalism.

In contrast, an extended formalism, especially the second order extension, is shown to be able to generate the first-order dynamic equations together with conservation principles without any additional pre-assumptions.

In this framework it is possible to define a generalized kinetic and potential energy for generalized degrees of freedom , as well as the total rest mass energy associated with the rest mass in the special relativity.

The approach has been formulated in a general case for a closed system with arbitrary number of DoF's.

The main object determining the *dynamics and causality* is the matrix S_{ik} of second order partial derivatives of entropy as well as its inverse.

Based on this concept, the *proper degrees of freedom* are determined, in terms of them this matrix is diagonal. Then, the sum $\sum_i \left(S_i^p/S_{ii}^p\right)^2$ in the *proper representation of entropy* is regarded as the inverse of the rest mass, related to the rest energy. This gives rise to assumption that the diagonal elements S_{ii} are responsible also for inertial properties of dynamics of the i-th degree of freedom .

A prolongation to the second order dynamics speaks also for this hypothesis. The second order equations in terms of the statistical degrees of freedom appear in the form of the eulerian equation of hydrodynamics, instead of the usual newtonian equation of mechanics. Since the corresponding general acceleration should be caused by non-vanishing off-diagonal elements S_{ik} - partial derivatives with respect to i-th degree of freedom mixed with other DoF's $k \neq i$, they are related to the interaction between i and k. Thus it gives rise to conclude that the *issue of interaction* phenomena is the *statistical correlation* between degrees of freedom .

Consequently, several contributions of these correlations, being collected together as a gradient of some scalar expression, can be defined and interpreted as the *interaction potential.*

Considered as a generalization of the framework of 2nd newtonian law, a factorization into potential gradient and inverse of inertial measure (e.g.mass)

as a pre-factor, is generally not unique, since such a pre-factor is generally not constant.

On the other hand, this evidence can be interpreted as the relativistic velocity-dependence of the inertial mass. This dependence results originally from a special form of the entropy function.

A restriction on the second order dynamics demands to omit constitutively all entropy derivatives of order higher than two not only in the generating functional δS but also in subsequent dynamic equations. In this case however, the second order dynamics vanishes, what means that only the massless dynamics like light propagation exist.

It implies the permanent momentum conservation as well as the conserved kinetic energy. Since the omitting the 3rd order entropy derivatives provide only constant potential energy, the dynamics correspond to the dynamics of non-interacting degrees of freedom .

The higher order dynamics should be constitutively derived by a prolongation of the entropy variation to contain higher order contributions with subsequent first and second order dynamics for a single degree of freedom determining the generalized velocity and generalized acceleration respectively.

Speculations about this scenario suggest that this *rest mass* related to the rest energy seems likely to be also interpreted as the generalized *inertial mass*.

A re-parametrization of a time scale makes suggestions to distinguish between true local conservation laws like (24 - 25) and fast relaxations down to a constant value like (26 - 32) for a short time scale $\Omega(\tau)$. Among the conventional true, (or exact) conservation, several quantities are subject to "statistical relaxation"- the exponential decay down to constant. The question how to distinguish between a true conservation and a fast relaxation is related to the resolution of the time reference DoF τ. A relaxation in fine-time scale appears as a conservation as being observed in a rough-time scale. For instance, a change of any quantity inside the interval of τ-values considerable in the quantum scale is observed as a quantum jump between constant values in a macroscopic scale. The instances between (eventually unobserved) jumps can be associated with the macroscopic conservation laws.

6.1 Concluding statement

The second order extension of the statistical EGM formalism is able to reproduce the structure and general relationships of mechanics. This formalism is

a promising alternative approach to conventional hamiltonian or lagrangian ones.

The approach is based on the only single governing principle - local maximization of the entropy variation. The producing functional is the scalar entropy function and no additional conditions are needed. It suggest that the *second law of thermodynamics* in form of the local entropy maximization is the general founding principle of physics. All physical relationships and postulates which are conventionally hold for fundamental, can be derived from this principle and are in fact corollaries of certain constructions entropy function.

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