THE PROBLEM OF APOLLONIUS AS AN OPPORTUNITY FOR TEACHING STUDENTS TO USE REFLECTIONS AND ROTATIONS TO SOLVE GEOMETRY PROBLEMS VIA GEOMETRIC (CLIFFORD) ALGEBRA

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ABSTRACT. The beautiful Problem of Apollonius from classical geometry ("*Construct all of the circles that are tangent, simultaneously, to three given coplanar circles*") does not appear to have been solved previously by vector methods. It is solved here via Geometric Algebra (GA, also known as Clifford Algebra) to show students how they can make use of GA's capabilities for expressing and manipulating rotations and reflections. As Viète did when deriving his ruler-and-compass solution, we first transform the problem by shrinking one of the given circles to a point. In the course of solving the transformed problem, guidance is provided to help students "see" geometric content in GA terms. Examples of the guidance that is given include (1) recognizing and formulating useful reflections and rotations that are present in diagrams; (2) using postulates on the equality of multivectors to obtain solvable equations; and (3) recognizing complex algebraic expressions that reduce to simple rotations of multivectors.

1. INTRODUCTION

"Stop wasting your time on GA! Nobody at Arizona State University [David Hestenes's institution] *uses it—students consider it too hard"*. This opinion, received in an email from one of Professor Hestenes's colleagues, might well disconcert both the uninitiated whom we hope to introduce to GA, and those of us who recognize the need to begin teaching GA at the high-school level ([1]–[3]). For some time to come, the high-school teachers whom we'll ask to "sell" GA to parents, administrators, and students will themselves be largely self-taught, and will at the same time need to learn how to teach GA to their students. A daunting task, given a typical teacher's workload. Those teachers (as well as their students) will need support from experts in using and teaching GA if it is to become an academic subject.

The present article is intended to give those experts a wider basis for understand the sort of support that may be most effective. The author is himself a self-taught student of GA who prepares instructional materials for other amateurs [4]. Because GA's capabilities for expressing and manipulating rotations and reflections are one of its most powerful features, he decided to use them to solve classic geometrical "construction" problems involving tangency between circles and other objects. Those experiences indicate that solving such problems via GA requires the student to "see" and "assemble" elements of diagrams in ways quite different from those which he had used previously.

Hestenes appeared to acknowledge that issue when he noted that students need "judicious guidance" to get through *New Foundations for Classical Mechanics* [5]. What the student appears to need is the training that Whitely [6] terms an "apprenticeship", in which the student is taught, consciously, how to see problems in GA terms. Whiteley, who applied that idea to mathematics generally, discussed the required apprenticeship in some detail. He explains that as we go about understanding and solving a mathematical problem, we are continually seeing and conceiving different elements of the problem and equations in different ways.

Whiteley advocates making that mental process—for example, how we see things at one moment, then shift consciously to another way to make use of another property—known explicitly to our students by modeling it for them as we teach. Of course, instructors must at the same time monitor how students are seeing and assembling the elements to which we are calling their attention. On that note, Whiteley cautions that dynamic geometry programs and other visuals, which can be invaluable tools for forming a shared understanding and for changing the way we see and think in mathematics, often fail to produce the desired results because of the gap between what the student focuses on and what the expert focuses on.

Whiteley, who was writing in 2002, was not the first to make such recommendations. The field has been active since then. Some of the proven ways of implementing his recommendations are described in [7], [8], [9], and [10]. Particularly effective is the narrative form of teaching, in which instructors ”think aloud” and model cognitive behaviors while working through problems in front of the class. Students also benefit when instructors develop and share diverse examples and diverse ways to see individual examples, along with the tools that let students experience what the instructor is seeing. From these many specific examples, students develop a basis from which to generalize and make fruitful conjectures, and thereby to understand how a subject ”fits together”.

Thus the purpose of this article is threefold. Firstly, to add to the store of solved problems contributed by authors such as González Calvet ([1], [2], and [3]). More importantly, we hope to show how Whiteley's recommendations might be followed to show students how to recognize opportunities for using GA's capabilities for handling rotations and reflections, using GA that's understandable at the high-school level. Finally, we provide experts with detailed examples of dedicated amateurs' thinking, so that those experts might guide us in learning and promoting GA. Chess instructor Jeremy Silman discusses the use of such a collaboration in [11].

Outline of the article

The article will use a form of first-person "teacher talk" ([8]) until the "Concluding Remarks" section. For reasons of space, the solution process will be somewhat streamlined, and will not enter into the sorts of dead ends that might be useful to present to students. The sequence of steps will be a standard one, recommended by many math-teaching researchers and supported by in-classroom experience:

- Initial examination of the problem, and preliminary work
- Conscious review of (probably) relevant experience
- Re-examination of problem
- Formulation of a solution strategy
- Solving the problem

The paper ends with brief concluding remarks.

2. Initial work on the Problem of Apollonius

2.1. Statement of the Problem of Apollonius, and transformation to the CCP Case. The problem statement is as follows: *"Construct all of the circles that are tangent, simultaneously, to three given coplanar circles."* Fig. 2.1 shows a solution.

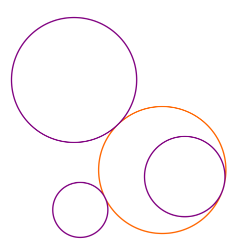

FIGURE 2.1. Example of a solution (orange circle) to the Problem of Apollonius. The blue circles are the givens.

If we were to examine Fig. 2.1 a bit, we'd soon recognize that the tangency relationships shown therein are preserved if we shrink the smallest of the given circles to a point.

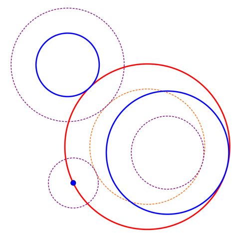

FIGURE 2.2. Solution (red circle) to the transformed version of the problem shown in Fig. 2.1 The smallest of the given circles has been reduced to a point. The diameter of the given circle that was internally tangent to the original version's solution circle has been increased, while that of the other given circle has decreased.

The problem statement for the transformed problem now becomes

"Given two circles and a point $\mathscr{P}$, all coplanar, construct the circles that pass through $\mathscr{P}$ and are tangent, simultaneously, to the given circles." (Fig. 2.3).

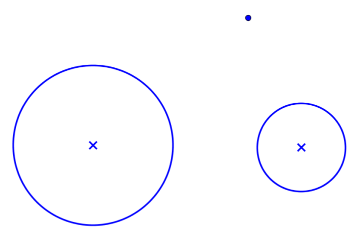

FIGURE 2.3. The CCP limiting case of the Problem of Apollonius: *"Given two circles and a point $\mathscr{P}$, all coplanar, construct the circles that pass through $\mathscr{P}$ and are tangent, simultaneously, to the given circles."*

This transformed version—known as the Circle-Circle-Point (CCP) limiting case of the Problem of Apollonius—is the problem on which we will work, because intuition suggests that it will be easier to solve than the original version.

2.2. **Preliminary examination of the transformed Problem.** Using a dynamic geometry program, we would see quickly that there are four solutions to the CCP problem. One of the solution circles encloses both of the givens; one encloses neither of the givens, and two enclose one, but not the other (Fig. 2.4) .

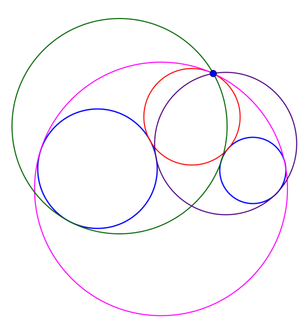

FIGURE 2.4. The four solutions to the CCP problem. One of the solution circles encloses both of the givens; one encloses neither of the givens and two enclose one of the givens, but not the other.

From experience, we know that in any problem involving points on circles, there will be equalities of distances, as well as relationships of rotation and reflection. We are using this problem as a vehicle for understanding how to make use of rotations and reflections, so for now we won't pay attention to the distance relationships. To help us identify the relationships of rotation and reflection that might exist in the CCP problem, let's follow a suggestion from [12], and ask ourselves, *"What can we introduce, that might help us understand the problem and frame its key features usefully?"* Reasonable candidates include some notation to identify important points in Fig. 2.4, along with a few segments to connect them. For now, we'll examine only the solution circle that encloses neither of the givens:

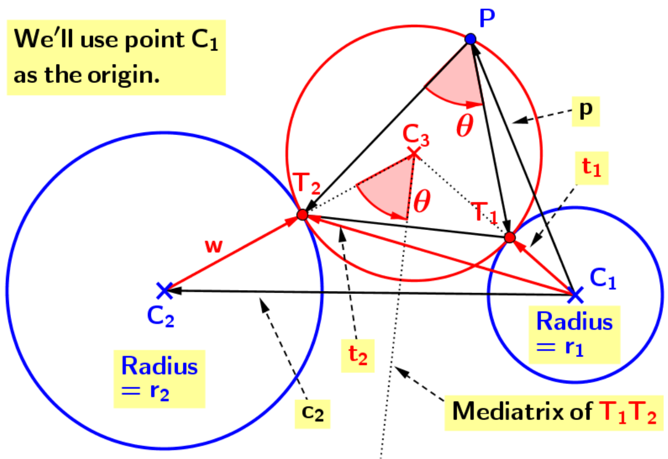

FIGURE 2.5. The diagram shown in Fig. 2.4, after adding elements that might help us identify useful relationships that involve rotations, reflections, and equalities of angles and distances.

We see now that the inscribed angle T_2PT_1 subtends the chord $\overline{T_2T_1}$, and that T_1 and $t2T_2$ are reflections of each other with respect to the bisector of the chord $\overline{T_2T_1}$, along which lies the center of the circle that we wish to find. That information should be useful. However, before we try to make use of it we'll try to identify relationships. Therefore, let's review our knowledge of the geometry of circles, and of how we have used that knowledge previously to solve possibly-related problems via GA.

3. REVIEW OF POSSIBLY RELEVANT KNOWLEDGE AND EXPERIENCE

In this section, we will review aspects of angle relationships in circles; the Geometric Algebra of reflections and rotations; and solutions to two other tangency problems.

3.1. **Review of angle relationships involving circles.** [13] and [14] provide an extensive background on equalities of angles in the context of tangencies. The relationships that appear to be most relevant at this point in our investigation of the problem are shown in Fig. 3.1.

3.2. **Review of Plane GA.** This review is not meant to be exhaustive; instead, we will review features of GA that we identified as potentially useful during our initial examination of the problem. We may need to revisit some of these features, and review others, as we formulate and implement a solution strategy.

We are especially interested in reviewing (1) how GA can be used to effect or express reflections and rotations that we might find useful; and (2) how to recognize when complex expressions simplify to reflections and rotations of geometric products, or of individual vectors. [14] gives additional details.

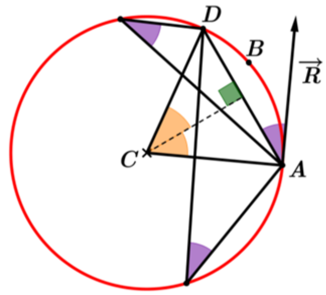

FIGURE 3.1. The angle relationships most likely to be useful to us for solving the CCP limiting case via rotations. All of the shaded angles are equal.

3.2.1. *Review of rotations.* One of the most important rotations—for our purposes—is the one that is produced when a vector is multiplied by the unit bivector, $\boldsymbol{i}$, for the plane: $\boldsymbol{vi}$ is $\boldsymbol{v}$'s 90-degree counter-clockwise rotation, while $\boldsymbol{iv}$ is $\boldsymbol{v}$'s 90-degree clockwise rotation.

Every geometric product $\boldsymbol{bc}$ is a rotation operator, whether we use it as such or not:

$$\boldsymbol{bc} = \|\boldsymbol{b}\|\|\boldsymbol{c}\|e^{\theta \boldsymbol{i}}.$$

where θ is the angle of rotation from $\boldsymbol{b}$ to $\boldsymbol{c}$. From that equation, we obtain the identity

$$e^{\theta \boldsymbol{i}} = \frac{\boldsymbol{bc}}{\|\boldsymbol{b}\|\|\boldsymbol{c}\|} = \left[\frac{\boldsymbol{b}}{\|\hat{\boldsymbol{b}}\|}\right]\left[\frac{\boldsymbol{c}}{\|\hat{\boldsymbol{c}}\|}\right] = \hat{\boldsymbol{b}}\hat{\boldsymbol{c}}.$$

A useful corollary is that any product of an odd number of vectors evaluates to a vector, while the product of an odd number of vectors evaluates to the sum of a scalar and a bivector.

3.2.2. *Review of reflections.* In plane geometry, the product $\hat{\boldsymbol{v}}\boldsymbol{a}\hat{\boldsymbol{v}}$ is $\boldsymbol{a}$'s reflection with respect to $\hat{\boldsymbol{v}}$, and $\hat{\boldsymbol{v}}\boldsymbol{ab}\hat{\boldsymbol{v}}$ is $\boldsymbol{ba}$. More generally, $\boldsymbol{vav}$ is v^2 times $\boldsymbol{a}$'s reflection with respect to $\hat{\boldsymbol{v}}$, and $\boldsymbol{vabv}$ is $v^2\boldsymbol{ba}$.

3.3. **Review of two problems that might be relevant.** We review these two problems to refresh our memories about details of their solutions, and to help us make useful conjectures about the form that the CCP problem's solutions might take take.

3.3.1. *Problem 1: "Given two coplanar circles, with a point Q on one of them, construct the circles that are tangent to both of the given circles, with point Q as one of the points of tangency"*. (Fig. 3.2). The solution to this problem should be useful to us because the points of tangency with the given circles in the CCP problem bear the same relationship to each other as do Q and T in Fig. 3.2.

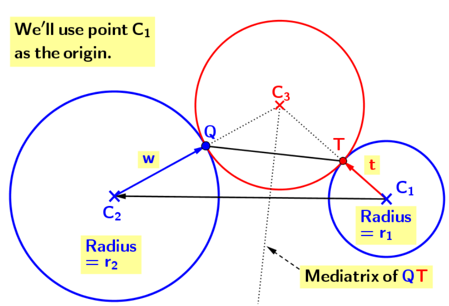

FIGURE 3.2. Diagram for Problem 1: *"Given two coplanar circles, with a point Q on one of them, construct the circles that are tangent to both of the given circles, with point Q as one of the points of tangency."*

Several solutions that use rotations are given by [14], but here we will use reflections. The triangle TQC_3 is isosceles, so $\hat{\boldsymbol{t}}$ is the reflection of $\hat{\boldsymbol{w}}$ with respect to the mediatrix of segment $\overline{QT}$. In order to make use of that fact, we need to express the direction of that mediatrix as a vector written in terms of known quantities. We can do so by constructing another isosceles triangle (C_1SC_3) that has the same mediatrix (Fig. 3.3).

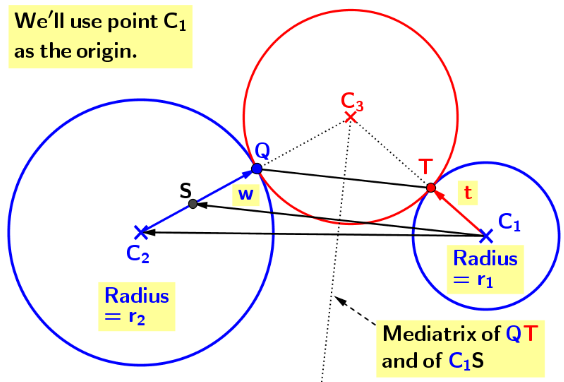

FIGURE 3.3. Adding segment $\overline{C_1S}$ to Fig. 3.2 to produce a new isosceles triangle with the same mediatrix as $\overline{QT}$.

The vector from C_1 to S is $\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}$, so the direction of the mediatrix of $\overline{QT}$ is the vector $[\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}]\,\boldsymbol{i}$. The unit vector with that direction is $\dfrac{[\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}]\,\boldsymbol{i}}{\|\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}\|}$. Therefore, to express $\hat{\boldsymbol{t}}$ as the reflection of $\hat{\boldsymbol{w}}$ with respect to the mediatrix, we write

$$
\begin{aligned}
\hat{\boldsymbol{t}} &= \left[\frac{[\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}]\,\boldsymbol{i}}{\|\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}\|}\right][\hat{\boldsymbol{w}}]\left[\frac{[\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}]\,\boldsymbol{i}}{\|\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}\|}\right] \\
&= \frac{\{[\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}]\,\boldsymbol{i}\}\,[\hat{\boldsymbol{w}}]\,\{[\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}]\,\boldsymbol{i}\}}{[\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}]^2} \\
&= \frac{[\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}]\,[\hat{\boldsymbol{w}}]\,[\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}]\,\boldsymbol{i}\boldsymbol{i}}{[\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}]^2},
\end{aligned}
$$

from which

$$
\boldsymbol{t}\ (= r_1\hat{\boldsymbol{t}}) = -r_1\left\{\frac{[\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}]\,[\hat{\boldsymbol{w}}]\,[\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}]}{[\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}]^2}\right\}. \tag{3.1}
$$

The geometric interpretation of that result is that $\hat{\boldsymbol{t}}$ and $-\hat{\boldsymbol{w}}$ are reflections of each other with respect to the vector $\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}$. After expanding and rearranging the numerator and denominator of (3.1), then using $\boldsymbol{w} = r_2\hat{\boldsymbol{w}}$, we obtain

$$
\boldsymbol{t} = r_1\left\{\frac{\left[{c_2}^2 - (r_2 - r_1)^2\right]\boldsymbol{w} - 2\,[\boldsymbol{c}_2\cdot\boldsymbol{w} + r_2\,(r_2 - r_1)]\,\boldsymbol{c}_2}{r_2{c_2}^2 + 2\,(r_2 - r_1)\,\boldsymbol{c}_2\cdot\boldsymbol{w} + r_2\,(r_2 - r_1)^2}\right\}. \tag{3.2}
$$

Because we still don't know how we will attempt to solve the CCP problem, we should keep our minds open as to which of (3.1) and (3.2) may be most useful to us.

3.3.2. *Given a circle and points A and B outside of it, construct the circles that are tangent to the given circle, and pass through A and B.* (Fig. 3.4) This problem is known as the "circle-point-point (CPP) limiting case of the Problem of Apollonius.

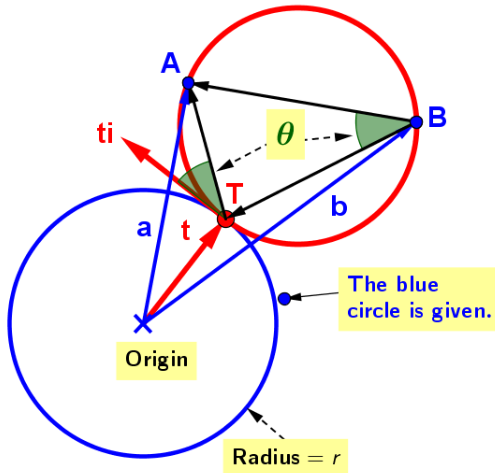

FIGURE 3.4. Diagram of the CPP limiting case of the Problem of Apollonius: *Given a circle and points A and B outside of it, construct the circles that are tangent to the given circle, and pass through A and B*. The red circle is the solution circle. Vectors and other elements shown are used in the solution.

Its GA solution, given in [14], may be useful to us because A and B are analogous to the point P in the CPP problem, in that they lie on the solution circle, while the unknown point T in Fig. 3.4 is analogous to either of the unknown points of tangency in the CCP problem.

To solve the CPP problem, we begin by equating two expressions for $e^{\theta \boldsymbol{i}}$. We'll use the fact that $\|\boldsymbol{t}\boldsymbol{i}\| = \|\boldsymbol{t}\| = r$:

$$\underbrace{\frac{(\boldsymbol{a}-\boldsymbol{b})(\boldsymbol{t}-\boldsymbol{b})}{\|\boldsymbol{a}-\boldsymbol{b}\|\|\boldsymbol{t}-\boldsymbol{b}\|} = \frac{(\boldsymbol{a}-\boldsymbol{t})\boldsymbol{t}\boldsymbol{i}}{\|\boldsymbol{a}-\boldsymbol{t}\|r}}_{\text{Both sides are expressions for } e^{\theta \boldsymbol{i}}}. \tag{3.3}$$

Next, we left-mutiply both sides by $(\boldsymbol{a}-\boldsymbol{t})$, and then by $\boldsymbol{t}$, to form an equation in which the left-hand side is a product of vectors, and the right-hand side is a bivector:

$$\boldsymbol{t}(\boldsymbol{a}-\boldsymbol{t})(\boldsymbol{a}-\boldsymbol{b})(\boldsymbol{t}-\boldsymbol{b}) = r\|\boldsymbol{a}-\boldsymbol{b}\|\|\boldsymbol{t}-\boldsymbol{b}\|\|\boldsymbol{a}-\boldsymbol{t}\|\boldsymbol{i}. \tag{3.4}$$

Why did we form an equation in which one side is a bivector? Because according the postulates for equality of multivectors, any two multivectors $\mathscr{M}$ and $\mathscr{N}$ are equal if and only if all of their respective parts are equal. In 2D, the product of an even number of vectors (like the left-hand side of (3.4)) must evaluate to the sum of a scalar and a multiple of the unit bivector $\boldsymbol{i}$. Because the right-hand side of (3.4) is a bivector, the scalar part of the left-hand side is zero. Therefore,

$$\langle \boldsymbol{t}(\boldsymbol{a}-\boldsymbol{t})(\boldsymbol{a}-\boldsymbol{b})(\boldsymbol{t}-\boldsymbol{b})\rangle_0 = 0. \tag{3.5}$$

Now here is where our knowledge of reflections can save us some work. Examining (3.5), we see the sequence $\boldsymbol{t}(\boldsymbol{a}-\boldsymbol{t})(\boldsymbol{a}-\boldsymbol{b})\boldsymbol{t}$, which from 3.2.2 is t^2 times the reflection of the product $(\boldsymbol{a}-\boldsymbol{t})(\boldsymbol{a}-\boldsymbol{b})$ with respect to $\hat{\boldsymbol{t}}$. Because $t^2 = r^2$, $\boldsymbol{t}(\boldsymbol{a}-\boldsymbol{t})(\boldsymbol{a}-\boldsymbol{b})\boldsymbol{t}$ simplifies to $r^2(\boldsymbol{a}-\boldsymbol{b})(\boldsymbol{a}-\boldsymbol{t})$. To take advantage of that simplification, we expand the left-hand side of (3.5) in a way that maintains that sequence intact:

$$\begin{aligned}
&\langle \boldsymbol{t}\,(\boldsymbol{a}-\boldsymbol{t})\,(\boldsymbol{a}-\boldsymbol{b})\,(\boldsymbol{t}-\boldsymbol{b})\rangle_0 = 0\\
&\langle \boldsymbol{t}\,(\boldsymbol{a}-\boldsymbol{t})\,(\boldsymbol{a}-\boldsymbol{b})\,\boldsymbol{t} - \boldsymbol{t}\,(\boldsymbol{a}-\boldsymbol{t})\,(\boldsymbol{a}-\boldsymbol{b})\,\boldsymbol{b}\rangle_0 = 0\\
&\langle r^2\,(\boldsymbol{a}-\boldsymbol{b})\,(\boldsymbol{a}-\boldsymbol{t}) - a^2\boldsymbol{t}\boldsymbol{b} + b^2\boldsymbol{t}\boldsymbol{a} + t^2\boldsymbol{a}\boldsymbol{b} - t^2b^2\rangle_0 = 0\\
&r^2\,(\boldsymbol{a}-\boldsymbol{b})\cdot(\boldsymbol{a}-\boldsymbol{t}) - a^2\boldsymbol{t}\cdot\boldsymbol{b} + b^2\boldsymbol{t}\cdot\boldsymbol{a} + r^2\boldsymbol{a}\cdot\boldsymbol{b} - r^2b^2 = 0,
\end{aligned} \tag{3.6}$$

which works out to

$$\boldsymbol{t}\cdot\left[\left(b^2-r^2\right)\boldsymbol{a} - \left(a^2-r^2\right)\boldsymbol{b}\right] = r^2\left(b^2-a^2\right). \tag{3.7}$$

As shown by analyses in [14], the geometric interpretation of that result is that there are two solution circles, whose points of tangency are reflections of each other with respect to the vector $\left(b^2-r^2\right)\boldsymbol{a} - \left(a^2-r^2\right)\boldsymbol{b}$ (Fig. 3.5).

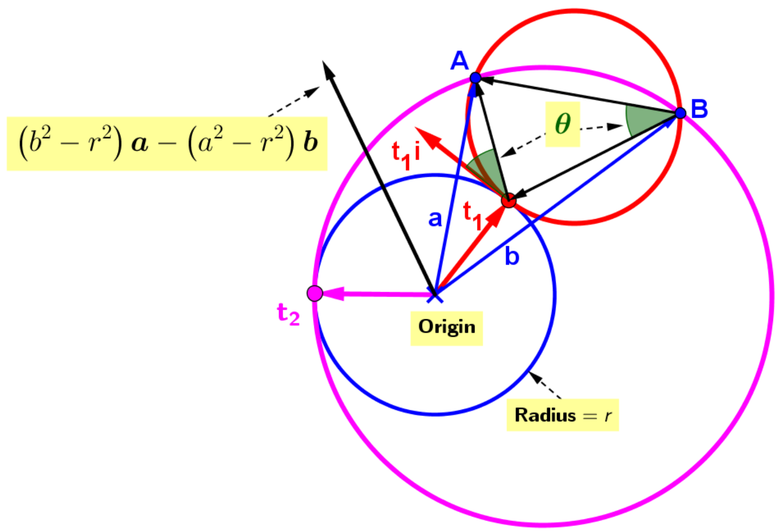

FIGURE 3.5. The two solutions to the CPP limiting case. The solution circles' points of tangency are symmetric with respect to the vector $\left(b^2-r^2\right)\boldsymbol{a} - \left(a^2-r^2\right)\boldsymbol{b}$

.

3.3.3. *What we have learned from our review of the solutions to these two problems.* Perhaps the most-important thing we have learned is the importance of recognizing and utilizing the interplay between rotations and reflections in tangency problems. In Problem 1, which we solved by using a reflection (3.3.1), we obtained an expression for the axis of reflection by using the unit bivector $\boldsymbol{i}$ as a rotation operator. We solved Problem 2 by using rotations, but we saved ourselves some work by recognizing that a product of four vectors in the resulting equation represented a scalar multiple of a simple geometric product.

We also learned that the solution to a tangency problem may take the form of the projection, upon a known vector, of the unknown vector for which we are trying to solve. Additional information (e.g., the radii of the given circles) would then be needed to identify the unknown vector or vectors uniquely.

4. SOLVING THE CCP PROBLEM

4.1. **Formulating a strategy.** For simplicity, we will limit ourselves for now to the solution circle that encloses neither of the given ones. Let's look again at our earlier diagram of that solution (Fig. 2.5).

Both of the problems that we reviewed in 3.3 were solved by identifying the point of tangency between the solution circle and a given circle. If we attempt to solve the CCP problem in the same way, then for each

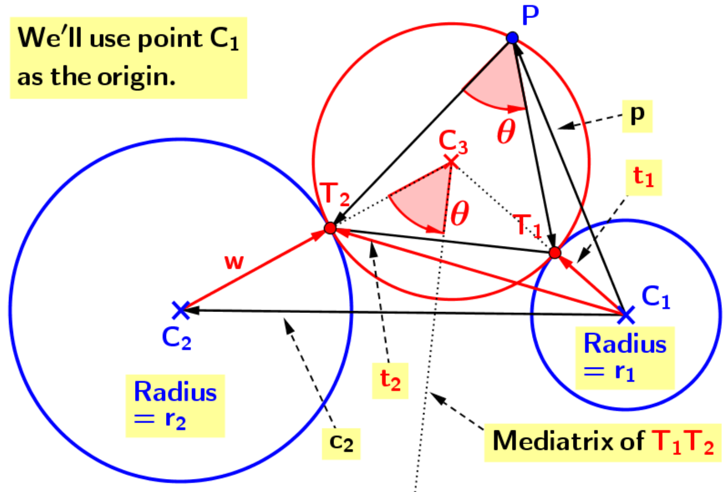

Fig. 2.5 (repeated for the reader's convenience). The diagram shown in Fig. 2.4, after adding elements that might help us identify useful relationships that involve rotations, reflections, and equalities of angles and distances.

solution circle there are two unknowns (the two points of tangency). To identify those points uniquely, we will need two independent equations that express the geometric relationships between them.

One of the necessary equations derives from a relationship that we noted in 2: the unknown points of tangency T_1 and T_2 in the CCP problem are related by reflection in the same way that the points Q and T were in the first problem that we reviewed (3.3.1). Specifically, T_1 and T_2 are reflections of each other with respect to the mediatrix of the chord $\overline{T_1T_2}$. Therefore, we can use the results from 3.3.1 directly, to write

$$\boldsymbol{t}_1 = -r_1 \left\{ \frac{[\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}]\,[\hat{\boldsymbol{w}}]\,[\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}]}{[\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}]^2} \right\} \tag{4.1}$$

and

$$\boldsymbol{t}_1 = r_1 \left\{ \frac{\left[c_2{}^2 - (r_2 - r_1)^2\right]\boldsymbol{w} - 2\,[\boldsymbol{c}_2\cdot\boldsymbol{w} + r_2\,(r_2 - r_1)]\,\boldsymbol{c}_2}{r_2 c_2{}^2 + 2\,(r_2 - r_1)\,\boldsymbol{c}_2\cdot\boldsymbol{w} + r_2\,(r_2 - r_1)^2} \right\}. \tag{4.2}$$

These two equations are not independent, because the second was derived from the first.

Now, how might we find a second equation that is independent of (4.1) and (4.2)? When we reviewed the CPP problem (), we noted that in each of the CCP problem's solution circles, the given point P is the vertex of an inscribed angle that subtends the chord $\overline{T_1T_2}$. That relationship between the points of tangency is independent of the reflection relationship between them, and can therefore give us the second equation that we need. Actually, we could write many "second" equations based upon that relationship. But which of them will be most useful to us?

One hint (which the author was slow to recognize) is that the angle T_1PT_2 is equal to the angle between the ray $\overrightarrow{C_3T_2}$ and the vector that gives the direction of the mediatrix of $\overline{T_1T_2}$. That fact enables us to relate T_1 and T_2 via a rotation, using the same vector that we used to relate them via reflection. Our review of previously-solved problems showed us that the interplay between rotations and reflections can have happy results; therefore, we have reason to choose, as our second equation, one that equates expressions for rotations through the angles θ:

$$\left[\frac{\boldsymbol{t}_2-\boldsymbol{p}}{\|\boldsymbol{t}_2-\boldsymbol{p}\|}\right]\left[\frac{\boldsymbol{t}_1-\boldsymbol{p}}{\|\boldsymbol{t}_1-\boldsymbol{p}\|}\right]=-\hat{\boldsymbol{w}}\left[\frac{[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}]\,\boldsymbol{i}}{\|\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}\|}\right], \tag{4.3}$$

in which $t_2=\boldsymbol{c}_2+r_2\hat{\boldsymbol{w}}$.

Thus, our strategy for solving the CPP problem will be to substitute either (4.1) or (4.2) for t_1 in (4.3), then solve for $\hat{\boldsymbol{w}}$.

4.1.1. *Finding the solution circle that encloses neither of the givens.* Examining (4.3), we can see its structural similarity to 3.3 in the solution of the CPP problem. So, let's follow the same route that we reviewed there. First, we'll multiply the right-hand side by $\hat{\boldsymbol{w}}$, and then by $\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}$, to form an equation in which the right-hand side is a bivector. Then, we'll set the scalar part of the right-hand side equal to zero:

$$\begin{aligned}&[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}]\,[\hat{\boldsymbol{w}}]\,[\boldsymbol{t}_2-\boldsymbol{p}]\,[\boldsymbol{t}_1-\boldsymbol{p}]=-\|\boldsymbol{t}_2-\boldsymbol{p}\|\|\boldsymbol{t}_1-\boldsymbol{p}\|\|\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}\|\boldsymbol{i}\\ \therefore\quad &\langle[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}]\,[\hat{\boldsymbol{w}}]\,[\boldsymbol{t}_2-\boldsymbol{p}]\,[\boldsymbol{t}_1-\boldsymbol{p}]\rangle_0=0.\end{aligned}$$

To get an idea as to how to proceed, let's make our substitutions for $\boldsymbol{t}_1$. Using the expression for $\boldsymbol{t}_1$ given by (4.1), we have

$$\langle[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}]\,[\hat{\boldsymbol{w}}]\,[\boldsymbol{t}_2-\boldsymbol{p}]\left[-r_1\left\{\frac{[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}]\,[\hat{\boldsymbol{w}}]\,[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}]}{[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}]^2}\right\}-\boldsymbol{p}\right]\rangle_0=0, \tag{4.4}$$

while by using the expression from (4.2), we find that

$$\langle[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}]\,[\hat{\boldsymbol{w}}]\,[\boldsymbol{t}_2-\boldsymbol{p}]\left[r_1\left\{\frac{\left[{c_2}^2-(r_2-r_1)^2\right]\boldsymbol{w}-2\,[\boldsymbol{c}_2\cdot\boldsymbol{w}+r_2\,(r_2-r_1)]\,\boldsymbol{c}_2}{r_2{c_2}^2+2\,(r_2-r_1)\,\boldsymbol{c}_2\cdot\boldsymbol{w}+r_2\,(r_2-r_1)^2}\right\}-\boldsymbol{p}\right]\rangle_0=0. \tag{4.5}$$

Although neither of the equations that we've just obtained looks very appealing, (4.5) looks less forbidding because $\boldsymbol{t}_1$ has been replaced by a linear combination of $\boldsymbol{w}$ and $\boldsymbol{t}_2$, rather than by a product of three vectors as in (4.4). However, let's not discard (4.4) without making a searching examination of its structure. In our minds, at least, we should clarify that structure by moving the scalar factor r_1 to the end of the product in which it occurs, then multiply both sides of the equation by -1 to "get rid of the minus sign and the subtraction". The result would be

$$\langle[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}]\,[\hat{\boldsymbol{w}}]\,[\boldsymbol{t}_2-\boldsymbol{p}]\left\{\frac{[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}]\,[\hat{\boldsymbol{w}}]\,[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}]\,r_1}{[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}]^2}+\boldsymbol{p}\right\}\rangle_0=0. \tag{4.6}$$

Now, starting from left to right, and "reading through" the square bracket on the left-hand end of the fraction, we see the sequence of factors

$$[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}]\,[\hat{\boldsymbol{w}}]\,[\boldsymbol{t}_2-\boldsymbol{p}]\,[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}]\,.$$

That sequence has the structure $\boldsymbol{vabv}$, which reduces to $v^2\boldsymbol{ba}$ (3.2.2). Because our "$\boldsymbol{v}$" is $\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}$, v^2 would cancel with the factor $[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}]^2$ in the denominator. That possibility might motivate us to examine (4.6) a bit further. Let's start by multiplying both sides by $[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}]^2$ so that we won't be dealing with fractions. That, too, is a way to "get rid of" the potentially bothersome denominator:

(4.7)
$$\langle[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}]\,[\hat{\boldsymbol{w}}]\,[\boldsymbol{t}_2-\boldsymbol{p}]\left\{[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}]\,[\hat{\boldsymbol{w}}]\,[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}]\,r_1+[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}]^2\,\boldsymbol{p}\right\}\rangle_0=0.$$

Now, let's read from left to right again. "Reading through the {", we see the following sequence of vectors:

$$[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}]\,[\hat{\boldsymbol{w}}]\,[\boldsymbol{t}_2-\boldsymbol{p}]\,[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}]\,[\hat{\boldsymbol{w}}]\,[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}]\,r_1\,.$$

That sequence has a striking overarching structure: $\boldsymbol{vabvav}$. Let's pause to make that explicit:

$$\underbrace{[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}]}_{\boldsymbol{v}}\underbrace{[\hat{\boldsymbol{w}}]}_{\boldsymbol{a}}\underbrace{[\boldsymbol{t}_2-\boldsymbol{p}]}_{\boldsymbol{b}}\underbrace{[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}]}_{\boldsymbol{v}}\underbrace{[\hat{\boldsymbol{w}}]}_{\boldsymbol{a}}\underbrace{[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}]}_{\boldsymbol{v}}.$$

We can simplify that sequence by conceiving it as a set of nested reflections:

$$\begin{aligned}\boldsymbol{vabvav} &= \boldsymbol{v}\,[\boldsymbol{a}\,(\boldsymbol{bv})\,\boldsymbol{a}]\,\boldsymbol{v}\\ &= \boldsymbol{v}\left[a^2\boldsymbol{vb}\right]\boldsymbol{v}\\ &= a^2\boldsymbol{v}\,[\boldsymbol{vb}]\,\boldsymbol{v}\\ &= a^2v^2\boldsymbol{bv}\\ &= [\hat{\boldsymbol{w}}]^2\,[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}]^2\,[\boldsymbol{t}_2-\boldsymbol{p}]\,[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}]\\ &= [\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}]^2\,[\boldsymbol{t}_2-\boldsymbol{p}]\,[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}]\,.\end{aligned} \tag{4.8}$$

That result seems useful, clearly: it will allow us to eliminate the scalar factor $[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}]^2$ in (4.7). Therefore, we would like to expand the left-hand side in a way that keeps the $\boldsymbol{vabvav}$ sequence intact. If we were working on a whiteboard, we could do that easily, then substitute the result from (4.8) for the sequence

$$[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}]\,[\hat{\boldsymbol{w}}]\,[\boldsymbol{t}_2-\boldsymbol{p}]\,[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}]\,[\hat{\boldsymbol{w}}]\,[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}]\,r_1\,,$$

after which we'd cancel the factor $[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}]^2$. However, we're working withing the size limits of a printed page, so we'll just present the equation that would result from that work:

$$\langle r_1\,[\boldsymbol{t}_2-\boldsymbol{p}]\,[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}]+[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}]\,[\hat{\boldsymbol{w}}]\,[\boldsymbol{t}_2-\boldsymbol{p}]\rangle_0=0.$$

This may be a good time to make the substitution $\boldsymbol{t}_2=\boldsymbol{c}_2+r_2\hat{\boldsymbol{w}}$:

$$\langle r_1\,[\boldsymbol{c}_2+r_2\hat{\boldsymbol{w}}-\boldsymbol{p}]\,[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}]+[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}]\,[\hat{\boldsymbol{w}}]\,[\boldsymbol{c}_2+r_2\hat{\boldsymbol{w}}-\boldsymbol{p}]\rangle_0=0.$$

There may be a clever way to expand the left-hand side, but we'll settle now for "brute force and ignorance":

$$\begin{aligned}&\langle r_1{c_2}^2+r_1\,(r_2-r_1)\,\boldsymbol{c}_2\hat{\boldsymbol{w}}+r_1r_2\hat{\boldsymbol{w}}\boldsymbol{c}_2+r_1r_2\hat{\boldsymbol{w}}\,(r_2-r_1)\,\hat{\boldsymbol{w}}\\ &-r_1\boldsymbol{pc}_2-r_1\,(r_2-r_1)\,\boldsymbol{p}\hat{\boldsymbol{w}}+\boldsymbol{c}_2\hat{\boldsymbol{w}}\boldsymbol{c}_2\boldsymbol{p}+\boldsymbol{c}_2\hat{\boldsymbol{w}}r_2\hat{\boldsymbol{w}}\boldsymbol{p}-\boldsymbol{c}_2\hat{\boldsymbol{w}}p^2\\ &+(r_2-r_1)\,\hat{\boldsymbol{w}}\hat{\boldsymbol{w}}\boldsymbol{c}_2\boldsymbol{p}+r_2\,(r_2-r_1)\,\hat{\boldsymbol{w}}\hat{\boldsymbol{w}}\hat{\boldsymbol{w}}\boldsymbol{p}-(r_2-r_1)\,\hat{\boldsymbol{w}}\hat{\boldsymbol{w}}p^2\rangle_0\\ &=0.\end{aligned}$$

The scalar part of the left-hand side is the sum of the scalar parts of the terms therein. The scalar part of $\boldsymbol{c}_2\hat{\boldsymbol{w}}\boldsymbol{c}_2\boldsymbol{p}$ can be found using he identity $\boldsymbol{ab}=2\boldsymbol{a}\cdot\boldsymbol{b}-\boldsymbol{ba}$, to transfrom $\langle\boldsymbol{c}_2\hat{\boldsymbol{w}}\boldsymbol{c}_2\boldsymbol{p}\rangle_0$ into $\langle(2\boldsymbol{c}_2\cdot\hat{\boldsymbol{w}}-\hat{\boldsymbol{w}}\boldsymbol{c}_2)\,\boldsymbol{c}_2\boldsymbol{p}\rangle_0$. Therefore,

$$\begin{aligned}&r_1{c_2}^2+r_1\,(r_2-r_1)\,\boldsymbol{c}_2\cdot\hat{\boldsymbol{w}}+r_1r_2\hat{\boldsymbol{w}}\cdot\boldsymbol{c}_2+r_1r_2\,(r_2-r_1)-r_1\boldsymbol{p}\cdot\boldsymbol{c}_2\\ &-r_1\,(r_2-r_1)\,\boldsymbol{p}\cdot\hat{\boldsymbol{w}}+2\,(\boldsymbol{c}_2\cdot\boldsymbol{p})\,(\boldsymbol{c}_2\cdot\hat{\boldsymbol{w}})-{c_2}^2\boldsymbol{p}\cdot\hat{\boldsymbol{w}}+r_2\boldsymbol{c}_2\cdot\boldsymbol{p}\\ &-p^2\boldsymbol{c}_2\cdot\hat{\boldsymbol{w}}+(r_2-r_1)\,\boldsymbol{c}_2\cdot\boldsymbol{p}+r_2\,(r_2-r_1)\,\hat{\boldsymbol{w}}\cdot\boldsymbol{p}-(r_2-r_1)\,p^2\\ &=0.\end{aligned}$$

The resemblance of that result to (3.6) from the CPP case is clear. To finish, we'll follow the procedure that we used there. First, we'll combine all of the terms that include $\hat{\boldsymbol{w}}$, into a dot product of $\hat{\boldsymbol{w}}$ with a linear combination of $\boldsymbol{p}$ and $\boldsymbol{c}_2$, to obtain

$$\begin{aligned}&\left\{\left[r_1\,(2r_2-r_1)+2\boldsymbol{c}_2\cdot\boldsymbol{p}-p^2\right]\boldsymbol{c}_2+\left[(r_2-r_1)^2-{c_2}^2\right]\boldsymbol{p}\right\}\cdot\hat{\boldsymbol{w}}\\ &=(r_2-r_1)\,p^2-r_1{c_2}^2-r_1r_2\,(r_2-r_1)-2\,(r_2-r_1)\,\boldsymbol{c}_2\cdot\boldsymbol{p}.\end{aligned}$$

We can put that result in a more-useful form by factoring, completing squares, and recalling that $\boldsymbol{w} = r_2\hat{\boldsymbol{w}}$:

$$\begin{aligned}&\left\{\left[(\boldsymbol{c}_2-\boldsymbol{p})^2 - c_2{}^2 + (r_2-r_1)^2 - r_2{}^2\right]\boldsymbol{c}_2 + \left[c_2{}^2 - (r_2-r_1)^2\right]\boldsymbol{p}\right\}\cdot\boldsymbol{w}\\ &= r_2{}^2\left[c_2{}^2 - \left(1-\frac{r_1}{r_2}\right)(\boldsymbol{c}_2-\boldsymbol{p})^2 + r_1(r_2-r_1)\right].\end{aligned} \tag{4.9}$$

Finally, we'll define

$$\left[(\boldsymbol{c}_2-\boldsymbol{p})^2 - c_2{}^2 + (r_2-r_1)^2 - r_2{}^2\right]\boldsymbol{c}_2 + \left[c_2{}^2 - (r_2-r_1)^2\right]\boldsymbol{p} = \boldsymbol{z},$$

and divide both sides of Eq. (4.9) by $|\boldsymbol{z}|$ to obtain

$$\boldsymbol{w}\cdot\hat{\boldsymbol{z}} = \frac{r_2{}^2\left[c_2{}^2 - \left(1-\frac{r_1}{r_2}\right)(\boldsymbol{c}_2-\boldsymbol{p})^2 + r_1(r_2-r_1)\right]}{\left|\left[(\boldsymbol{c}_2-\boldsymbol{p})^2 - c_2{}^2 + (r_2-r_1)^2 - r_2{}^2\right]\boldsymbol{c}_2 + \left[c_2{}^2 - (r_2-r_1)^2\right]\boldsymbol{p}\right|}. \tag{4.10}$$

The geometric interpretation of that result is that there are two vectors $\boldsymbol{w}$, both of which have the same projection upon $\hat{\boldsymbol{z}}$. Because $w^2 = r_2{}^2$ for both of them, their components perpendicular to $\hat{\boldsymbol{z}}$ are given by

$$\boldsymbol{w}_\perp = \pm\sqrt{r_2{}^2 - \left\{\frac{r_2{}^2\left[c_2{}^2 - \left(1-\frac{r_1}{r_2}\right)(\boldsymbol{c}_2-\boldsymbol{p})^2 + r_1(r_2-r_1)\right]}{\left|\left[(\boldsymbol{c}_2-\boldsymbol{p})^2 - c_2{}^2 + (r_2-r_1)^2 - r_2{}^2\right]\boldsymbol{c}_2 + \left[c_2{}^2 - (r_2-r_1)^2\right]\boldsymbol{p}\right|}\right\}^2}. \tag{4.11}$$

As shown in Fig. 4.1, we attempted to find the solution circle that encloses neither of the givens, and thereby also found the one that encloses both of them.

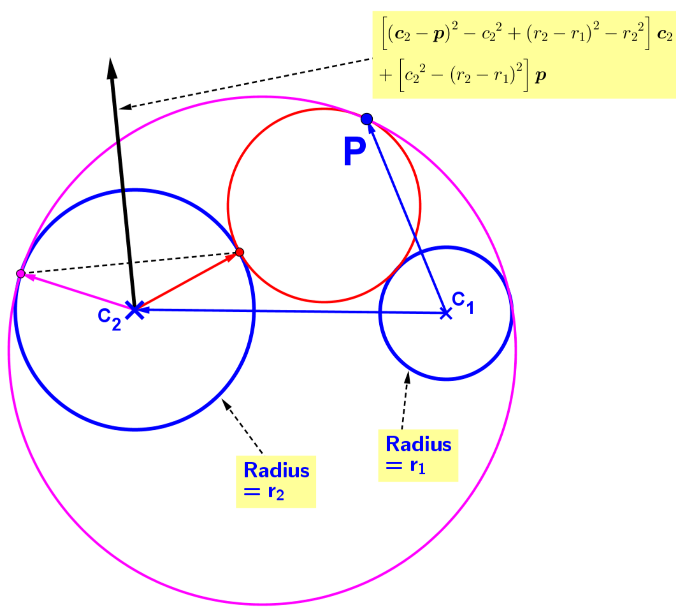

FIGURE 4.1. The solution circle that encloses both of the given circles, and the solution circle that encloses neither. The vectors to their points of tangency are reflections of each other with respect to the vector $\left[(\boldsymbol{c}_2-\boldsymbol{p})^2 - c_2{}^2 + (r_2-r_1)^2 - r_2{}^2\right]\boldsymbol{c}_2 + \left[c_2{}^2 - (r_2-r_1)^2\right]\boldsymbol{p}$.

4.1.2. *Finding the solution circles that enclose only one of the givens.* We've now identified the points where the two externally-tangent solution circles are tangent to the given circle that's centered on C_2. The points at which the other two solution circles (those that enclose one of the givens, and are tangent externally to the other) are tangent to the circle centered on C_2 can be found by using the same method. As indicated in Fig. 4.2, we'd start by noting that the vectors from C_1 and C_2 to their respective circles points of tangency are reflections of each other with respect to the vector $\boldsymbol{i}\left[\boldsymbol{c}_2+(r_1+r_2\hat{\boldsymbol{y}})\right]$:

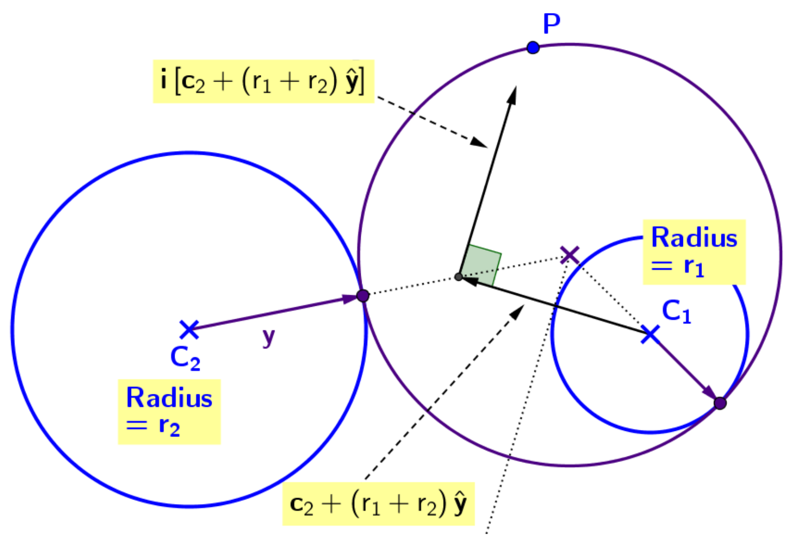

FIGURE 4.2. Identification of key features for the solution circles that enclose only one of the given circles.

The analog of Eq. (4.9) for this pair of solution circles is

$$\begin{aligned}&\left\{\left[(\boldsymbol{c}_2-\boldsymbol{p})^2-c_2{}^2+(r_1+r_2)^2-r_2{}^2\right]\boldsymbol{c}_2+\left[c_2{}^2-(r_1+r_2)^2\right]\boldsymbol{p}\right\}\cdot\boldsymbol{y}\\ &=r_2{}^2\left[c_2{}^2-\left(1+\frac{r_1}{r_2}\right)(\boldsymbol{c}_2-\boldsymbol{p})^2-r_1\left(r_2+r_1\right)\right].\end{aligned} \tag{4.12}$$

In Fig. 4.3, we see again that in attempting to identify one of the circles that encloses only one of the givens, we identified both.

4.2. **The four solutions to the CCP limiting case.** The four solution circles, along with the vectors to their points of tangency, are presented in Fig. 4.4.

5. CONCLUDING REMARKS

Solutions via rotations are aesthetically appealing, and show some of GA's most-important capabilities to good advantage. They are also a good opportunity to help students develop the "eyes" needed to translate geometry problems into GA expressions, and to simplify complex geometric products expeditiously.

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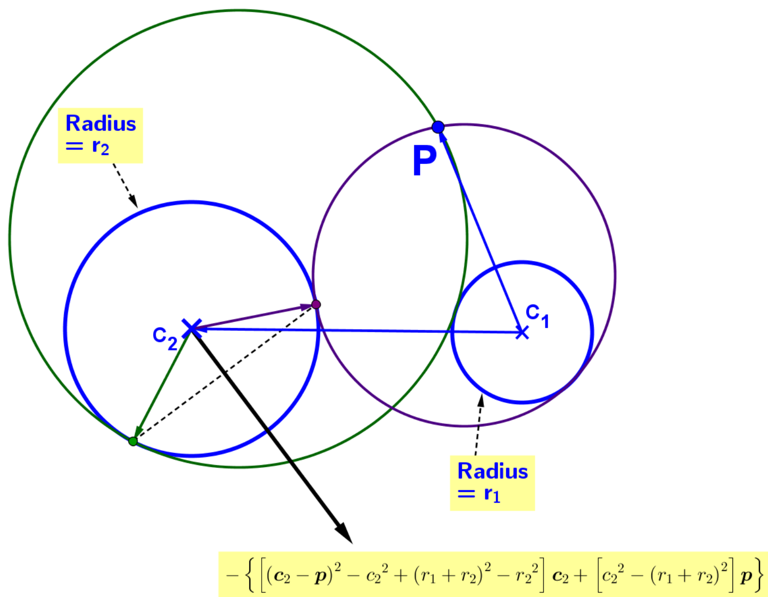

FIGURE 4.3. The two solution circles that enclose only one of the given circles. The vectors to their points of tangency are reflections of earch other with respect to the vector $\left[(\boldsymbol{c}_2-\boldsymbol{p})^2-c_2{}^2+(r_1+r_2)^2-r_2{}^2\right]\boldsymbol{c}_2+\left[c_2{}^2-(r_1+r_2)^2\right]\boldsymbol{p}$.

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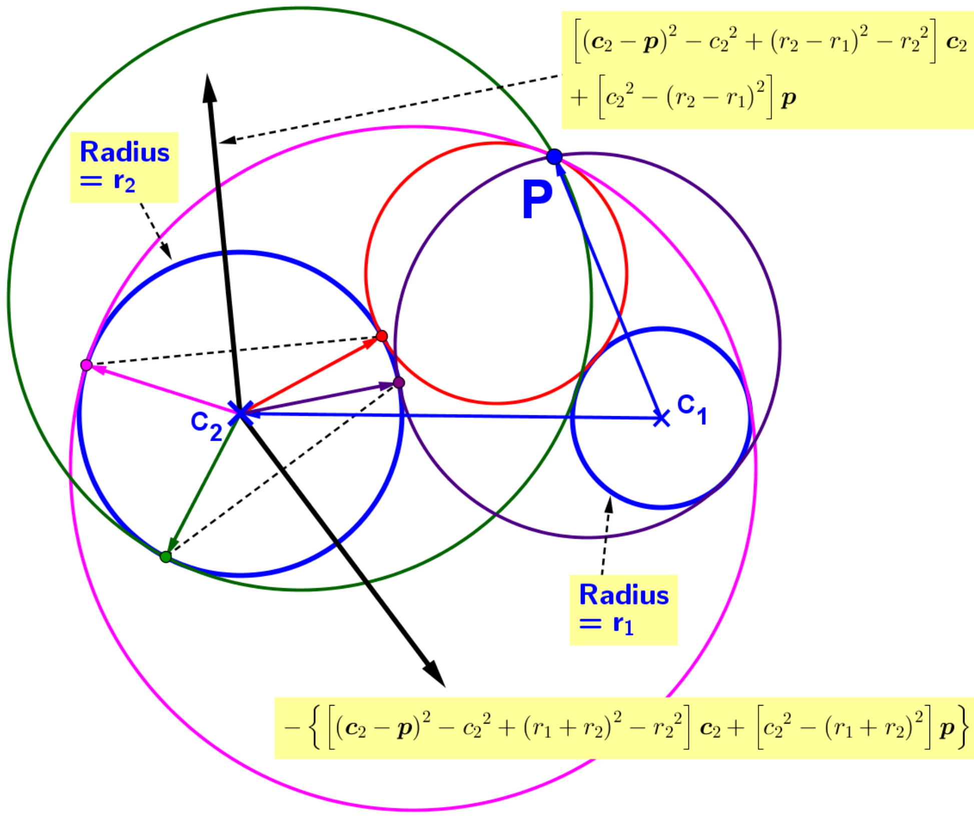

FIGURE 4.4. The four solution circles, along with the vectors to their points of tangency.