

Euler's proof of Fermat's Last Theorem for $n = 3$ is incorrect

Nguyen Van Quang
Hue - Vietnam, 05 - 2016

Abstract

We have spotted an error of Euler's proof, so that the used infinite descent is impossible in his proof.

1 Euler's proof for $n = 3$

First, we rewrite a proof for $n = 3$, which was proven by Euler in 1770 as follows:

As Fermat did for the case $n = 4$, Euler used the technique of infinite descent. The proof assumes a solution (x, y, z) to the equation $x^3 + y^3 + z^3 = 0$, where the three non-zero integers x, y, z are pairwise coprime and not all positive. One of three must be even, whereas the other two are odd. Without loss of generality, z may be assumed to be even.

Since x and y are both odd, they cannot be equal, if $x = y$, then $2x^3 = -z^3$, which implies that x is even, a contradiction.

Since x and y are both odd, their sum and difference are both even numbers.

$$2u = x + y$$

$$2v = x - y$$

Where the non-zero integers u and v are coprime and have different parity (one is even, the other odd). Since $x = u + v$ and $y = u - v$, it follows that

$$-z^3 = (u + v)^3 + (u - v)^3 = 2u(u^2 + 3v^2)$$

Since u and v have opposite parity, $u^2 + 3v^2$ is always an odd number. Therefore, since z even, u is even and v is odd. Since u and v are coprime, the greatest common divisor of $2u$ and $u^2 + 3v^2$ is either 1 (case A) or 3 (case B).

Proof for Case A

In this case, the two factors of $-z^3$ are coprime. This implies that 3 does not divide u and the two factors are cubes of two smaller numbers, r and s .

$$2u = r^3$$

$$u^2 + 3v^2 = s^3$$

Since $u^2 + 3v^2$ is odd, so is s . Then Euler claimed that it is possible to write:

$$s = e^2 + 3f^2$$

which e and f integers, so that

$$u = e(e^2 - 9f^2)$$

$$v = 3f(e^2 - f^2)$$

Since u is even and v is odd, then e is even and f is odd. Since

$$r^3 = 2u = 2e(e - 3f)(e + 3f)$$

The factors $2e$, $(e - 3f)$, $(e + 3f)$ are coprime, since 3 can not divide e : if e were divisible by 3, then 3 would divide u , violating the designation of u and v as coprime. Since the three factors on the right-hand side are coprime, they must individually equal cubes of smaller integers

$$-2e = k^3$$

$$e - 3f = l^3$$

$$e + 3f = m^3$$

Which yields a smaller solution $k^3 + l^3 + m^3 = 0$. Therefore, by the argument of infinite descent, the original solution (x, y, z) was impossible.

Proof for Case B

In this case, the greatest common divisor of $2u$ and $u^2 + 3v^2$ is 3. That implies that 3 divides u , and one may express $u = 3w$ in terms of a smaller integer w . Since u is divisible by 4, so is w , hence, w is also even. since u and v are coprime, so are v and w . Therefore, neither 3 nor 4 divide v .

Substituting u by w in the equation for z^3 yields

$$-z^3 = 6w(9w^2 + 3v^2) = 18w(3w^2 + v^2)$$

Because v and w are coprime, and because 3 does not divide v , then $18w$ and $3w^2 + v^2$ are also coprime. Therefore, since their product is a cube, they are each the cube of smaller integers, r and s

$$18w = r^3$$

$$3w^2 + v^2 = s^3$$

By the step as in case A, it is possible to write :

$$s = e^2 + 3f^2$$

which e and f integer, so that

$$v = e(e^2 - 9f^2)$$

$$w = 3f(e^2 - f^2)$$

Thus, e is odd and f is even, because v is odd. The expression for $18w$ then becomes

$$r^3 = 18w = 54f(e^2 - f^2) = 54f(e + f)(e - f)$$

Since 3^3 divides r^3 we have that 3 divides r , so $(r/3)^3$ is an integer that equals $2f(e + f)(e - f)$. Since e and f are coprime, so are the three factors $2e$, $e + f$, and $e - f$, therefore, they are each the cube of smaller integers k , l , and m .

$$-2e = k^3$$

$$e + f = l^3$$

$$e - f = m^3$$

which yields a smaller solution $k^3 + l^3 + m^3 = 0$. Therefore, by the argument of infinite descent, the original solution (x, y, z) was impossible.

2 Disproof

Lemma. *if the equation $x^3 + y^3 + z^3 = 0$ is satisfied in integers, then one of the numbers x , y , and z must be divisible by 3*

proof. From the equation $x^3 + y^3 + z^3 = 0$, we obtain:

$$(x + y + z)^3 = 3(z + x)(z + y)(x + y)$$

Then, $x + y + z$ is divisible by 3, $(x + y + z)^3$ is divisible by 3^3

So $(z + x)(z + y)(x + y)$ must be divisible by 3:

If $z + x$ is divisible by 3, then y is divisible by 3

If $z + y$ is divisible by 3, then x is divisible by 3

If $x + y$ is divisible by 3, then z is divisible by 3

Hence, one of x , y , and z must be divisible by 3.

Mistake in Euler's proof

For the case A

Since step,

$$u = e(e^2 - 9f^2)$$

$$v = 3f(e^2 - f^2)$$

Euler already considered only u , and passed over v , and it was a gap of proof as follows :

Since $v = 3f(e^2 - f^2)$, then v is divisible by 3.

Since

$$2v = x - y$$

Then, $x - y$ is divisible by 3, hence, both of them are divisible by 3, or both not divisible by 3. Since x and y are coprime, then x and y have not common divisor, so both x and y are not divisible by 3. By lemma above, z must be divisible by 3, which implies that $2u$ and $u^2 + 3v^2$ have common divisor 3, a contradiction. Case A is impossible.

For the case B

In the case B, u is divisible by 3, v is not divisible by 3, since

$$v = e(e^2 + 9f^2)$$

$$-2e = k^3$$

Then e is not divisible by 3, so k is not divisible by 3, it returns to the case A (k as z in the case A) that was shown impossible.

References

1. Proof of Fermat's Last Theorem for specific exponents- Wikipedia.
2. Quang N V, A proof of the four color theorem by induction. Vixra:1601.0247(CO)

Email:

nguyenvquang67@gmail.com

quangnhu67@yahoo.com.vn