Uncountable Sums in $\mathbb{R}$?

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Dedicated to Marie-Louise Nykamp

Abstract

It is shown that infinite series of strictly positive numbers cannot converge to a positive number, unless the series has at most a countable number of terms.

Given an arbitrary infinite set I and a mapping $f : I \longrightarrow (0, \infty)$, we say that the series

(1) $\quad \sum_{i \in I} f(i)$

converges to a certain $s \in \mathbb{R}$, if and only if

(2) $$\begin{array}{l} \forall \ \epsilon > 0 \ : \\ \exists \ J \subset I, \ J \text{ finite } : \\ \forall \ K \subset I, \ K \text{ finite } : \\ \quad J \subseteq K \quad \Longrightarrow \quad |\, s - \sum_{i \in K} f(i) \,| \leq \epsilon \end{array}$$

in which case we write

(3) $\quad \sum_{i \in I} f(i) = s$

We note that the above definition in (2) of a convergent series clearly seems to be by far the most natural one, and not only in case of real valued terms $f(i)$, but as well in the case of values in arbitrary metric spaces. Also, and as such, this definition is well known in the literature.

What appears to be less well known, at least as far as the author of the present paper is concerned, is the

Lemma

The following implication is true :

$$(4) \qquad \begin{array}{l} \forall \; I \text{ an infinite set, } \; f : I \longrightarrow (0, \infty), \; s \in (0, \infty) \; : \\ \quad (\; \sum_{i \in I} f(i) = s \;) \quad \Longrightarrow \quad I \text{ countable} \end{array}$$

Proof

Obviously, (4) is equivalent with

$$(5) \qquad \begin{array}{l} \forall \; I \text{ infinite, } \; f : I \longrightarrow (0, \infty) \; : \\ \quad (\; \sum_{i \in I} f(i) = 1 \;) \quad \Longrightarrow \quad I \text{ countable} \end{array}$$

Let us assume that (5) does not hold. Then

$$(6) \qquad \begin{array}{l} \exists \; I \text{ uncountable, } \; f : I \longrightarrow (0, \infty) \; : \\ \quad \sum_{i \in I} f(i) = 1 \end{array}$$

But

$$(7) \qquad \exists \; n_0 \geq 2 \; : \; I_{n_0} = \{ \; i \in I \mid 1/n_0 \leq f(i) \; \} \text{ uncountable}$$

since otherwise, $I = \bigcup_{n \geq 2} I_n$ is countable.

Thus

(8) $\qquad I_{n_0} \subseteq I_{n_0+1} \subseteq \dots$

are all uncountable.

Now let $\epsilon = 1$, then (2) gives a finite subset $J_1 \subset I$, such that

(9) $\qquad \forall\ J \subset I,\ J \text{ finite},\ J_1 \subseteq J\ :\ s - 1 \leq \sum_{i \in J} f(i) \leq s + 1$

In view of (8), for every $m \geq 1$, we can take a finite subset $J_m \subset I_{n_0}$ with m elements which is disjoint from J. Then $J^* = J \bigcup J_m$ will be finite, further, we shall have $J_1 \subseteq J^*$, and

(10) $\qquad \sum_{i \in J^*} f(i) = \sum_{i \in J} f(i) + \sum_{i \in I_{n_0}} f(i)\ \geq\ \sum_{i \in J^*} f(i) + m/n_0$

and since m is arbitrary, the relations (9) and (10) obviously conflict.