Two Theorems on Solutions in Eulerian Description

Valdir Monteiro dos Santos Godoi

valdir.msgodoi@gmail.com

Abstract – We present two proofs of theorems needed to the major work we are doing on existence and breakdown solutions of the Navier-Stokes equations for incompressible case without external force in n = 3 spatial dimensions.
Keywords – Navier-Stokes equations, velocity, pressure, Eulerian description, Lagrangian description, formulation, equivalent equations, exact solutions, existence, inexistence.

Let $u^0(x,y,z)$ and $p^0(x,y,z)$ be respectively the initial velocity and initial pressure of the three-dimensional incompressible $(\nabla \cdot u = \nabla \cdot u^0 = 0)$ Navier-Stokes equations without external force

$$(1) \qquad \frac{\partial p(X,t)}{\partial x_i} + \frac{\partial u_i(X,t)}{\partial t} + \sum_{j=1}^{3} u_j(X,t)\frac{\partial u_i(X,t)}{\partial x_j} = \nu\nabla^2 u_i(X,t),$$

$1 \leq i \leq 3$, $X = (x_1, x_2, x_3) \in \mathbb{R}^3$, $x_1 \equiv x$, $x_2 \equiv y$, $x_3 \equiv z$, $x_i, t \in \mathbb{R}$, $t \geq 0$.

Then in $t = 0$ is valid, for each integer i belongs to $1 \leq i \leq 3$,

$$(2) \qquad \frac{\partial p^0(X)}{\partial x_i} + \frac{\partial u_i(X,t)}{\partial t}\big|_{t=0} + \sum_{j=1}^{3} u_j^0(X)\frac{\partial u_i^0(X)}{\partial x_j} = \nu\nabla^2 u_i^0(X).$$

Supposing that $u(x,y,z,t) = u^0(x+t, y+t, z+t)$ and $p(x,y,z,t) = p^0(x+t, y+t, z+t)$ is a solution (u,p) for (1), we have

$$(3) \qquad \frac{\partial p^0(\xi)}{\partial x_i} + \frac{\partial u_i^0(\xi)}{\partial t} + \sum_{j=1}^{3} u_j^0(\xi)\frac{\partial u_i^0(\xi)}{\partial x_j} = \nu\nabla^2 u_i^0(\xi),$$

where $\xi = (\xi_1, \xi_2, \xi_3)$ and $\xi_i = \xi_i(X,t) = x_i + t, 1 \leq i \leq 3$.

For $t = 0$ the equations (2) and (3) are equals, because in $t = 0$ we have $\xi_i = x_i$ and therefore $\xi = (\xi_1, \xi_2, \xi_3) = (x_1, x_2, x_3) = X$.

For $t > 0$, if (2) is valid for any $X = (x,y,z) \in \mathbb{R}^3$ then (3) is valid for any $\xi \in \mathbb{R}^3$ substituting $x \mapsto \xi_1 = x + t$, $y \mapsto \xi_2 = y + t$, $z \mapsto \xi_3 = z + t$, $x,y,z \in \mathbb{R}, t \geq 0$, so $u(x,y,z,t) = u^0(x+t, y+t, z+t)$ and $p(x,y,z,t) = p^0(x+t, y+t, z+t)$, i.e., $u(X,t) = u^0(\xi)$ and $p(X,t) = p^0(\xi)$, solve equation (3) and therefore the Navier-Stokes equation (1).

The initial motivation to prove it is as follows. Let $A(x), B(x), C(x)$ and $D(x)$ functions such that is always valid, for any $x \in \mathbb{R}$, the relation

1

(4) $$A(x) + B(x) + C(x) = D(x).$$

Then, as $(x + t) \in \mathbb{R}, x, t \in \mathbb{R}, t \geq 0$, need be valid too the relation

(5) $$A(x + t) + B(x + t) + C(x + t) = D(x + t).$$

The same argument can be used for functions of two and three spatial dimensions (or better, for n spatial dimensions), for example, $\forall x, y, z, t \in \mathbb{R}, t \geq 0$,

(6) $$A_i(x, y, z) + B_i(x, y, z) + C_i(x, y, z) = D_i(x, y, z)$$
$$\Rightarrow A_i(x + t, \ y + t, \ z + t) + B_i(x + t, \ y + t, \ z + t) +$$
$$+ C_i(x + t, \ y + t, \ z + t) = D_i(x + t, \ y + t, \ z + t).$$

Applying the previous relation (6) to the Navier-Stokes equations (2) for $t = 0$, if

(7.1) $$A_i(x, y, z) = \frac{\partial p^0(X)}{\partial x_i},$$

(7.2) $$B_i(x, y, z) = \frac{\partial u_i(X,t)}{\partial t}\Big|_{t=0},$$

(7.3) $$C_i(x, y, z) = \sum_{j=1}^{3} u_j^0(X) \frac{\partial u_i^0(X)}{\partial x_j},$$

(7.4) $$D_i(x, y, z) = \nu \nabla^2 u_i^0(X),$$

(7.5) $$A_i(x, y, z) + B_i(x, y, z) + C_i(x, y, z) = D_i(x, y, z),$$

$X = (x, y, z)$, then, using $\xi = \xi(X, t) = (x + t, y + t, z + t)$, need be valid too the equalities

(8.1) $$A_i(x + t, \ y + t, \ z + t) = \frac{\partial p^0(\xi)}{\partial x_i},$$

(8.2) $$B_i(x + t, \ y + t, \ z + t) = \left(\frac{\partial u_i(X,t)}{\partial t}\Big|_{t=0}\right)(\xi),$$

(8.3) $$C_i(x + t, \ y + t, \ z + t) = \sum_{j=1}^{3} u_j^0(\xi) \frac{\partial u_i^0(\xi)}{\partial x_j},$$

(8.4) $$D_i(x + t, \ y + t, \ z + t) = \nu \nabla^2 u_i^0(\xi),$$

(8.5) $$A_i(x + t, \ y + t, \ z + t) + B_i(x + t, \ y + t, \ z + t) +$$
$$+ C_i(x + t, \ y + t, \ z + t) = D_i(x + t, \ y + t, \ z + t).$$

The expression $\left(\frac{\partial u_i(X,t)}{\partial t}\big|_{t=0}\right)(\xi)$ in (8.2) means that first is calculated the value of $\frac{\partial u_i(X,t)}{\partial t}$, next we assign the value $t = 0$ in this result and then we substitute $x \mapsto \xi_1 = x + t,\ y \mapsto \xi_2 = y + t,\ z \mapsto \xi_3 = z + t$, i.e., $X \mapsto \xi$.

Note that the right side of the relations (8.1) to (8.4) corresponds to each parcel of the Navier-Stokes equations (8.5) with the solution (u, p) such that

(9.1) $u(X, t) = u^0(\xi)$,
(9.2) $p(X, t) = p^0(\xi)$,

$X = (x, y, z)$, $\xi = \xi(X, t) = (x + t,\ y + t,\ z + t)$, then (9) is a solution for (1) if $u^0(X)$ and $p^0(X)$ are initial conditions.

We will now prove that if the variables (9.1) and (9.2) solve (1) for $t \geq 0$ then $u^0(x, y, z)$ and $p^0(x, y, z)$ solve (1) for $t = 0$, i.e., then both $u^0(x, y, z)$ and $p^0(x, y, z)$ solve (2). This is an important result of this section. We'll use the chain rule[1].

Proof: Starting from (1), the three-dimensional incompressible Navier-Stokes equations, where $\nabla \cdot u = \nabla \cdot u^0 = 0$,

(10) $\frac{\partial p(X,t)}{\partial x_i} + \frac{\partial u_i(X,t)}{\partial t} + \sum_{j=1}^3 u_j(X, t) \frac{\partial u_i(X,t)}{\partial x_j} = \nu \nabla^2 u_i(X, t)$,

$1 \leq i \leq 3$, $X = (x, y, z)$, if a solution (u, p) for them is (9), i.e.,

(11.1) $u(X, t) = u^0(\xi)$,
(11.2) $p(X, t) = p^0(\xi)$,

$\xi = \xi(X, t) = (x + t,\ y + t,\ z + t)$, then we have, according (3),

(12) $\frac{\partial p^0(\xi)}{\partial x_i} + \frac{\partial u_i^0(\xi)}{\partial t} + \sum_{j=1}^3 u_j^0(\xi) \frac{\partial u_i^0(\xi)}{\partial x_j} = \nu \nabla^2 u_i^0(\xi)$.

How $\xi_i = x_i + t$ then $\frac{\partial \xi_i}{\partial x_i} = \frac{\partial \xi_i}{\partial t} = 1$ and $\frac{\partial \xi_i}{\partial x_j} = 0$ if $i \neq j$, so using the chain rule[1] we have, for each parcel in (10) and (12),

(13.1) $\frac{\partial p(X,t)}{\partial x_i} = \frac{\partial p^0(\xi)}{\partial x_i} = \sum_{j=1}^3 \frac{\partial p^0(\xi)}{\partial \xi_j} \frac{\partial \xi_j}{\partial x_i} = \frac{\partial p^0(\xi)}{\partial \xi_i}$

(13.2) $\frac{\partial u_i(X,t)}{\partial t} = \frac{\partial u_i^0(\xi)}{\partial t} = \sum_{j=1}^3 \frac{\partial u_i^0(\xi)}{\partial \xi_j} \frac{\partial \xi_j}{\partial t} = \sum_{j=1}^3 \frac{\partial u_i^0(\xi)}{\partial \xi_j}$

(13.3) $u_j(X, t) \frac{\partial u_i(X,t)}{\partial x_j} = u_j^0(\xi) \frac{\partial u_i^0(\xi)}{\partial x_j} = u_j^0(\xi) \frac{\partial u_i^0(\xi)}{\partial \xi_j} \frac{\partial \xi_j}{\partial x_j} =$

$$= u_j^0(\xi)\frac{\partial u_i^0(\xi)}{\partial \xi_j}$$

$$(13.4) \qquad \nabla^2 u_i(X,t) = \nabla^2 u_i^0(\xi) = \left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}\frac{\partial}{\partial x_3}\right)u_i^0(\xi) =$$

$$= \sum_{j=1}^{3}\left(\frac{\partial}{\partial \xi_j}\frac{\partial \xi_j}{\partial x_j}\frac{\partial}{\partial \xi_j}\frac{\partial \xi_j}{\partial x_j}\right)u_i^0(\xi) = \sum_{j=1}^{3}\left(\frac{\partial}{\partial \xi_j}\frac{\partial}{\partial \xi_j}\right)u_i^0(\xi) =$$

$$= \nabla_\xi^2 u_i^0(\xi)$$

Adding the parcels (13), with (13.3) for each integer $j = 1, 2, 3$ and the multiplication of (13.4) by viscosity coefficient ν, we come to

$$(14) \qquad \frac{\partial p^0(\xi)}{\partial \xi_i} + \sum_{j=1}^{3}\frac{\partial u_i^0(\xi)}{\partial \xi_j} + \sum_{j=1}^{3} u_j^0(\xi)\frac{\partial u_i^0(\xi)}{\partial \xi_j} = \nu\nabla_\xi^2 u_i^0(\xi),$$

which is equivalent to previous Navier-Stokes equations (10) and (12) with the solution (11), although it is not a conventional Navier-Stokes equation because the time derivative disappears, i.e.,

$$(15) \qquad \frac{\partial u_i(X,t)}{\partial t} \mapsto \sum_{j=1}^{3}\frac{\partial u_i^0(\xi)}{\partial \xi_j}.$$

Note that the right side of (15) is not $\frac{\partial u_i^0(\xi)}{\partial t} + \sum_{j=1}^{3}\frac{\partial u_i^0(\xi)}{\partial \xi_j}$, because here u_i^0 is, initially, a function only of $\xi = (\xi_1, \xi_2, \xi_3)$, not including t, but each ξ_i is a function of t and for this reason here is $\frac{\partial u_i(X,t)}{\partial t} = \frac{\partial u_i^0(\xi)}{\partial t} = \sum_{j=1}^{3}\frac{\partial u_i^0(\xi)}{\partial \xi_j}\frac{\partial \xi_j}{\partial t} =$ $\sum_{j=1}^{3}\frac{\partial u_i^0(\xi)}{\partial \xi_j}$, with $\xi_j = x_j + t$, $\frac{\partial \xi_j}{\partial t} = 1$.

In $t = 0$, when $\xi_i = x_i$, the equation (14) became

$$(16) \qquad \frac{\partial p^0(X)}{\partial x_i} + \sum_{j=1}^{3}\frac{\partial u_i^0(X)}{\partial x_j} + \sum_{j=1}^{3} u_j^0(X)\frac{\partial u_i^0(X)}{\partial x_j} = \nu\nabla^2 u_i^0(X).$$

If this equation is equivalent to (2) then

$$(17) \qquad \frac{\partial u_i(X,t)}{\partial t}\Big|_{t=0} = \sum_{j=1}^{3}\frac{\partial u_i^0(X)}{\partial x_j},$$

which is thereby a good manner of define or choose the temporal derivative of velocity at $t = 0$ when the solution for velocity is $u(X,t) = u^0(\xi)$.

Similarly, for $t > 0$ we have

$$(18) \qquad \frac{\partial u_i(X,t)}{\partial t} = \sum_{j=1}^{3}\frac{\partial u_i^0(\xi)}{\partial \xi_j},$$

4

$X = (x, y, z)$, $\xi = (\xi_1, \xi_2, \xi_3)$, $\xi_i = \xi_i(X, t) = x_i + t$, $1 \leq i \leq 3$.

Concluding, assuming that (9), identical to (11), is a solution for (1), identical to (10), we come to (16) for $t = 0$, which is equivalent to (2) with the additional initial condition (17) and it has a solution $(u^0(X), p^0(X))$. This is what we wanted to prove. □

Next, we will prove the opposite way of the previous demonstration: if $u^0(x, y, z)$ and $p^0(x, y, z)$ solve (1) for $t = 0$, i.e., if both $u^0(x, y, z)$ and $p^0(x, y, z)$ solve (2), then the variables (u, p) given in (9.1) and (9.2) solve (1) for $t \geq 0$. This is the fundamental result of this section. The proof basically follows what we write from beginning of this section until the equations (9), increasing the transformations (13) and the conditions (17) and (18). We'll use the chain rule[1] again.

Proof: If $u^0(x, y, z)$ and $p^0(x, y, z)$ solve the three-dimensional incompressible $(\nabla \cdot u = \nabla \cdot u^0 = 0)$ Navier-Stokes equations

(19) $\qquad \dfrac{\partial p(X,t)}{\partial x_i} + \dfrac{\partial u_i(X,t)}{\partial t} + \sum_{j=1}^{3} u_j(X, t) \dfrac{\partial u_i(X,t)}{\partial x_j} = \nu \nabla^2 u_i(X, t)$

for $t = 0$, with $1 \leq i \leq 3$, $X = (x_1, x_2, x_3) \in \mathbb{R}^3$, $x_1 \equiv x$, $x_2 \equiv y$, $x_3 \equiv z$, $x_i, t \in \mathbb{R}$, $t \geq 0$, then in $t = 0$ is valid, for each integer i belongs to $1 \leq i \leq 3$,

(20) $\qquad \dfrac{\partial p^0(X)}{\partial x_i} + \dfrac{\partial u_i(X,t)}{\partial t}\Big|_{t=0} + \sum_{j=1}^{3} u_j^0(X) \dfrac{\partial u_i^0(X)}{\partial x_j} = \nu \nabla^2 u_i^0(X)$.

Supposing that $u(x, y, z, t) = u^0(x + t, y + t, z + t)$ and $p(x, y, z, t) = p^0(x + t, y + t, z + t)$ is a solution (u, p) for (19), we have

(21) $\qquad \dfrac{\partial p^0(\xi)}{\partial x_i} + \dfrac{\partial u_i^0(\xi)}{\partial t} + \sum_{j=1}^{3} u_j^0(\xi) \dfrac{\partial u_i^0(\xi)}{\partial x_j} = \nu \nabla^2 u_i^0(\xi)$,

using $\xi = (\xi_1, \xi_2, \xi_3)$ and $\xi_i = \xi_i(X, t) = x_i + t$, $1 \leq i \leq 3$.

For $t = 0$ the equations (20) and (21) are equals, because in $t = 0$ we have $\xi_i = x_i$ and therefore $\xi = (\xi_1, \xi_2, \xi_3) = (x_1, x_2, x_3) = X$.

For $t > 0$, if (20) is valid for any $X = (x, y, z) \in \mathbb{R}^3$ then (21) is valid for any $\xi \in \mathbb{R}^3$ substituting $x \mapsto \xi_1 = x + t$, $y \mapsto \xi_2 = y + t$, $z \mapsto \xi_3 = z + t$, $x, y, z \in \mathbb{R}, t \geq 0$, according transformations (22) below, so $u(x, y, z, t) = u^0(x + t, y + t, z + t)$ and $p(x, y, z, t) = p^0(x + t, y + t, z + t)$, i.e., $u(X, t) = u^0(\xi)$ and $p(X, t) = p^0(\xi)$, solve equation (21) and therefore the Navier-Stokes equation (19).

How $\xi_i = x_i + t$ then $\dfrac{\partial \xi_i}{\partial x_i} = \dfrac{\partial \xi_i}{\partial t} = 1$ and $\dfrac{\partial \xi_i}{\partial x_j} = 0$ if $i \neq j$, so using the chain rule[1] we have, for each parcel in (21),

5

(22.1) $\quad \dfrac{\partial p^0(\xi)}{\partial x_i} = \dfrac{\partial p(\xi(X,t))}{\partial x_i} = \sum_{j=1}^{3} \dfrac{\partial p^0(\xi)}{\partial \xi_j} \dfrac{\partial \xi_j}{\partial x_i} = \dfrac{\partial p^0(\xi)}{\partial \xi_i}$

(22.2) $\quad \dfrac{\partial u_i^0(\xi)}{\partial t} = \dfrac{\partial u_i(\xi(X,t))}{\partial t} = \sum_{j=1}^{3} \dfrac{\partial u_i^0(\xi)}{\partial \xi_j} \dfrac{\partial \xi_j}{\partial t} = \sum_{j=1}^{3} \dfrac{\partial u_i^0(\xi)}{\partial \xi_j}$

(22.3) $\quad u_j^0(\xi) \dfrac{\partial u_i^0(\xi)}{\partial x_j} = u_j(\xi(X,t)) \dfrac{\partial u_i(\xi(X,t))}{\partial x_j} = u_j^0(\xi) \dfrac{\partial u_i^0(\xi)}{\partial \xi_j} \dfrac{\partial \xi_j}{\partial x_j} =$

$\qquad = u_j^0(\xi) \dfrac{\partial u_i^0(\xi)}{\partial \xi_j}$

(22.4) $\quad \nabla^2 u_i^0(\xi) = \nabla^2 u_i(\xi(X,t)) = \sum_{j=1}^{3} \left(\dfrac{\partial}{\partial x_j} \dfrac{\partial}{\partial x_j} \right) u_i^0(\xi(X,t)) =$

$\qquad = \sum_{j=1}^{3} \left(\dfrac{\partial}{\partial \xi_j} \dfrac{\partial \xi_j}{\partial x_j} \dfrac{\partial}{\partial \xi_j} \dfrac{\partial \xi_j}{\partial x_j} \right) u_i^0(\xi) = \sum_{j=1}^{3} \left(\dfrac{\partial}{\partial \xi_j} \dfrac{\partial}{\partial \xi_j} \right) u_i^0(\xi) =$

$\qquad = \nabla_\xi^2 u_i^0(\xi)$

The equation (21) transformed through by (22) gives

(23) $\quad \dfrac{\partial p^0(\xi)}{\partial \xi_i} + \sum_{j=1}^{3} \dfrac{\partial u_i^0(\xi)}{\partial \xi_j} + \sum_{j=1}^{3} u_j^0(\xi) \dfrac{\partial u_i^0(\xi)}{\partial \xi_j} = \nu \nabla_\xi^2 u_i^0(\xi),$

that is, we transform $X \longmapsto \xi$ and from $\xi_i = x_i + t$ we have $\dfrac{\partial \xi_i}{\partial x_i} = 1$ and therefore $\partial x_i = \partial \xi_i$.

The unexpected transformation is

(24) $\quad \dfrac{\partial u_i^0(\xi)}{\partial t} = \dfrac{\partial u_i(\xi(X,t))}{\partial t} = \sum_{j=1}^{3} \dfrac{\partial u_i^0(\xi)}{\partial \xi_j},$

making (23) not be in the form of a standard Navier-Stokes equation. In $t = 0$ the transformation (24) becomes

(25) $\quad \dfrac{\partial u_i^0(\xi)}{\partial t}\Big|_{t=0} = \dfrac{\partial u_i(\xi(X,t))}{\partial t}\Big|_{t=0} = \dfrac{\partial u_i(X,t)}{\partial t}\Big|_{t=0} = \sum_{j=1}^{3} \dfrac{\partial u_i^0(X)}{\partial x_j},$

$\xi_j = x_j$, $\xi = X$, for $t = 0$, thus we need to assume the additional initial condition

(26) $\quad \dfrac{\partial u_i(X,t)}{\partial t}\Big|_{t=0} = \sum_{j=1}^{3} \dfrac{\partial u_i^0(X)}{\partial x_j}$

when the solution for Navier-Stokes equation (1), identical to (19), is given by (9), i.e.,

(27.1) $\quad u(X,t) = u^0(\xi),$
(27.2) $\quad p(X,t) = p^0(\xi),$

$X = (x, y, z), \; \xi = \xi(X, t) = (x + t, \; y + t, \; z + t)$.

Concluding, if $(u^0(X), p^0(X))$ solve (2), identical to (20), substituting in (20) the transformation $X \mapsto \xi, \; X = (x, y, z), \; \xi = (\xi_1, \xi_2, \xi_3), \; \xi_i = x_i + t$, we come to (23),

$$(28) \qquad \frac{\partial p^0(\xi)}{\partial \xi_i} + \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j} + \sum_{j=1}^{3} u_j^0(\xi) \frac{\partial u_i^0(\xi)}{\partial \xi_j} = \nu \nabla_\xi^2 u_i^0(\xi),$$

assuming the additional initial condition (26)

$$(29) \qquad \frac{\partial u_i(X,t)}{\partial t} \Big|_{t=0} = \sum_{j=1}^{3} \frac{\partial u_i^0(X)}{\partial x_j}$$

due to transformation (24),

$$(30) \qquad \frac{\partial u_i^0(\xi)}{\partial t} = \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j}.$$

Using (30) in (28) we come to

$$(31) \qquad \frac{\partial p^0(\xi)}{\partial \xi_i} + \frac{\partial u_i^0(\xi)}{\partial t} + \sum_{j=1}^{3} u_j^0(\xi) \frac{\partial u_i^0(\xi)}{\partial \xi_j} = \nu \nabla_\xi^2 u_i^0(\xi),$$

the Navier-Stokes equations with the solution $(u^0(\xi), p^0(\xi))$, i.e., $(u(X, t), p(X, t))$, according (27), identical to (9).

Using (27) and $\partial \xi_i = \partial x_i$ in (31) we come finally to

$$(32) \qquad \frac{\partial p(X,t)}{\partial x_i} + \frac{\partial u_i(X,t)}{\partial t} + \sum_{j=1}^{3} u_j(X,t) \frac{\partial u_i(X,t)}{\partial x_j} = \nu \nabla_X^2 u_i(X,t),$$

the Navier-Stokes equations (1) with the solution $(u(X, t), p(X, t))$. This is what we wanted to prove. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

What we see in the two previous proofs can be applied, with the obvious adaptations, to solutions of the form

(33.1) $\qquad u(X, t) = u^0(\xi)$,
(33.2) $\qquad p(X, t) = p^0(\xi)$,

$X = (x, y, z), \; \xi = (\xi_1, \xi_2, \xi_3), \; \xi_i = x_i + T_i(t), \; T_i(0) = 0, \; 1 \le i \le 3$,

with the conditions

$$(34) \qquad \frac{\partial u_i(X,t)}{\partial t} = \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j} \frac{\partial \xi_j}{\partial t} = \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j} T_j'(t),$$

and

(35) $\qquad \frac{\partial u_i(X,t)}{\partial t}\big|_{t=0} = \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j} T_j'(0) = \sum_{j=1}^{3} \frac{\partial u_i^0(X)}{\partial x_j} T_j'(0),$

being the functions $T_i(t)$ differentiable of class $C^1([0,\infty))$. In this case the equations (23) and (28) are

(36) $\qquad \frac{\partial p^o(\xi)}{\partial \xi_i} + \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j} T_j'(t) + \sum_{j=1}^{3} u_j^0(\xi) \frac{\partial u_i^0(\xi)}{\partial \xi_j} =$

$$= \frac{\partial p^o(\xi)}{\partial \xi_i} + \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j} \left[T_j'(t) + u_j^0(\xi) \right] = \nu \nabla_\xi^2 u_i^0(\xi).$$

Note that the equation (34) implies

(37) $\qquad u_i(X,t) = u_i^0(X) + \int_0^t \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j} T_j'(t) \, dt =$

$$= u_i^0(\xi_1, \xi_2, \xi_3) = u_i^0\big(x_1 + T_1(t), \ x_2 + T_2(t), \ x_3 + T_3(t)\big),$$

that must be true for all differentiable function $u_i^0(\xi)$ with $\xi_i = x_i + T_i(t)$, $T_i(t)$ differentiable, $T_i(0) = 0$, $1 \leq i \leq 3$.

It is clear that in the Eulerian description[2] the computational and analytical challenges will be, more than solving the Navier-Stokes equations for $t > 0$, solve these equations for $t = 0$, the initial instant. Unfortunately, it is not for all pair of values (u^0, p^0) that exists solution to the equation (28) and related equations, so or u^0 is a function of p^0, or p^0 is a function of u^0, or both u^0 and p^0 are functions of another functions, for example, a potential function ϕ such that $u^0 = \nabla\phi(t = 0)$, $u = \nabla\phi$, resulting in the known Bernouilli's law. From Lagrangian description[2] yet there is a great problem: the collision of particles of the fluid, which always are forgotten or overlooked. The adoption of the continuity equation do not solves it.

NOTE: A few days ago I realized the possibility of proving the invariance of waves and Maxwell's equations with respect to Galilean transformations with this method, without the need to introduce the famous coefficient $\sqrt{1 - \frac{v^2}{c^2}}$ of Einstein, including the Schrodinger equation. This also seems to be able to reach to the General Relativity.

References

[1] Apostol, Tom M., *Calculus*, vol. II, chap. 8. New York: John Wiley & Sons (1969).

[2] Childress, Stephen, *An Introduction to Theoretical Fluid Mechanics*, Courant Lecture Notes in Mathematics, 19. New York: Courant Institute of Mathematical Sciences (2009).