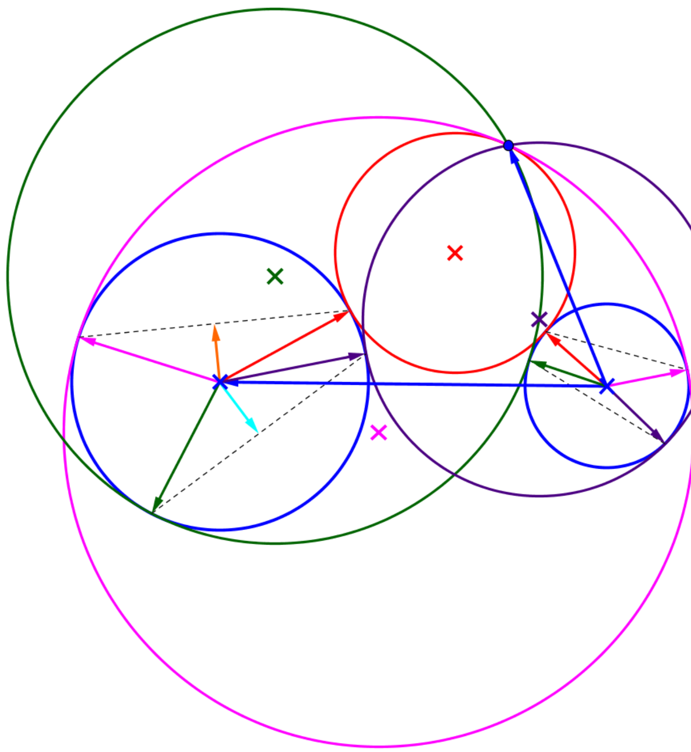
Solution of the "CCP" Case of the Problem of Apollonius

via Vector Rotations and Reflections using Geometric Algebra



Solution of the "CCP" Case of the Problem of Apollonius via Vector Rotations and Reflections using Geometric Algebra

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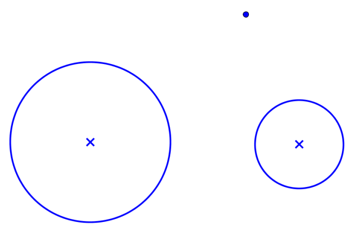
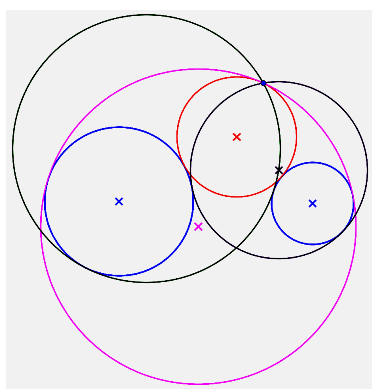
The "Problem of Apollonius", in plane geometry, is to construct all of circles that are tangent, simultaneously, to three given circles. As Viète showed, that problem can be solved by reducing it to the so-called "Circle-Circle-Point" (CCP) case:

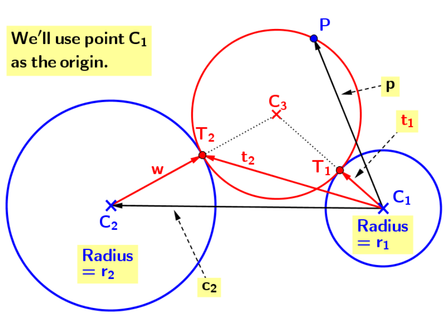
Given two circles and a point, all of them in the same plane, construct the circles that pass through the given point and are tangent to both of the given circles.



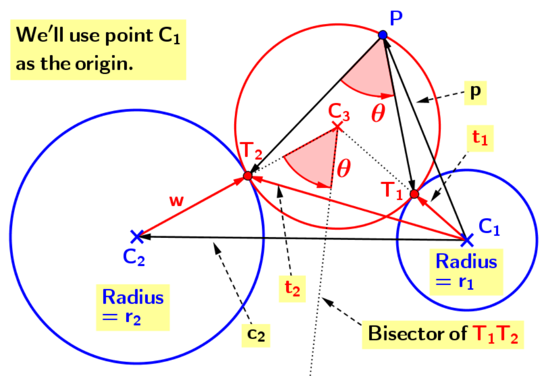
There are four solution circles, two of which are tangent externally to both of the given circles. Each of the other solution circles encloses one of the givens, but is tangent externally to the other:



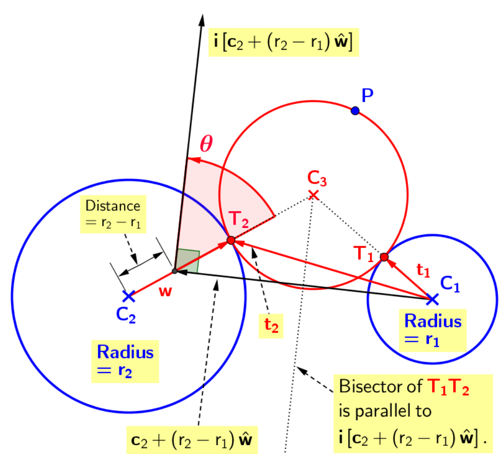
We'll begin by finding the red circle shown above. First, we'll label important points, vectors, and dimensions:



Our strategy will be to identify the vector $\boldsymbol{w}$, thereby dtermining the red circle: its center will be the intersection of the line C_2T_2 and the mediatrix of the segment T_2P_2. The two keys to our solution are that the angles labeled θ are equal,



and that the points of tangency T_1 and T_2 are reflections of each other with respect to the vector $\boldsymbol{i}\,[\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}]$.



More importantly, the vector $\boldsymbol{t}_1$ is the reflection of the vector $r_1\hat{\boldsymbol{w}}$ with respect to $\boldsymbol{i}\,[\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}]$. For any two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, the product $\boldsymbol{uvu}$ is u^2 times the reflection of $\boldsymbol{v}$ with respect to $\boldsymbol{u}$. Therefore,

$$\begin{aligned}
\boldsymbol{t}_1 &= \left\{\left[\frac{\boldsymbol{i}\,[\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}]}{|\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}|}\right]\right\}[r_1\hat{\boldsymbol{w}}]\left\{\left[\frac{\boldsymbol{i}\,[\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}]}{|\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}|}\right]\right\} \\
&= -r_1\left\{\frac{[\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}]\,\hat{\boldsymbol{w}}\,[\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}]}{[\boldsymbol{c}_2 + (r_2 - r_1)\,\hat{\boldsymbol{w}}]^2}\right\}. \qquad (1)
\end{aligned}$$

To make use of the equality of the angles θ, we write

$$\left[\frac{\boldsymbol{t}_2-\boldsymbol{p}}{|\boldsymbol{t}_2-\boldsymbol{p}|}\right]\left[\frac{\boldsymbol{t}_1-\boldsymbol{p}}{|\boldsymbol{t}_1-\boldsymbol{p}|}\right]=\hat{\boldsymbol{w}}\left[\frac{\boldsymbol{i}\left[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}\right]}{|\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}|}\right]\left(=e^{\theta\boldsymbol{i}}\right). \tag{2}$$

Next, in order to make use of postulates about equality of multivectors, we left-multiply both sides of Eq. (2) by $\hat{\boldsymbol{w}}$, and right-multiply by $\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}$, to obtain

$$\hat{\boldsymbol{w}}\left[\boldsymbol{t}_2-\boldsymbol{p}\right]\left[\boldsymbol{t}_1-\boldsymbol{p}\right]\left[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}\right]=\underbrace{|\boldsymbol{t}_2-\boldsymbol{p}|\,|\boldsymbol{t}_1-\boldsymbol{p}|\,|\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}|\,\boldsymbol{i}}_{A\ bivector};$$

$$\therefore\quad \langle\hat{\boldsymbol{w}}\left[\boldsymbol{t}_2-\boldsymbol{p}\right]\left[\boldsymbol{t}_1-\boldsymbol{p}\right]\left[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}\right]\rangle_0=0.$$

For convenience, we'll rewrite that result as

$$\langle\hat{\boldsymbol{w}}\left[\boldsymbol{t}_2-\boldsymbol{p}\right]\boldsymbol{t}_1\left[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}\right]\rangle_0-\langle\hat{\boldsymbol{w}}\left[\boldsymbol{t}_2-\boldsymbol{p}\right]\boldsymbol{p}\left[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}\right]\rangle_0=0. \tag{3}$$

Now, we'll treat each term of the left-hand side separately. First, we note that $\boldsymbol{t}_2=\boldsymbol{c}_2+r_2\hat{\boldsymbol{w}}$. Making that substitution, and using the expression derived for $\boldsymbol{t}_1$ in Eq. (1), the first term of Eq. (3) becomes

$$\begin{aligned}
&\langle\hat{\boldsymbol{w}}\left[\boldsymbol{c}_2+r_2\hat{\boldsymbol{w}}-\boldsymbol{p}\right]\left\{-r_1\frac{\left[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}\right]\hat{\boldsymbol{w}}\left[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}\right]}{\left[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}\right]^2}\right\}\left[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}\right]\rangle_0\\
&=-r_1\langle\hat{\boldsymbol{w}}\left[\boldsymbol{c}_2+r_2\hat{\boldsymbol{w}}-\boldsymbol{p}\right]\left[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}\right]\hat{\boldsymbol{w}}\rangle_0\\
&=-r_1\langle\left[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}\right]\left[\boldsymbol{c}_2+r_2\hat{\boldsymbol{w}}-\boldsymbol{p}\right]\rangle_0,
\end{aligned} \tag{4}$$

because for any three vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$, $\boldsymbol{abca}=a^2\boldsymbol{cb}$. After expansion and simplification, the expression in Eq. (4) becomes

$$-r_1\left(2r_2-r_1\right)\boldsymbol{c}_2\cdot\hat{\boldsymbol{w}}+r_1\left(r_2-r_1\right)\boldsymbol{p}\cdot\hat{\boldsymbol{w}}++r_1\boldsymbol{c}_2\cdot\boldsymbol{p}-r_1{c_2}^2-r_1r_2\left(r_2-r_1\right). \tag{5}$$

Turning now to the second term on the left-hand side of Eq. (3), and making the substitution $\boldsymbol{t}_2=\boldsymbol{c}_2+r_2\hat{\boldsymbol{w}}$,

$$\langle\hat{\boldsymbol{w}}\left[\boldsymbol{t}_2-\boldsymbol{p}\right]\boldsymbol{p}\left[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}\right]\rangle_0=\langle\hat{\boldsymbol{w}}\left[\boldsymbol{c}_2+r_2\hat{\boldsymbol{w}}-\boldsymbol{p}\right]\boldsymbol{p}\left[\boldsymbol{c}_2+(r_2-r_1)\,\hat{\boldsymbol{w}}\right]\rangle_0.$$

After expansion and simplification, that result becomes

$$\left[2\boldsymbol{c}_2\cdot\boldsymbol{p}-p^2\right]\boldsymbol{c}_2\cdot\hat{\boldsymbol{w}}+\left[r_2\left(r_2-r_1\right)-{c_2}^2\right]\boldsymbol{p}\cdot\hat{\boldsymbol{w}}+\left(2r_2-r_1\right)\boldsymbol{c}_2\cdot\boldsymbol{p}-\left(r_2-r_1\right)p^2. \tag{6}$$

Next, as indicated by Eq. (3), we subtract the expression in (6) from (5), and set the result equal to zero. After simplifying and rearranging, we find that

$$\begin{aligned}
&\left\{\left[r_1\left(2r_2-r_1\right)+2\boldsymbol{c}_2\cdot\boldsymbol{p}-p^2\right]\boldsymbol{c}_2+\left[\left(r_2-r_1\right)^2-{c_2}^2\right]\boldsymbol{p}\right\}\cdot\hat{\boldsymbol{w}}\\
&=\left(r_2-r_1\right)p^2-r_1{c_2}^2-r_1r_2\left(r_2-r_1\right)-2\left(r_2-r_1\right)\boldsymbol{c}_2\cdot\boldsymbol{p}.
\end{aligned}$$

We can put that result in a more-useful form by factoring, completing squares, and recalling that $\boldsymbol{w}=r_2\hat{\boldsymbol{w}}$:

$$\begin{aligned}
&\left\{\left[\left(\boldsymbol{c}_2-\boldsymbol{p}\right)^2-{c_2}^2+\left(r_2-r_1\right)^2-{r_2}^2\right]\boldsymbol{c}_2+\left[{c_2}^2-\left(r_2-r_1\right)^2\right]\boldsymbol{p}\right\}\cdot\boldsymbol{w}\\
&={r_2}^2\left[{c_2}^2-\left(1-\frac{r_1}{r_2}\right)\left(\boldsymbol{c}_2-\boldsymbol{p}\right)^2+r_1\left(r_2-r_1\right)\right].
\end{aligned} \tag{7}$$

Finally, we'll define

$$\left[(\boldsymbol{c}_2-\boldsymbol{p})^2-{c_2}^2+(r_2-r_1)^2-{r_2}^2\right]\boldsymbol{c}_2+\left[{c_2}^2-(r_2-r_1)^2\right]\boldsymbol{p}=\boldsymbol{z},$$

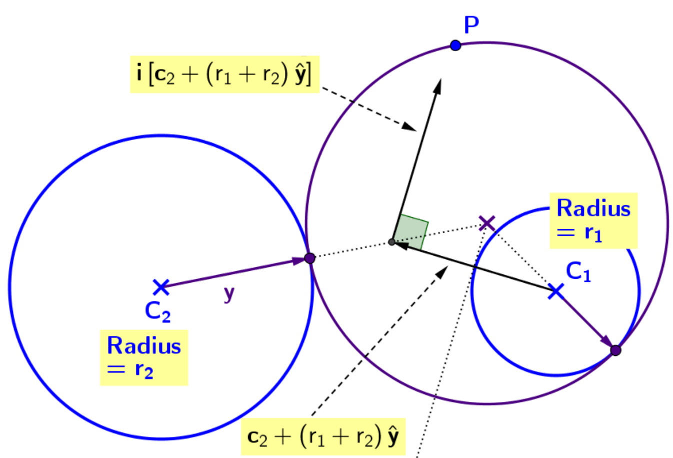
and divide both sides of Eq. (7) by $|\boldsymbol{z}|$ to obtain

$$\boldsymbol{w}\cdot\hat{\boldsymbol{z}}=\frac{{r_2}^2\left[{c_2}^2-\left(1-\frac{r_1}{r_2}\right)(\boldsymbol{c}_2-\boldsymbol{p})^2+r_1(r_2-r_1)\right]}{\left|\left[(\boldsymbol{c}_2-\boldsymbol{p})^2-{c_2}^2+(r_2-r_1)^2-{r_2}^2\right]\boldsymbol{c}_2+\left[{c_2}^2-(r_2-r_1)^2\right]\boldsymbol{p}\right|}. \tag{8}$$

The geometric interpretation of that result is that there are two vectors $\boldsymbol{w}$, both of which have the same projection upon $\hat{\boldsymbol{z}}$. Because $w^2={r_2}^2$ for both of them, their components perpendicular to $\hat{\boldsymbol{z}}$ are given by

$$\boldsymbol{w}_\perp=\pm\sqrt{{r_2}^2-\left\{\frac{{r_2}^2\left[{c_2}^2-\left(1-\frac{r_1}{r_2}\right)(\boldsymbol{c}_2-\boldsymbol{p})^2+r_1(r_2-r_1)\right]}{\left|\left[(\boldsymbol{c}_2-\boldsymbol{p})^2-{c_2}^2+(r_2-r_1)^2-{r_2}^2\right]\boldsymbol{c}_2+\left[{c_2}^2-(r_2-r_1)^2\right]\boldsymbol{p}\right|}\right\}^2}. \tag{9}$$

We've now identified the points where the two externally-tangent solution circles are tangent to the given circle that's centered on C_2. The points at which the other two solution circles (those that enclose one of the givens, and are tangent externally to the other) are tangent to the circle centered on C_2 can be found by using the same method. As indicafted in the figure shown below, ee'd start by noting that the vectors from C_1 and C_2 to their respective circles points of tangency are reflections of each other with respect to the vector $\boldsymbol{i}\left[\boldsymbol{c}_2+(r_1+r_2\hat{\boldsymbol{y}})\right]$:



The analog of Eq. (7) for this pair of solution circles is

$$\left\{\left[(\boldsymbol{c}_2-\boldsymbol{p})^2-{c_2}^2+(r_1+r_2)^2-{r_2}^2\right]\boldsymbol{c}_2+\left[{c_2}^2-(r_1-r_2)^2\right]\boldsymbol{p}\right\}\cdot\boldsymbol{y}$$
$$={r_2}^2\left[{c_2}^2-\left(1+\frac{r_1}{r_2}\right)(\boldsymbol{c}_2-\boldsymbol{p})^2-r_1(r_2+r_1)\right]. \tag{10}$$