ON GOLDBACH'S CONJECTURE

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ABSTRACT

An equivalent form of the Goldbach Conjecture is stated using manipulation of characteristic equations and simple logical arguments that lead to an equation which restates the conjecture. A new form of the number of unordered partitions of an even number into two primes is presented.

Theorem 1:

$$\left\lfloor 1 - \frac{sin^2(\pi\frac{f}{g})}{(\pi\frac{f}{g})^2} \right\rfloor = \begin{cases} 1 \; if \; g|f \\ 0 \; otherwise \end{cases}$$

Proof:

For n>0, $f(n) = \frac{sin^2(\pi n)}{(\pi n)^2}$ is clearly 0 if n is an integer and $0 < f(n) < 1$ otherwise.

Therefore, $1 - f(n)$ is 1 if n is an integer and $0 < f(n) < 1$ if n is not an integer. So by the property of the floor function, $\left\lfloor 1 - \frac{sin^2(\pi n)}{\pi n^2} \right\rfloor = \begin{cases} 1 \; if \; n \; is \; an \; integer \\ 0 \; othewise \end{cases}$ end proof

It is well known that $\lfloor x + n \rfloor = n + \lfloor x \rfloor$ for an integer n. So, $\left\lfloor 1 - \frac{sin^2(\pi\frac{f}{g})}{(\pi\frac{f}{g})^2} \right\rfloor = 1 + \left\lfloor - \frac{sin^2(\pi\frac{f}{g})}{(\pi\frac{f}{g})^2} \right\rfloor$.

By Wilson's theorem, for a natural number n >1, $(n-1)! + 1 \equiv mod \; n \;\; iff \; n \; is \; prime.$

So by the proof of the previous theorem, $1 + \left\lfloor - \frac{sin^2(\pi\frac{(n-1)!+1}{n})}{(\pi\frac{(n-1)!+1}{n})^2} \right\rfloor = \begin{cases} 1 \; if \; n \; is \; prime \\ 0 \; otherwise \end{cases} \;\; for \; n > 1.$

This is clearly the characteristic equation of the primes.

Theorem 2:

$$\sum_{j=1}^{n-1}\left(1+\left\lfloor -\frac{\sin^2\left(\pi\frac{(2n-2-j)!+1}{2n-1-j}\right)}{\left(\pi\frac{(2n-2-j)!+1}{2n-1-j}\right)^2}\right\rfloor\right)\left(1+\left\lfloor -\frac{\sin^2\left(\pi\frac{j!+1}{j+1}\right)}{\left(\pi\frac{j!+1}{j+1}\right)^2}\right\rfloor\right)=\widetilde{p_{2n}}$$

where $\widetilde{p_{2n}}$ is the number of unordered partitions of $2n$ into two primes.

Proof:

For an even number > 2, I will use 16 in this example, the numbers from 1 to $2n$ may be written in order and then one may write the numbers backwards, offset by 1, directly above as follows,

16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16

It is clear to see that each column will be equal to $2n$, 16 in this example. Where the top and bottom both have primes, this is a solution for $2n$ of the Goldbach Conjecture. To avoid counting a solution twice and noting that 1 is not in the solution set, it is clear to see that it is only necessary to count solutions between $2n-2$ and $2n$ inclusive. The primes may be counted backwards using the characteristic equation of the primes at the value $2n-1-j$ and ranging j from 1 to $n-1$, the primes may be counted forward by using the characteristic equation of the primes at the value $j+1$ and ranging j from 1 to $n-1$. This will include any possible solution. A characteristic equation can only be 1 or 0, so multiplying these two characteristic equations together ensures that both must be one for the product to be 1, otherwise it will be 0. In this case, multiplying these two characteristic equations together and summing will count only primes which sum to $2n$ and thus,

The form of the number of partitions of $2n$ into two primes, which I will denote as $\widetilde{p_{2n}}$,

$$\sum_{j=1}^{n-1}\left(1+\left\lfloor -\frac{\sin^2\left(\pi\frac{(2n-2-j)!+1}{2n-1-j}\right)}{\left(\pi\frac{(2n-2-j)!+1}{2n-1-j}\right)^2}\right\rfloor\right)\left(1+\left\lfloor -\frac{\sin^2\left(\pi\frac{j!+1}{j+1}\right)}{\left(\pi\frac{j!+1}{j+1}\right)^2}\right\rfloor\right)=\widetilde{p_{2n}}$$

End Proof

Theorem 3:

$$\widetilde{p_{2n}} = \widetilde{c_{2n}} + \pi(n) + \big(\pi(2n-2) - \pi(n-1)\big) - n + 1$$

Where $\widetilde{p_n}$ is the number of unordered partitions of $2n$ into 2 primes, $\widetilde{c_n}$ is the number of unordered partitions of n into 2 composites, and $\pi(n)$ is the prime counting function.

Proof:

As I have shown, $\sum_{j=1}^{n-1}\left(1+\left\lfloor -\frac{\sin^2\left(\pi\frac{(2n-2-j)!+1}{2n-1-j}\right)}{\left(\pi\frac{(2n-2-j)!+1}{2n-1-j}\right)^2}\right\rfloor\right)\left(1+\left\lfloor -\frac{\sin^2\left(\pi\frac{j!+1}{j+1}\right)}{\left(\pi\frac{j!+1}{j+1}\right)^2}\right\rfloor\right)=\widetilde{p_{2n}}$

Expanding the sum,

$$\widetilde{p_{2n}} = \sum_{j=1}^{n-1} 1 + \sum_{j=1}^{n-1} \left\lfloor -\frac{sin^2(\pi\frac{j!+1}{j+1})}{(\pi\frac{j!+1}{j+1})^2} \right\rfloor + \sum_{j=1}^{n-1} \left\lfloor -\frac{sin^2(\pi\frac{(2n-2-j)!+1}{2n-1-j})}{(\pi\frac{(2n-2-j)!+1}{2n-1-j})^2} \right\rfloor + \sum_{j=1}^{n-1} \left\lfloor -\frac{sin^2(\pi\frac{j!+1}{j+1})}{(\pi\frac{j!+1}{j+1})^2} \right\rfloor \times \left\lfloor -\frac{sin^2(\pi\frac{(2n-2-j)!+1}{2n-1-j})}{(\pi\frac{(2n-2-j)!+1}{2n-1-j})^2} \right\rfloor$$

The first of these sums is clearly $n-1$. In the second, $\sum_{j=1}^{n-1} \left\lfloor -\frac{sin^2(\pi\frac{j!+1}{j+1})}{(\pi\frac{j!+1}{j+1})^2} \right\rfloor$ the 1 that was added to the floor function has been omitted, so the sum has been subtracted from $n-1$. The sum with 1 added to the floor function would have been $\pi(n)$, so this is simply the number of composites less than or equal to n minus 1 which is $n-\pi(n)-1$. With this and noting that the sum is now negative in its form, $\sum_{j=1}^{n-1} \left\lfloor -\frac{sin^2(\pi\frac{j!+1}{j+1})}{(\pi\frac{j!+1}{j+1})^2} \right\rfloor = \pi(n)-n+1$.

The second sum in the theorem, $\sum_{j=1}^{n-1} \left\lfloor -\frac{sin^2(\pi\frac{(2n-2-j)!+1}{2n-1-j})}{(\pi\frac{(2n-2-j)!+1}{2n-1-j})^2} \right\rfloor$ using the same approach as before would have been $\pi(2n-2)-\pi(n-1)$ if 1 was still added to the floor function, because the characteristic equation would have been counting primes in this interval. So this sum is the negative of $n-1-(\pi(2n-2)-\pi(n-1))$.

Therefore, $\sum_{j=1}^{n-1} \left\lfloor -\frac{sin^2(\pi\frac{(2n-2-j)!+1}{2n-1-j})}{(\pi\frac{(2n-2-j)!+1}{2n-1-j})^2} \right\rfloor = 1-n+\pi(2n-2)-\pi(n-1)$.

The third sum in the theorem, $\sum_{j=1}^{n-1} \left\lfloor -\frac{sin^2(\pi\frac{j!+1}{j+1})}{(\pi\frac{j!+1}{j+1})^2} \right\rfloor \times \left\lfloor -\frac{sin^2(\pi\frac{(2n-2-j)!+1}{2n-1-j})}{(\pi\frac{(2n-2-j)!+1}{2n-1-j})^2} \right\rfloor$ is still counting partitions of $2n$, but now it is clearly counting composites. So this sum is equal to the number of unordered partitions of $2n$ into two composites, which I will denote as $\widetilde{c_{2n}}$. This sum is positive because the characteristic equations are multiplied together.

Adding the sums together therefore gives,
$\widetilde{p_{2n}} = \widetilde{c_{2n}} + \pi(n) + (\pi(2n-2)-\pi(n-1)) - n + 1$ End Proof

Theorem 4:

$$\widetilde{c_{2n+1}} = n - 1 - \pi(2n-1),$$

where $\widetilde{c_{2n+1}}$ is the number of unordered partitions of $2n+1$ into 2 composites.

Proof:

List the partitions of a particular odd number $2n+1$. I will use 17 in this example.

17

1,16

2,15

3,14

4,13

5,12

6,11

7,10

8,9

From this list it is clear that every partition is an odd number paired with an even number. Any even number > 2 is composite. Therefore, a pair will be an unordered partition of $2n + 1$ into two composites if the odd number in the pair is not prime. Excluding of course $2n - 1$, which will always be paired with 2 and excluding 1, because it is not in the solution set of composites. There are exactly n unordered partitions total, because the partitions range from $1 + (n - 1)$ to $n + n$. Now, $n - \pi(2n - 2)$ will count all odd composites $\leq 2n - 2$. Subtracting 1 from this will exclude $2n - 1$ itself, since it will always be paired with 2 in a partition. Since we can count all odd composite solutions within the partitions, and these will always be paired with even numbers, this count includes all possible solutions. Therefore, $\widetilde{c_{2n+1}} = n - 1 - \pi(2n - 2)$.

End Proof

Recall the equation proven in theorem 3, which was

$$\widetilde{p_{2n}} \;=\; \widetilde{c_{2n}} + \pi(n) + \big(\pi(2n-2) - \pi(n-1)\big) - n + 1\,.$$

We can now replace $\pi(2n - 2) - n + 1$ in this equation with $-\widetilde{c_{2n+1}}$. The equation therefore becomes,

$$\widetilde{p_{2n}} \;=\; \widetilde{c_{2n}} - \widetilde{c_{2n+1}} + \pi(n) - \pi(n-1).$$

Now, $\pi(n) - \pi(n - 1)$ is just the characteristic equation of the primes. So this may be simplified further as,

$$\widetilde{p_{2n}} \;=\; \widetilde{c_{2n}} - \widetilde{c_{2n+1}} + \chi_p(n).$$

Where $\chi_p(n)$ is the characteristic function of the primes which is 1 if n is a prime and 0 otherwise.

From this, the Goldbach Conjecture can be restated as I will show in the following theorem.

Theorem 5:

The Goldbach Conjecture is equivalent to proving that $\widetilde{c_{2n}} > \widetilde{c_{2n+1}}$ for composite n.

Proof:

By Proof of theorem 4, the number of unordered partitions of $2n$ into 2 primes has the form,

$$\widetilde{p_{2n}} = \widetilde{c_{2n}} - \widetilde{c_{2n+1}} + \chi_p(n).$$

If n is prime, a solution will always exist in the form $n + n$. So, it suffices to prove there is a solution for all composite n. But, for composite n, $\chi_p(n) = 0$. So for composite n,

$$\widetilde{p_{2n}} = \widetilde{c_{2n}} - \widetilde{c_{2n+1}}.$$

Now, if the Goldbach Conjecture is true, than the left side of this equation must be greater than 0 and so the right side must be as well. This means of course that we cannot have $\widetilde{c_{2n}} = \widetilde{c_{2n+1}}$. Therefore, we must have $\widetilde{c_{2n}} > \widetilde{c_{2n+1}}$ for composite n. End Proof

By proof of theorem 4, this of course implies that we must have $\widetilde{c_{2n}} > n - 1 - \pi(2n-2)$ for composite n. By conjecture, this statement is true for all $n \geq 4$.

CONCLUSION

It can now be seen that proof of even a close approximation to the number of unordered partitions of $2n$ into 2 composites may be a key factor in proving that the number of unordered partitions of $2n$ into 2 primes cannot be 0 for $n > 1$. From this it is clear that studying $\widetilde{c_{2n}}$ is just as important as studying $\widetilde{p_{2n}}$. They will hereby forever be connected by their relationship in the form, $\widetilde{p_{2n}} = \widetilde{c_{2n}} - \widetilde{c_{2n+1}} + \chi_p(n)$. This can only add to the complexity of such a simple and beautifully impossible question. Is every even number > 2 the sum of two primes?