Abstract theorems on exchange of limits and preservation of (semi)continuity of functions and measures in the filter convergence setting

A. Boccuto and X. Dimitriou

Abstract

We give necessary and sufficient conditions for exchange of limits of double-indexed families, taking values in sets endowed with an abstract structure of convergence, and for preservation of continuity or semicontinuity of the limit family, with respect to filter convergence. As consequences, we give some filter limit theorems and some characterization of continuity and semicontinuity of the limit of a pointwise convergent family of set functions.

1 Introduction

A widely investigated problem in convergence theory and topology is to find necessary and/or sufficient conditions for continuity and/or semicontinuity of the limit of a pointwise convergent net of functions or measures. There have been many recent related studies in abstract structures, like topological spaces, lattice groups, metric semigroups, cone metric spaces, with respect to usual, statistical or filter/ideal convergence, and associated with the notions of equicontinuity, filter exhaustiveness and filter continuous convergence. The study of semicontinuous functions is associated with quasi-metric spaces, that is spaces endowed with an asymmetric distance function.

A concept associated with these topics is that of *strong uniform continuity*, which is used to study the problem of finding a topology with respect to which the set of the continuous functions is closed, and pointwise convergence of continuous functions implies convergence in this topology.

Another related field is the study of convergence theorems for measures taking values in abstract structures. When it is dealt with the classical convergence, it is possible to prove σ-additivity, (s)-boundedness and absolute continuity of the limit measure directly from pointwise convergence (with respect to a single order sequence of regulator) of the involved measures, without requiring additional hypotheses. This is not always true in the setting of filter convergence.

We present a unified axiomatic approach and extend results of this kind to double-indexed families, taking values in abstract structures, whose particular cases are lattice groups, topological groups, (quasi-)metric semigroups and cone (quasi-)metric spaces. To include both continuity and semicontinuity, we assume the existence of a "generalized distance" function, which is assumed to satisfy only the triangular property, and takes values in a group endowed with a suitable system of "intervals" or "halflines" containing its neutral element 0. Thus, both topological groups and lattice groups endowed with (r)-, (D)- or order convergence are particular cases of these abstract structures. We prove some results on exchange of limits in the setting of filter convergence. Observe that the involved "distance" can be symmetric or asymmetric. Furthermore, in our setting both sequences and nets of functions/measures are included, and note that it is possible to consider them as families endowed with filters.

As applications, we give some necessary and sufficient conditions for continuity from above/below, absolute continuity and semicontinuity of the limit measure in the context of filter convergence, which include the cases of σ-additivity and (s)-boundedness, showing, by means of related examples, that they are not always satisfied, differently from the classical case.

2 Assumptions and examples

We begin with giving our axiomatic approach, which deals with abstract convergence with respect to filters, without using necessarily nets.

Definitions 2.1. (a) Let Λ be any nonempty set, and $\mathcal{P}(\Lambda)$ be the class of all subsets of Λ. A family of sets $\mathcal{F} \subset \mathcal{P}(\Lambda)$ is called a *filter* of Λ iff $\mathcal{F} \neq \emptyset$, $\emptyset \notin \mathcal{F}$, $A \cap B \in \mathcal{F}$ for each A, $B \in \mathcal{F}$, and $B \in \mathcal{F}$ whenever $B \supset A$ and $A \in \mathcal{F}$.

(b) Let R be a nonempty set, $Y = (Y, +)$ be an abelian group with neutral element 0. Given $k \in \mathbb{N}$ and $U_1, U_2, \ldots, U_k \subset Y$, put $U_1 + U_2 + \cdots + U_k := \{u_1 + u_2 + \ldots + u_k \colon u_j \in U_j,\ j = 1, 2, \ldots, k\}$, and $k\,U := U + \cdots + U$ (k times).

(c) Let Π be a nonempty set. A Π-*system* $\mathcal{U}$ is a class of families $\mathbf{U} = (U_\pi)_{\pi\in\Pi}$ of subsets of Y, with $0 \in \bigcap_{\pi\in\Pi} U_\pi$ for each $\mathbf{U} = (U_\pi)_{\pi\in\Pi}$, and such that for every $\mathbf{U} = (U_\pi)_{\pi\in\Pi}$, $\mathbf{V} = (V_\pi)_{\pi\in\Pi} \in \mathcal{U}$ there is $\mathbf{W} = (W_\pi)_{\pi\in\Pi} \in \mathcal{U}$ such that $U_\pi + V_\pi \subset W_\pi$ for every $\pi \in \Pi$. Let $\rho : R \times R \to Y$ be a function, and suppose that

$\mathcal{H}1)$ for every $\mathbf{U} = (U_\pi)_\pi$, $\mathbf{V} = (V_\pi)_\pi \in \mathcal{U}$, for each $\pi \in \Pi$ and a, b, $c \in R$, if $\rho(a, b) \in U_\pi$ and $\rho(b, c) \in V_\pi$, then $\rho(a, c) \in U_\pi + V_\pi$.

(d) Fix a Π-system $\mathcal{U}$ on Y and a filter $\mathcal{F}$ of Λ. A family b_λ, $\lambda \in \Lambda$, of elements of R is said to $(\mathcal{UF})$-*backward* (resp. $(\mathcal{UF})$-*forward*) *converge* to $b \in R$ iff there is a family $(U_\pi)_{\pi\in\Pi} \in \mathcal{U}$, such that for every $\pi \in \Pi$ there is a set $F \in \mathcal{F}$ with $\rho(b_\lambda, b) \in U_\pi$ (resp. $\rho(b, b_\lambda) \in U_\pi$) for any $\lambda \in F$. We say that $(b_\lambda)_\lambda$ $(\mathcal{UF})$-*converges* to $b \in R$ iff it $(\mathcal{UF})$-converges both backward and forward to b, and in this case we write $(\mathcal{UF}) \lim_{\lambda\in\Lambda} b_\lambda = b$.

(e) Let Ξ be a nonempty set. Given two families $(a_{\lambda,\xi})_{\lambda\in\Lambda,\xi\in\Xi}$ and $(a_\xi)_{\xi\in\Xi}$ of elements of R, we say that $(a_{\lambda,\xi})_{\lambda,\xi}$ $(\Xi\mathcal{UF})$-*backward* (resp. $(\Xi\mathcal{UF})$-*forward*) converges to $(a_\xi)_\xi$ iff there is a family $(U_\pi)_{\pi\in\Pi} \in \mathcal{U}$, such that for each $\pi \in \Pi$ and $\xi \in \Xi$ there is $F \in \mathcal{F}$ with $\rho(a_{\lambda,\xi}, a_\xi) \in U_\pi$ (resp. $\rho(a_\xi, a_{\lambda,\xi}) \in U_\pi$) for any $\lambda \in F$. Analogously as above it is possible to formulate the notions of $(\Xi\mathcal{UF})$-convergence and $(\Xi\mathcal{UF})$-limit.

Remark 2.2. Observe that, in our context, we will consider filters without dealing explicitly with nets, and this is not a restriction. A *net* on R is a function $\mathcal{N}:\Lambda\to R$, where $\Lambda=(\Lambda,\geq)$ is a *directed* set, namely a partially ordered set such that for any λ_1, $\lambda_2\in\Lambda$ there exists $\lambda_0\in\Lambda$ with $\lambda_0\geq\lambda_j$, $j=1,2$. Given a directed set $(\Lambda,\geq)$, it is possible to associate the filter $\mathcal{F}_\Lambda$ generated by the family $\mathcal{C}':=\{\{\lambda'\in\Lambda\colon\lambda'\geq\lambda\}\colon\lambda\in\Lambda\}$. Note that $\mathcal{C}'$ is a *filter base* of Λ, that is for every A, $B\in\mathcal{C}'$ there is an element $C\in\mathcal{C}'$ with $C\subset A\cap B$. The filter generated by a filter base $\mathcal{C}$ is the family $\{A\subset\Lambda\colon$ there is $B\in\mathcal{C}$ with $B\subset A\}$. Conversely, given a filter base $\mathcal{C}:=\{C_\lambda:\lambda\in\Lambda\}$, it is possible to associate a directed partial order $\geq$ on Λ, by setting $\lambda_1\geq\lambda_2$ if and only if $C_{\lambda_1}\subset C_{\lambda_2}$, λ_1, $\lambda_2\in\Lambda$.

Examples 2.3. We now present some kinds of abstract space in which our approach can be applied, including both symmetric and asymmetric distance functions.

(a) Let R be a Dedekind complete lattice group, $Y=R$, and let $\rho(a,b):=|a-b|$, a, $b\in R$, be the *absolute value* of $a-b$. Let $\Pi_1:=\mathbb{R}^+$ be endowed with the usual order, $\mathcal{U}_1:=\{([-\varepsilon\,u,\varepsilon\,u])_{\varepsilon\in\mathbb{R}^+}\colon u\in R,\ u>0\}$ (*(r)-convergence*); $\Pi_2:=\mathbb{N}$ be with the usual order, $\mathcal{U}_2:=\{([-\sigma_p,\sigma_p])_{p\in\mathbb{N}}\colon(\sigma_p)_p$ is an (O)-sequence$\}$, where an *(O)-sequence* is a decreasing sequence in R whose infimum is equal to 0 (*order convergence* of *(O)-convergence*); $\Pi_3:=\mathbb{N}^{\mathbb{N}}$ be directed with the pointwise order, $\mathcal{U}_3:=\Big\{\Big(\Big[-\bigvee_{t=1}^{\infty}a_{t,\varphi(t)},\bigvee_{t=1}^{\infty}a_{t,\varphi(t)}\Big]\Big)_{\varphi\in\mathbb{N}^{\mathbb{N}}}:(a_{t,l})_{t,l}$ is a (D)-sequence$\Big\}$, where a *(D)-sequence* or *regulator* is a bounded double sequence in R such that $(a_{t,l})_l$ is an (O)-sequence for each $t\in\mathbb{N}$ (*(D)-convergence*).

It is not difficult to check that $\mathcal{U}_j$, $j=1,2,3$, are Π_j-systems, satisfying $\mathcal{H}1$).

(b) We can extend the examples given in (a) to the case in which R is a *cone metric space* (with respect to Y), that is R is a nonempty set and $(Y,+)$ is a Dedekind complete lattice group endowed with a *distance function* $\rho:R\times R\to Y$, satisfying the following axioms:

- $\rho(a,b)\geq 0$ and $\rho(a,b)=0$ if and only if $a=b$;
- $\rho(a,b)=\rho(b,a)$ (*symmetric property*);
- $\rho(a,c)\leq\rho(a,b)+\rho(b,c)$ (*triangular property*) for every a, b, $c\in R$.

When a cone metric space R is a semigroup, we say that R is a *cone metric semigroup.* A cone metric semigroup in which $Y=\mathbb{R}$ is said to be a *metric semigroup.* If ρ satisfies the first and the third of the above axioms, but not the symmetric property, then we say that ρ is an *asymmetric distance function* and that R is a *cone asymmetric metric space* or *cone quasi-metric space.* For example, let $\mathcal{T}$ be a nonempty set, $R=\{f:\mathcal{T}\to\mathbb{R},\ f$ is bounded$\}$, $a_0\neq 1$ be a fixed positive real number and u be a fixed element of R with $u>0$. For each f_1, $f_2\in R$ and $t\in\mathcal{T}$, set

$$d^{(u)}_{a_0,t}(f_1(t),f_2(t))=\begin{cases}(f_2(t)-f_1(t))\,u, & \text{if } f_1(t)\leq f_2(t),\\[2ex] a_0(f_1(t)-f_2(t))\,u, & \text{if } f_1(t)>f_2(t),\end{cases}\tag{1}$$

and let $\rho^{(u)}_{a_0}(f_1,f_2)=\bigvee_{t\in\mathcal{T}}d^{(u)}_{a_0,t}(f_1(t),f_2(t))$. It is not difficult to see that ρ_{a_0} is an asymmetric distance function.

(c) When R is a lattice group and $Y = R$, it is advisable to deal not only with continuity, but also with upper or lower semicontinuity. In this setting we take $\rho(a, b) := b - a$, a, $b \in R$, Π_j, $j = 1, 2, 3$, as in (a), $\mathcal{U}_1^{(0)} := \{(\{r \in R : r \leq \varepsilon\, u\})_{\varepsilon \in \mathbb{R}^+} : u \in R, u > 0\}$, $\mathcal{U}_2^{(0)} := \{(\{r \in R : r \leq \sigma_p\})_{p \in \mathbb{N}}$: $(\sigma_p)_p$ is an (O)-sequence$\}$, $\mathcal{U}_3^{(0)} := \Big\{ \Big(\Big\{ r \in R : r \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \Big\} \Big)_{\varphi \in \mathbb{N}^{\mathbb{N}}} : (a_{t,l})_{t,l}$ is a (D)-sequence$\Big\}$.

(d) Let R be a Hausdorff topological group with neutral element 0 satisfying the first axiom of countability, $Y = R$, $\Pi^* = \mathbb{N}$, $\mathcal{U}^* := \{(U_p)_{p \in \mathbb{N}}$: $(U_p)_{p \in \mathbb{N}}$ is a base of closed symmetric neighborhoods of $0\}$, and $\rho(a, b) = b - a$. It is not difficult to see that $\mathcal{U}^*$ is a Π^*-system.

(e) Let $\mathcal{F}$ be a filter of Λ. When we consider (r)-convergence and R is a cone quasi-metric space, a family $(b_\lambda)_\lambda$ of elements of R is said to *$(r\mathcal{F})$-backward converge to b* iff there is $u \in Y$, $u > 0$, with $\{\lambda \in \Lambda : \rho(b_\lambda, b) \leq \varepsilon\, u\} \in \mathcal{F}$ for all $\varepsilon > 0$. When we deal with (O)-sequences, we say that $(b_\lambda)_\lambda$ *$(O\mathcal{F})$-backward converges to b* iff there exists an (O)-sequence $(\sigma_p)_p$ in Y with $\{\lambda \in \Lambda : \rho(b_\lambda, b) \leq \sigma_p\} \in \mathcal{F}$ for every $p \in \mathbb{N}$. When we consider (D)-sequences, we say that the net $(b_\lambda)_\lambda$ *$(D\mathcal{F})$-backward converges to b* iff there exists a regulator $(a_{t,l})_{t,l}$ in Y with

$$\Big\{ \lambda \in \Lambda : \rho(b_\lambda, b) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \Big\} \in \mathcal{F} \quad \text{for each } \varphi \in \mathbb{N}^{\mathbb{N}}.$$

When $\Lambda = \mathbb{N}$ and $\mathcal{F} = \mathcal{F}_{\text{cofin}}$, we have the classical (r)-, (O)-, (D)-(backward, forward) convergence. If $(R, +)$ is a Hausdorff topological group and $Y = R$, then we say that a net b_λ, $\lambda \in \Lambda$ in R, *$\mathcal{F}$-backward converges to $b \in R$* iff $\{\lambda \in \Lambda : b_\lambda - b \in U\} \in \mathcal{F}$ for each neighborhood U of 0. Similarly as above it is possible to formulate the corresponding notions of $(r\mathcal{F})$-, $(O\mathcal{F})$-, $(D\mathcal{F})$-(forward) convergences and limits.

(f) When R is a Dedekind complete lattice group, $(a_{\lambda,\xi})_{\lambda \in \Lambda, \xi \in \Xi}$ and $(a_\xi)_{\xi \in \Xi}$ are two families in R and $\mathcal{U}$ is the Π-system associated with (r)-convergence (resp. (O)-convergence, (D)-convergence), we say that $(\Xi r\mathcal{F}) \lim\limits_{\lambda \in \Lambda} a_{\lambda,\xi} = a_\xi$ (resp. $(\Xi O\mathcal{F}) \lim\limits_{\lambda \in \Lambda} a_{\lambda,\xi} = a_\xi$, $(\Xi D\mathcal{F}) \lim\limits_{\lambda \in \Lambda} a_{\lambda,\xi} = a_\xi$) iff $(\Xi \mathcal{U}\mathcal{F}) \lim\limits_{\lambda \in \Lambda} a_{\lambda,\xi} = a_\xi$. Analogously it is possible to formulate the corresponding concepts of backward and forward convergences. In particular, when $R = \mathbb{R}$ endowed with the usual convergence, since it coincides with (r)- (O)- and (D)-convergence, we will denote by $(\mathcal{F})$- and $(\Xi\mathcal{F})$-(backward, forward) convergence the usual filter (backward, forward) convergence and the ordinary pointwise filter (backward, forward) convergence. When R is a Hausdorff topological group, $\mathcal{U}^*$, Π^* are as in (d), we get that the $(\Xi\mathcal{U}^*\mathcal{F})$-convergence is equivalent to the pointwise $(\mathcal{F})$-convergence, and hence we write $(\mathcal{F}) \lim\limits_{\lambda \in \Lambda} a_{\lambda,\xi} = a_\xi$ for every $\xi \in \Xi$, or $(\Xi\mathcal{F}) \lim\limits_{\lambda \in \Lambda} a_{\lambda,\xi} = a_\xi$.

(g) Observe that, in general, a family $(b_\lambda)_\lambda$ can be backward (resp. forward) convergent to more than one element. For example, if R is a Dedekind complete lattice group, Λ is a nonempty set, $\mathcal{F}$ is any filter of Λ, $\rho(a, b) = b - a$ for every a, $b \in R$, $b_\lambda = 0$ for every $\lambda \in \Lambda$ and b is *any* element of R with $b \leq 0$ (resp. $b \geq 0$), then it is not difficult to see that $(b_\lambda)_\lambda$ $(r\mathcal{F})$-backward (resp. $(r\mathcal{F})$-forward) converges to b.

(h) In general, backward and forward convergence are not equivalent. For example, similarly as in (1), let $\mathcal{T}$ be a nonempty set, $\Lambda := [1, +\infty[$ be endowed with the usual order, $\mathcal{F}$ be a filter of Λ

containing all halflines $[c,+\infty[$ with $c \geq 1$, pick $R = \{f : \mathcal{T} \to \mathbb{R},\ f \text{ is bounded}\}$, and let $\mathbf{0}$, $\mathbf{1}$ be those functions which associate to every element of $\mathcal{T}$ the real constants 0, 1, respectively. For any f_1, $f_2 \in R$ and $t \in \mathcal{T}$, set

$$d'_t(f_1(t), f_2(t)) = \begin{cases} (f_2(t) - f_1(t)) \cdot \mathbf{1}, & \text{if } f_1(t) \leq f_2(t), \\ \\ \mathbf{1}, & \text{if } f_1(t) > f_2(t), \end{cases} \tag{2}$$

and put $\rho'(f_1, f_2) := \bigvee_{t \in \mathcal{T}} d'_t(f_1(t), f_2(t))$. It is not difficult to check that ρ' is an asymmetric distance function For each $\lambda \in \Lambda$, set $f_\lambda := \dfrac{1}{\lambda} \cdot \mathbf{1}$, $h_\lambda := -f_\lambda = -\dfrac{1}{\lambda} \cdot \mathbf{1}$. Note that $d(\mathbf{0}, f_\lambda) = f_\lambda$, $d(f_\lambda, \mathbf{0}) = \mathbf{1}$, $d(h_\lambda, \mathbf{0}) = f_\lambda$, $d(\mathbf{0}, h_\lambda) = \mathbf{1}$, From this it is not difficult to deduce that the family $(f_\lambda)_\lambda$ $(r\mathcal{F})$-forward converges to $\mathbf{0}$ and $(h_\lambda)_\lambda$ $(r\mathcal{F})$-backward converges to $\mathbf{0}$, while $(f_\lambda)_\lambda$ does not $(r\mathcal{F})$-backward converge to $\mathbf{0}$ and $(h_\lambda)_\lambda$ does not $(r\mathcal{F})$-forward converge to $\mathbf{0}$.

However, if Λ is any nonempty set, $\mathcal{F}$ is any filter of Λ, $\rho^{(u)}_{a_0}$ is as in (1) and $C_{a_0} = \max\Big\{a_0, \dfrac{1}{a_0}\Big\}$, then it is not difficult to see that $\rho^{(u)}_{a_0}(f_1, f_2) \leq C_{a_0}\, \rho^{(u)}_{a_0}(f_2, f_1)$ whenever f_1, $f_2 \in R$. From this it follows that a family $(f_\lambda)_{\lambda \in \Lambda}$ in R is $(r\mathcal{F})$-backward convergent if and only if it is $(r\mathcal{F})$-forward convergent. We claim that, in this case, the involved limit coincide. Indeed, if $(f_\lambda)_\lambda$ $(r\mathcal{F})$-backward converges to f_0 and $(r\mathcal{F})$-forward converges to h_0 with respect to $\rho^{(u)}_{a_0}$, then there exist v, $w \in R$ such that for every $\varepsilon > 0$ there are F_1, $F_2 \in \mathcal{F}$ with $\rho^{(u)}_{a_0}(h_0, f_\lambda) \leq \varepsilon\, v$ for every $\lambda \in F_1$, $\rho^{(u)}_{a_0}(f_\lambda, f_0) \leq \varepsilon\, w$ whenever $\lambda \in F_2$. Note that $F_1 \cap F_2 \in \mathcal{F}$. If λ_0 is any fixed element of $F_1 \cap F_2$, then from the triangular property of $\rho^{(u)}_{a_0}$ we deduce that

$$\rho^{(u)}_{a_0}(h_0, f_0) \leq \rho^{(u)}_{a_0}(h_0, f_{\lambda_0}) + \rho^{(u)}_{a_0}(f_{\lambda_0}, f_0) \leq \varepsilon\,(v + w).$$

Thus, by arbitrariness of ε, we get $\rho^{(u)}_{a_0}(h_0, f_0) = 0$, and hence $h_0 = f_0$, getting the claim.

3 The main results

We give the fundamental results of the paper in our unified setting, which includes lattice groups, cone metric spaces, metric groups and topological groups, symmetric and asymmetric distances, continuity and semicontinuity of the limit, families of functions and of measures. We first present the notion of weak filter backward and forward exhaustiveness in our abstract context, which extends the corresponding ones given in the literature and the classical concept of equicontinuity.

Definitions 3.1. (a) Let Ξ is a nonempty set, fix $\xi \in \Xi$ and let $\mathcal{S}_\xi$ be a filter of Ξ. We say that the family $(a_{\lambda,\xi})_{\lambda,\xi}$ is *weakly* $(\mathcal{UF})$*-backward* (resp. *forward*) *exhaustive at* ξ iff there exists a family $(U_\pi)_{\pi \in \Pi} \in \mathcal{U}$ such that for each $\pi \in \Pi$ there is a set $S \in \mathcal{S}_\xi$ such that for every $\zeta \in S$ there is a set $F_\zeta \in \mathcal{F}$ with $\rho(a_{\lambda,\zeta}, a_{\lambda,\xi}) \in U_\pi$ (resp. $\rho(a_{\lambda,\xi}, a_{\lambda,\zeta}) \in U_\pi$) for any $\lambda \in F_\zeta$. The family $(a_{\lambda,\xi})_{\lambda,\xi}$ is said to be *weakly* $(\mathcal{UF})$*-exhaustive at* ξ iff it is both weakly $(\mathcal{UF})$-backward and weakly $(\mathcal{UF})$-forward exhaustive at ξ.

(b) Let $\mathcal{S}_\xi$, $\xi \in \Xi$, be a family of filters of Ξ. We say that $(a_{\lambda,\xi})_{\lambda,\xi}$ is *weakly* $(\mathcal{UF})$*- (backward, forward) exhaustive on* Ξ iff it is weakly $(\mathcal{UF})$- (backward, forward) exhaustive at every $\xi \in \Xi$ with respect to a single family $\mathbf{U} \in \mathcal{U}$, independent of ξ.

Example 3.2. We now show that, in general, weak $(\mathcal{UF})$-backward and forward exhaustiveness do not coincide. Let $\Lambda = R = \Xi = \mathbb{R}$, $Y = \mathbb{R}$ be equipped with the usual convergence, that is $\Pi := \mathbb{R}^+$ be endowed with the usual order, and $\mathcal{U} := \{([-\varepsilon\, u, \varepsilon\, u])_{\varepsilon \in \mathbb{R}^+} \colon u \in \mathbb{R}^+\}$. Let us define $\rho : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$\rho(\xi,\zeta) := \begin{cases} \zeta - \xi, & \text{if } \xi \le \zeta, \\ 1, & \text{if } \xi > \zeta, \end{cases} \qquad \xi, \zeta \in \mathbb{R}.$$

It is not difficult to see that ρ is an asymmetric distance function. Let $\mathcal{F}$ be any filter of Λ, and for every $\xi \in \Xi$, let $\mathcal{S}_\xi$ be the filter of all neighborhoods of ξ with respect to the topology generated by ρ. Set $a_{\lambda,\xi} := \xi + \lambda$, ξ, $\lambda \in \mathbb{R}$. We claim that the family $(a_{\lambda,\xi})_{\lambda,\xi}$ is weakly $(\mathcal{UF})$-forward exhaustive at ξ. Indeed, in correspondence with $\varepsilon > 0$, take $\eta := \min\Big\{\varepsilon, \dfrac{1}{2}\Big\}$, and set $F_\zeta := \Lambda$ for any $\zeta \in [\xi, \xi + \eta] = S_\rho(\xi, \eta)$, where $S_\rho(\xi,\eta)$ denotes the ball of center ξ and radius η with respect to ρ. For every $\lambda \in \Lambda$ and $\zeta \in [\xi, \xi+\eta]$ we get $\rho(a_{\lambda,\xi}, a_{\lambda,\zeta}) = \zeta + \lambda - (\xi + \lambda) = \zeta - \xi \in [-\eta, \eta]$, getting the claim.

Now, in correspondence with every $\xi \in \mathbb{R}$ and $\theta > 0$, let $\eta = \min\{\theta, 1\}$ and take $\zeta = \xi + \eta$. Note that $\zeta \in S_\rho(\xi, \theta)$. Choose arbitrarily $F \in \mathcal{F}$. It is not hard to see that $\rho(a_{\lambda,\zeta}, a_{\lambda,\xi}) = \rho(a_{\lambda,\xi+\eta}, a_{\lambda,\xi}) = 1$ for every $\lambda \in F$. Hence, the family $(a_{\lambda,\xi})_{\lambda,\xi}$ is not weakly $(\mathcal{UF})$-backward exhaustive at ξ. Furthermore note that, analogously as in (2), it is not difficult to check that $(\mathcal{UF})$-forward (resp. backward) convergence does not imply $(\mathcal{UF})$-backward (resp. forward) convergence with respect to ρ.

The following result deals with characterizations and properties of the limit family.

Theorem 3.3. *Assume that $(a_{\lambda,\xi})_{\lambda,\xi}$ $(\Xi\mathcal{UF})$-converges to $(a_\xi)_\xi$, fix $\xi \in \Xi$ and let $\mathcal{S}_\xi$ be a filter of Ξ. Then the following are equivalent:*

(i) $(a_{\lambda,\xi})_{\lambda,\xi}$ is weakly $(\mathcal{UF})$-backward (resp. forward) exhaustive at ξ;

(ii) $(a_\zeta)_\zeta$ $(\mathcal{US}_\xi)$-backward (resp. forward) converges to a_ξ as $\zeta \to \xi$.

Remark 3.4. Observe that Theorem 3.3 holds also if $(\Xi\mathcal{UF})$-convergence is replaced by $(\Xi\mathcal{UF})$-forward convergence, under the hypothesis that forward convergence implies backward convergence. In general this last condition is essential. Indeed, let $\Lambda := [1, +\infty[$ be endowed with the usual order, $\mathcal{F}$ be a filter of Λ containing all halflines $[c, +\infty[$ with $c \ge 1$, $\Xi := [0,1]$ be equipped with the usual distance, $\mathcal{S}_\xi$, $\xi \in \Xi$, be the filter of all neighborhoods of ξ, $Y = \mathbb{R}$ be endowed with the usual convergence, $R = [0,1] \times [0,1]$ and $\rho^* : R \times R \to \mathbb{R}$ be defined by

$$\rho^*((\xi_1,\xi_2),(\zeta_1,\zeta_2)) = \begin{cases} 0, & \text{if } (\xi_1,\xi_2) = (\zeta_1,\zeta_2), \\ \max\{|\xi_1 - \zeta_1|, |\xi_2 - \zeta_2|\}, & \text{if } \xi_1 \le \zeta_1 \text{ and } \zeta_1 > 0, \\ 1, & \text{otherwise.} \end{cases}$$

It is not difficult to check that ρ^* is an asymmetric distance function. For every $\lambda \in \Lambda$ and $\xi \in \Xi$, set $a^*_{\lambda,\xi} := \Big(\dfrac{1}{\lambda}, \xi\Big)$. Observe that $\rho^*\Big((0,\xi), \Big(\dfrac{1}{\lambda}, \xi\Big)\Big) = \dfrac{1}{\lambda}$ and $\rho^*\Big(\Big(\dfrac{1}{\lambda}, \xi\Big), (0,\xi)\Big) = 1$ for every $\lambda \in \Lambda$ and $\xi \in \Xi$. It is not difficult to see that the family $(a^*_{\lambda,\xi})_{\lambda,\xi}$ $(\Xi\mathcal{UF})$-forward converges to $(a^*_\xi)_{\xi\in\Xi}$, where $a^*_\xi = (0,\xi)$, $\xi \in \Xi$, but does not $(\Xi\mathcal{UF})$-backward converges. Moreover, since $\rho^*((0,\zeta),(0,0)) = \rho^*((0,0),(0,\zeta)) = 1$ for every $\zeta \in \Xi$, $\zeta \neq 0$, the family $(a^*_\zeta)_{\zeta\in\Xi}$ is neither $(\mathcal{US}_\xi)$-backward nor $(\mathcal{US}_\xi)$-forward convergent to a^*_0 as $\zeta \to 0$. Furthermore, we get

$$\begin{aligned} \rho^*(a^*_{\lambda,\zeta}, a^*_{\lambda,0}) &= \rho^*\Big(\Big(\frac{1}{\lambda},\zeta\Big), \Big(\frac{1}{\lambda},0\Big)\Big) = \\ = \quad \zeta &= \rho^*\Big(\Big(\frac{1}{\lambda},0\Big), \Big(\frac{1}{\lambda},\zeta\Big)\Big) = \rho^*(a^*_{\lambda,0}, a^*_{\lambda,\zeta}) \end{aligned} \tag{3}$$

for every $\lambda \in \Lambda$ and $\zeta \in \Xi$. From (3) it is not difficult to deduce that the family $(a^*_{\lambda,\xi})_{\lambda,\xi}$ is both weakly $(\mathcal{UF})$-forward and weakly $(\mathcal{UF})$-backward exhaustive at 0.

We now give some kinds of convergences for families, which are some necessary and sufficient conditions for exchange of limits. We extend to our setting the concepts of Arzelà, Alexandroff and strong uniform convergence.

Definitions 3.5. (a) Fix $\xi \in \Xi$, and let $\mathcal{S}_\xi$ be a filter of Ξ. We say that $(a_{\lambda,\xi})_{\lambda,\xi}$ *$(\mathcal{UF})$-forward strongly uniformly converges to $(a_\xi)_\xi$ at ξ* (shortly, $a_{\lambda,\xi} \overset{\mathcal{UF}\, fw-\mathcal{T}^s}{\longrightarrow} a_\xi$) iff there exists a family $(U_\pi)_\pi \in \mathcal{U}$ such that for each $\pi \in \Pi$ there is $F \in \mathcal{F}$ such that for every $\lambda \in F$ there is a set $S_\lambda \in \mathcal{S}_\xi$ with $\rho(a_\zeta, a_{\lambda,\zeta}) \in U_\pi$ whenever $\zeta \in S_\lambda$.

(b) We say that $(a_{\lambda,\xi})_{\lambda,\xi}$ is *$(\mathcal{UF})$-forward Arzelà convergent to $(a_\xi)_\xi$ at ξ* (in brief, $a_{\lambda,\xi} \overset{\mathcal{UF}\, fw-Arz.}{\longrightarrow} a_\xi$) iff there exists a family $(U_\pi)_{\pi\in\Pi} \in \mathcal{U}$ such that for every $\pi \in \Pi$, $F \in \mathcal{F}$ there are a finite set $\{\lambda_1, \lambda_2, \dots, \lambda_q\} \subset F$ and a set $S \in \mathcal{S}_\xi$, such that for each $\zeta \in S$ there is $j \in [1,q]$ with $\rho(a_\zeta, a_{\lambda_j,\zeta}) \in U_\pi$.

(c) If $\mathcal{S}_\xi$, $\xi \in \Xi$, is a family of filters of Ξ, then we say that a *finitely uniform cover* of Ξ is a family $\mathcal{V}$ of subsets of Ξ such that $\Xi = \bigcup_{V\in\mathcal{V}} V$, and for every $\xi \in \Xi$ there are a set $S_\xi \in \mathcal{S}_\xi$ and a finite subset $\mathcal{Y} := \{V_{l_1}, \dots, V_{l_q}\}$ of $\mathcal{V}$, such that for each $\zeta \in S_\xi$ there exists $j \in [1,q]$ with $\zeta \in V_{l_j}$.

(d) The family $(a_{\lambda,\xi})_{\lambda,\xi}$ is said to *$(\mathcal{UF})$-forward strongly uniformly* (resp. *$(\mathcal{UF})$-forward Arzelà) converge to $(a_\xi)_\xi$ on Ξ* iff it $(\mathcal{UF})$-strongly uniformly (resp. $(\mathcal{UF})$-Arzelà) converges to $(a_\xi)_\xi$ at ξ for every $\xi \in \Xi$ with respect to a single family $\mathbf{U} \in \mathcal{U}$, independent of ξ.

(e) We say that $(a_{\lambda,\xi})_{\lambda,\xi}$ is *$(\mathcal{UF})$-forward Alexandroff convergent to $(a_\xi)_\xi$ on Ξ* (shortly, $a_{\lambda,\xi} \overset{\mathcal{UF}\, fw-Al.}{\longrightarrow} a_\xi$ on Ξ) iff there exists a family $(U_\pi)_\pi \in \mathcal{U}$ such that for each $\pi \in \Pi$ and $F \in \mathcal{F}$ there are a nonempty set $\Lambda_0 \subset F$ and a finitely uniform cover $\{V_\lambda : \lambda \in \Lambda_0\}$ of Ξ with $\rho(a_\zeta, a_{\lambda,\zeta}) \in U_\pi$ for any $\lambda \in \Lambda_0$ and $\zeta \in V_\lambda$.

Note that, analogously as above, it is possible to formulate the corresponding concepts of (backward) filter strong uniform, Arzelà and Alexandroff convergence.

Theorem 3.6. *Let $\xi \in \Xi$ be fixed, $\mathcal{S}_\xi$ be a filter of Ξ, and suppose that*

3.6.1) $(\Lambda\mathcal{U}\mathcal{S}_\xi)\lim\limits_{\zeta\to\xi} a_{\lambda,\zeta} = a_{\lambda,\xi}$;

3.6.2) *the family* $(a_{\lambda,\zeta})_{\lambda,\zeta}$ $(\Xi\mathcal{U}\mathcal{F})$*-converges to* $(a_\zeta)_\zeta$.

Then the following are equivalent:

(i) $(a_\zeta)_\zeta$ $(\mathcal{U}\mathcal{S}_\xi)$*-backward converges to* a_ξ *as* $\zeta\to\xi$;

(ii) $a_{\lambda,\xi} \stackrel{\mathcal{U}\mathcal{F}\,fw-\mathcal{T}^s}{\longrightarrow} a_\xi$ *at* ξ;

(iii) $a_{\lambda,\xi} \stackrel{\mathcal{U}\mathcal{F}\,fw-Arz.}{\longrightarrow} a_\xi$ *at* ξ.

Remarks 3.7. (a) In general, Theorem 3.6 does not hold, when the involved "forward" convergences are replaced by the corresponding "backward" ones. Indeed, for example, let $\Lambda:=\mathbb{N}$, $\mathcal{F}$ be any filter of $\mathbb{N}$, $\Xi=[0,1]$ be endowed with the usual metric, $\xi=1$, $\mathcal{S}_\xi$ be the filter of all neighborhoods of 1 contained in $[0,1]$, $R=Y=\mathbb{R}$, $\mathcal{U}:=\{(\{\zeta\in\mathbb{R}\colon\ \zeta\le\varepsilon\,u\})_{\varepsilon\in\mathbb{R}^+} : u\in\mathbb{R}^+\}$, $\rho(a,b)=b-a$, a, $b\in\mathbb{R}$. Put $a_{n,\zeta}:=\zeta^n$, $n\in\mathbb{N}$, $\zeta\in[0,1]$. We get $\lim\limits_{\zeta\to\xi} a_{n,\zeta}=1$ for every $n\in\mathbb{N}$, and

$$a_\zeta := \lim_n a_{n,\zeta} = \begin{cases} 0, & \text{if } 0\le\zeta<1,\\[2ex] 1, & \text{if } \zeta=1.\end{cases}$$

Note that for each $\varepsilon>0$ and $n\in\mathbb{N}$ we get

$$\rho(a_{n,\zeta},a_\zeta) = a_\zeta - a_{n,\zeta} = \begin{cases} -\zeta^n<\varepsilon, & \text{if } 0\le\zeta<1,\\[2ex] 0<\varepsilon, & \text{if } \zeta=1.\end{cases}$$

Hence, $a_{n,\xi} \stackrel{\mathcal{U}\mathcal{F}\,bw-\mathcal{T}^s}{\longrightarrow} a_\xi$ at ξ. On the other hand, for every $n\in\mathbb{N}$ and for each neighborhood S of 1 contained in $[0,1]$ there is a real number $\zeta\in S\cap]0,1[$, close enough to 1, with $\zeta>\dfrac{1}{2^{1/n}}$, and hence

$$\rho(a_\zeta,a_{n,\zeta}) = a_{n,\zeta}-a_\zeta = \zeta^n > \frac{1}{2}.$$

Thus, $a_{n,\xi} \stackrel{\mathcal{U}\mathcal{F}\,fw-\mathcal{T}^s}{\not\longrightarrow} a_\xi$ at ξ. The family $(a_\zeta)_\zeta$ $(\mathcal{U}\mathcal{S}_\xi)$-forward, but not backward converges, to a_ξ as $\zeta\to\xi$: indeed for every $\zeta\in[0,1[$ we have $\rho(a_\xi,a_\zeta)=a_\zeta-a_\xi=-1<\varepsilon$ for each $\varepsilon>0$, but $\rho(a_\zeta,a_\xi)=a_\xi-a_\zeta=1$. Note that the function $\zeta\mapsto a_\zeta$, $\zeta\in[0,1]$, is upper semicontinuous, but not lower semicontinuous at 1.

(b) Observe that Theorem 3.6 does not hold, where in 3.6.1) the involved convergence is replaced by the corresponding backward or forward convergence.

Let Λ, $\mathcal{F}$, R, Y, $\mathcal{U}$, ρ be as in (a), $\Xi:=\mathbb{R}$ be endowed with the usual metric, $\xi=0$ and $\mathcal{S}_\xi$ be the filter of all neighborhoods of 0. Set

$$a_{n,\zeta} := \begin{cases} 0, & \text{if } \zeta\in\left]-\infty,-\dfrac{1}{n}\right]\cup\{0\}\cup\left[\dfrac{1}{n},+\infty\right[,\\[2ex] 1, & \text{otherwise.}\end{cases}$$

Observe that $a_\zeta := \lim_n a_{n,\zeta} = 0$ for every $\zeta \in \mathbb{R}$, so that 3.6.2) holds, and the condition (i) of Theorem 3.6 is fulfilled. Moreover it is not difficult to see that, for each $n \in \mathbb{N}$, $a_{n,\zeta}$ converges backward, but not forward, to $a_{n,0} = 0$ as ζ tends to 0, and hence 3.6.1) is not verified. However, note that for every $n \in \mathbb{N}$ and for every neighborhood U of 0 there is $\zeta \in U$ with $a_{n,\zeta} = 1$, and hence $\rho(a_\zeta, a_{n,\zeta}) = a_{n,\zeta} - a_\zeta = 1$. Thus, the condition (ii) of Theorem 3.6 is not satisfied.

Furthermore, if we define $b_{n,\zeta}$, $n \in \mathbb{N}$, $\zeta \in \mathbb{R}$, by

$$b_{n,\zeta} := \begin{cases} 1, & \text{if } \zeta \in \left]-\infty, -\frac{1}{n}\right] \cup \left[\frac{1}{n}, +\infty\right[, \\ 2, & \text{if } \zeta = 0, \\ 0, & \text{otherwise,} \end{cases}$$

then

$$b_\zeta := \lim_n b_{n,\zeta} = \begin{cases} 1, & \text{if } \zeta \neq 0, \\ \\ 2, & \text{if } \zeta = 0. \end{cases}$$

Hence, 3.6.2) is satisfied, but the condition (i) of Theorem 3.6 does not hold. Observe that, for any $n \in \mathbb{N}$, $b_{n,\zeta}$ converges forward, but not backward, to $b_{n,0} = 2$ as ζ tends to 0, and hence 3.6.1) is not satisfied. On the other hand, since $\rho(b_\zeta, b_{n,\zeta}) = b_{n,\zeta} - b_\zeta \leq 0$ for any $n \in \mathbb{N}$ and $\zeta \in \mathbb{R}$, we get that condition (ii) of Theorem 3.6 is fulfilled.

We now turn to the main theorem in our abstract setting.

Theorem 3.8. *Let $\mathcal{S}_\xi$, $\xi \in \Xi$, be a family of filters of Ξ, with the property that $\xi \in S$ for every $\xi \in \Xi$ and $S \in \mathcal{S}_\xi$. Suppose that* 3.6.2) *holds, and that*

3.8.1) $(\Lambda\mathcal{U}\mathcal{S}_\xi) \lim_{\zeta \to \xi} a_{\lambda,\zeta} = a_{\lambda,\xi}$ *for each $\xi \in \Xi$ with respect to a single family $\mathbf{Y} \in \mathcal{U}$, independent both of λ and ξ.*

Then the following are equivalent:

(i) $(a_\zeta)_\zeta$ $(\mathcal{U}\mathcal{S}_\xi)$*-backward converges to a_ξ as $\zeta \to \xi$ for every $\xi \in \Xi$, with respect to a single family $\mathbf{U} \in \mathcal{U}$, independent of ξ;*

(ii) $a_{\lambda,\xi} \overset{\mathcal{U}\mathcal{F}\, fw-\mathcal{T}^s}{\longrightarrow} a_\xi$ *on Ξ;*

(iii) $a_{\lambda,\xi} \overset{\mathcal{U}\mathcal{F}\, fw-Al.}{\longrightarrow} a_\xi$ *on Ξ;*

(iv) $a_{\lambda,\xi} \overset{\mathcal{U}\mathcal{F}\, fw-Arz.}{\longrightarrow} a_\xi$ *on Ξ;*

(v) $(a_{\lambda,\xi})_{\lambda,\xi}$ *is weakly $(\mathcal{U}\mathcal{F})$-backward exhaustive on Ξ.*

Remark 3.9. Observe that, when the function ρ is symmetric, Theorems 3.3, 3.6 and 3.8 can be viewed as necessary and sufficient conditions in order to have exchange of limits.

4 Applications to set functions

In this section, as consequences of Theorems 3.3, 3.6 and 3.8, we will give some necessary and sufficient conditions for some kind of continuity and semicontinuity of the limit of set functions. We givea result on continuity from below of the limit measure. Note that, thanks to the limit theorems existing in the literature, these conditions are often fulfilled. However, we give an example in which these properties do not hold in the setting of filter convergence.

Let Λ be any nonempty set, $\mathcal{F}$ be any filter of Λ, G be any infinite set, Σ be a σ-algebra of subsets of G, (R, Y) be a (symmetric) cone metric semigroup, $\Xi := \mathbb{N} \cup \{+\infty\}$, $\xi := +\infty$, $\mathcal{S}_\xi := \{F \cup \{+\infty\}\colon F \in \mathcal{F}_{\text{cofin}}\}$. It is not difficult to check that $\mathcal{S}_\xi$ is a filter of Ξ. Moreover, let $\mathcal{U}$ be a fixed Π-system associated with (R, Y).

A set function $m : \Sigma \to R$ is said to be *$\mathcal{U}$-continuous from below* (resp. *from above*) on Σ iff $(\mathcal{U}\mathcal{S}_\xi) \lim\limits_k \rho(m(C_k), m(C)) = 0$ for every increasing (resp. decreasing) sequence $(C_k)_k$ in Σ whose union (resp. intersection) is equal to C. A consequence of Theorems 3.3 and 3.6 is the following

Theorem 4.1. *Let $m_\lambda : \Sigma \to R$, $\lambda \in \Lambda$, be a family of set functions, $\mathcal{U}$-continuous from below on Σ with respect to a family $\mathbf{U} \in \mathcal{U}$ independent of λ. Suppose that*

4.1.1) *$m(E) := (\mathcal{U}\mathcal{F}) \lim\limits_\lambda m_\lambda(E)$, $E \in \Sigma$, exists in R with respect to a family $\mathbf{V} \in \mathcal{U}$ independent of $E \in \Sigma$.*

Then the following are equivalent:

(i) m is $\mathcal{U}$-continuous from below on Σ;

(ii) for every increasing sequence $(C_k)_k$ in Σ there is a family $(W_\pi)_\pi \in \mathcal{U}$ such that for any $\pi \in \Pi$ there is $\overline{k} \in \mathbb{N}$ such that, for every $k \geq \overline{k}$, there is a set $F \in \mathcal{F}$ with $\rho\left(m_\lambda\left(C_k\right), m_\lambda\left(C\right)\right) \in W_\pi$ for each $\lambda \in F$;

(iii) for any increasing sequence $(C_k)_k$ in Σ there is a family $(U_\pi)_\pi \in \mathcal{U}$ such that for every $\pi \in \Pi$ there is $F \in \mathcal{F}$ such that for each $\lambda \in F$ there exists a positive integer k_λ with $\rho\left(m(C_k), (m_\lambda\left(C_k\right)\right)$ for any $k \geq k_\lambda$;

(iv) for every increasing sequence $(C_k)_k$ in Σ there is a family $(Y_\pi)_\pi \in \mathcal{U}$ such that for each $\pi \in \Pi$ and $F \in \mathcal{F}$ there are $\lambda_1, \ldots, \lambda_q \in F$ and $\overline{k} \in \mathbb{N}$ such that for each $k \geq \overline{k}$ there exists $j \in [1, q]$ with $\rho(m(C_k), m_{\lambda_j}(C_k)) \in Y_\pi$.

Remarks 4.2. (a) Observe that results analogous to Theorem 4.1 hold when the involved set functions m_λ, $\lambda \in \Lambda$, are $\mathcal{U}$-continuous from above or *$\mathcal{U}$-(s)-bounded* on Σ, that is if $(\mathcal{U}) \lim\limits_k \rho(m_\lambda(A_k), 0) = 0$ for every disjoint sequence $(A_k)_k$ in Σ.

(b) Note that the conditions (ii)-(iv) of Theorem 4.1 are just satisfied, for example when $R = Y$ is a Dedekind complete lattice group, $\rho(a, b) = |a - b|$, a, $b \in \mathbb{R}$, $\Lambda = \mathbb{N}$, $\mathcal{F} = \mathcal{F}_{\text{cofin}}$ and $(m_n)_n$ is a sequence of σ-additive positive R-valued measures, thanks to the classical limit theorems.

The next step is to give necessary and sufficient conditions for absolute continuity of the limit measure.

Let $\nu : \Sigma \to \mathbb{R}_0^+$ be a finitely additive measure. We endow Σ with the Fréchet-Nikodým topology generated by the pseudometric $\rho_\nu(D, E) := |\nu(D) - \nu(E)|$, D, $E \in \Sigma$. Pick now $\Xi = \Sigma$, and for each $E \in \Sigma$ let $\mathcal{S}_E$ be the filter generated by the base $\mathcal{W} := \{\{D \in \Sigma : \rho_\nu(D, E) < \eta\}\colon \eta > 0\}$.

We say that $(m_\lambda)_\lambda$ is *weakly* $(\mathcal{UF})$*-ν-exhaustive at* $E \in \Sigma$ iff there is a family $(U_\pi)_\pi \in \mathcal{U}$ (depending on E) such that for each $\pi \in \Pi$ there is $\eta > 0$ such that for every $D \in \Sigma$ with $\rho_\nu(D, E) < \eta$ there is a set $F_D \in \mathcal{F}$ with $\rho(m_\lambda(D), m_\lambda(E)) \in U_\pi$ whenever $\lambda \in F$. We say that $(m_\lambda)_\lambda$ is *weakly* $(\mathcal{UF})$*-ν-exhaustive on* Σ iff it is weakly $(\mathcal{UF})$-ν-exhaustive at every $E \in \Sigma$ with respect to a family $\mathbf{X} \in \mathcal{U}$ independent of $E \in \Sigma$.

A measure $m : \Sigma \to R$ is said to be *$\mathcal{U}$-ν-continuous at* $E \in \Sigma$ iff there is a family $(U_\pi)_\pi \in \mathcal{U}$ (depending on E) such that for every $\pi \in \Pi$ there is $\eta > 0$ with $\rho(m(D), m(E)) \in U_\pi$ whenever $\rho_\nu(D, E) < \eta$. We say that m is *globally $\mathcal{U}$-ν-continuous on* Σ with respect to ν iff it is $\mathcal{U}$-ν-continuous at E with respect to ν for each $E \in \Sigma$, relatively to a family $\mathbf{T} \in \mathcal{U}$, independent of $E \in \Sigma$.

The next result is a consequence of Theorem 3.8.

Theorem 4.3. *Let $m_\lambda : \Sigma \to R$, $\lambda \in \Lambda$, be a family of measures, $\mathcal{U}$-ν-continuous at a fixed set $E \in \Sigma$ (resp. globally $\mathcal{U}$-ν-continuous on Σ) with respect to a family $\mathbf{Z} \in \mathcal{U}$ independent of λ, and $(\Xi\mathcal{UF})$-convergent to a measure $m_0 : \Sigma \to R$. Then the following are equivalent:*

(i) the limit measure m_0 is $\mathcal{U}$-ν-continuous at E (resp. globally $\mathcal{U}$-ν-continuous on Σ);

(ii) the net m_λ, $\lambda \in \Lambda$, is weakly $(\mathcal{UF})$-exhaustive at E (resp. on Σ);

(iii) there is a family $(U_\pi)_\pi \in \mathcal{U}$, depending on $E \in \Sigma$ (resp. independent of $E \in \Sigma$), such that for each $\pi \in \Pi$ there is $F \in \mathcal{F}$ such that for every $\lambda \in F$ there is $\eta > 0$ with $\rho(m_0(D), m_\lambda(D)) \in U_\pi$ for each $D \in \Sigma$ with $\rho_\nu(D, E) < \eta$.

(iv) There is a family $(Y_\pi)_\pi \in \mathcal{U}$, depending on $E \in \Sigma$ (resp. independent of $E \in \Sigma$), such that for every $\pi \in \mathbb{N}$ and $F \in \mathcal{F}$ there are $\lambda_1, \lambda_2, \ldots, \lambda_q \in F$ and a positive real number η such that for any $D \in \Sigma$ with $\rho_\nu(D, E) < \eta$ there exists $j \in [1, q]$ with $\rho(m_0(D), m_{\lambda_j}(D)) \in Y_\pi$.

Moreover, if the m_λ's are globally $\mathcal{U}$-ν-continuous, the statements (i)*-*(iv) *are equivalent to the following:*

(v) there is a family $(W_\pi)_\pi \in \mathcal{U}$ such that for any $\pi \in \Pi$ and $F \in \mathcal{F}$ there exist a nonempty set $\Lambda_0 \subset F$ and a finitely uniform cover $\{V_\lambda : \lambda \in \Lambda_0\}$ of Σ with $\rho(m_0(D), m_\lambda(D)) \in W_\pi$ whenever $\lambda \in \Lambda_0$ and $D \in V_\lambda$.

Remarks 4.4. (a) Observe that, when $\Lambda = \mathbb{N}$, $\mathcal{F} = \mathcal{F}_{\text{cofin}}$, m_n, $n \in \mathbb{N}$, are positive σ-additive measures, R is a Dedekind complete lattice group, $Y = R$, $\rho(a, b) = |b - a|$, a, $b \in R$, we get that the conditions (ii)-(v) of Theorem 4.3 are fulfilled, thanks to the limit theorems existing in the literature.

(b) Let $\Sigma = \mathcal{P}(\mathbb{N})$ be the class of all subsets of $\mathbb{N}$, $\mathcal{F}$ be a filter containing $\mathcal{F}_{\text{cofin}}$ and $\nu(A) = \sum_{k \in A} \frac{1}{2^k}$, $A \in \Sigma$. For each $n \in \mathbb{N}$, let us define the Dirac measure $\delta_n : \Sigma \to \mathbb{R}$ by

$$\delta_n(A) := \begin{cases} 1, & \text{if } n \in A, \\ \\ 0, & \text{if } n \in \mathbb{N} \setminus A. \end{cases} \tag{4}$$

It is not difficult to see that δ_n is σ-additive on Σ. Moreover, δ_n is ν-continuous at $\emptyset$ (that is, ν-absolutely continuous): indeed, if $\vartheta_n = \frac{1}{2^n}$ and $\nu(A) < \vartheta_n$, then $n \notin A$, and hence $\delta_n(A) = 0$. We claim that the sequence $(\delta_n)_n$ is not weakly $\mathcal{F}$-exhaustive at $\emptyset$. Indeed, observe that for each $\vartheta > 0$ there is a cofinite set $D_\vartheta \subset \mathbb{N}$ with $\nu(D_\vartheta) < \vartheta$. Note that, since $\mathcal{F}$ contains $\mathcal{F}_{\text{cofin}}$, every element of $\mathcal{F}$ is infinite, otherwise $\emptyset \in \mathcal{F}$, which is impossible. Furthermore, observe that for every infinite subset $F \subset \mathbb{N}$, and a fortiori for any $F \in \mathcal{F}$, there are a sufficiently large integer $\overline{n} \in F \cap D_\vartheta$, so that $\delta_{\overline{n}}(D_\vartheta) = 1$. From this we deduce that the sequence $(\delta_n)_n$ is not weakly $\mathcal{F}$-exhaustive at $\emptyset$. If $\mathcal{F}$ is an ultrafilter of $\mathbb{N}$ containing $\mathcal{F}_{\text{cofin}}$, then for every $A \subset \mathbb{N}$ we have

$$\delta'(A) := (\mathcal{F}) \lim_n \delta_n(A) = \begin{cases} 1, & \text{if } A \in \mathcal{F}, \\ \\ 0, & \text{if } A \notin \mathcal{F}. \end{cases} \tag{5}$$

We claim that δ' is not ν-continuous at $\emptyset$. Indeed, fix arbitrarily $\eta > 0$ and let $\overline{k} \in \mathbb{N}$ be such that $\frac{1}{2^{\overline{k}-1}} \leq \eta$. Let A be any element of $\mathcal{F}$ and set $A^* := A \cap ([\overline{k}, +\infty[)$, then $A^* \in \mathcal{F}$. We get $\nu(A^*) \leq \sum_{k=\overline{k}}^{\infty} \frac{1}{2^k} = \frac{1}{2^{\overline{k}-1}} \leq \eta$ and $\delta'(A^*) = 1$, getting the claim.

Furthermore, in this case, the conditions (i)-(iv) in Theorem 4.1 do not hold. Indeed, choose a filter $\mathcal{F}$ of $\mathbb{N}$ containing $\mathcal{F}_{\text{cofin}}$, and let $C_k := [1, k]$, $k \in \mathbb{N}$. Observe that, as said before, every element of $\mathcal{F}$ is infinite. For every k and for any infinite set $F \subset \mathbb{N}$ there is $\overline{n} \in F \setminus C_k$, and hence we get $\delta_{\overline{n}}(\mathbb{N}) - \delta_{\overline{n}}(C_k) = 1$. Thus, in this case, the condition (ii) of Theorem 4.1 is not fulfilled. If $\mathcal{F}$ is an ultrafilter of $\mathbb{N}$, then the measure δ' defined in (5) is not σ-additive on Σ. Indeed, if A is any element of $\mathcal{F}$, then we get $\sum_{n \in A} \delta'(\{n\}) = 0$ and $\delta'(A) = 1$.

When R is a Dedekind complete lattice group, $Y = R$, $\rho(a, b) = b - a$ and $\mathcal{U}_j^{(0)}$, $j = 1, 2, 3$, are as in Example 2.3 (c), we obtain some results similar to the previous ones also for semicontinuous set functions. In this setting, the concepts of weak backward (resp. forward) filter exhaustiveness and lower (resp. upper) semicontinuity are formulated as follows.

Definitions 4.5. (a) We say that $(m_\lambda)_\lambda$ is *weakly* $(\mathcal{UF})$-ν-*backward* (resp. (*forward*) *exhaustive at* $E \in \Sigma$ iff there is a family $(U_\pi)_\pi \in \mathcal{U}$ (depending on E) such that for each $\pi \in \Pi$ there is $\eta > 0$ such that for every $D \in \Sigma$ with $\rho_\nu(D, E) < \eta$ there is a set $F_D \in \mathcal{F}$ with $m_\lambda(E) - m_\lambda(D)$ (resp. $m_\lambda(D) - m_\lambda(E)$)$\in U_\pi$ whenever $\lambda \in F$.

(b) We say that $(m_\lambda)_\lambda$ is *weakly* $(\mathcal{UF})$*-ν-backward* (resp. *forward*) *exhaustive on* Σ iff it is weakly $(\mathcal{UF})$-ν-backward (resp. forward) exhaustive at every $E \in \Sigma$ with respect to a family $\mathbf{X} \in \mathcal{U}$ independent of $E \in \Sigma$.

(c) We say that $(m_\lambda)_\lambda$ is *weakly* $(\mathcal{UF})$*-ν-exhaustive at* E (resp. *on* Σ) iff it is weakly $(\mathcal{UF})$-ν-backward and forward exhaustive at E (resp. on Σ).

(d) We say that $m : \Sigma \to R$ is $\mathcal{U}$*-ν-lower* (resp. *upper*) *semicontinuous at* $E \in \Sigma$ iff there is a family $(U_\pi)_\pi \in \mathcal{U}$ (depending on E) such that for every $\pi \in \Pi$ there is $\eta > 0$ with $m(E) - m(D)$ (resp. $m(D) - m(E)$) $\in U_\pi$ whenever $\rho_\nu(D, E) < \eta$. We say that m is *globally* $\mathcal{U}$*-ν-lower* (resp. *upper*) *semicontinuous on* Σ iff it is $\mathcal{U}$-ν-lower (resp. upper) semicontinuous at E for each $E \in \Sigma$ with respect to a family $\mathbf{T} \in \mathcal{U}$, independent of $E \in \Sigma$.

Similarly as Theorem 4.3, it is possible to prove the following result about semicontinuity of the limit set function. The next theorem is given in the case of lower semicontinuity; an analogous result holds in the setting of upper semicontinuity.

Theorem 4.6. *Suppose that $m_\lambda : \Sigma \to R$, $\lambda \in \Lambda$, are globally $\mathcal{U}$-ν-continuous on Σ with respect to a family $\mathbf{S} \in \mathcal{U}$, independent of λ, and $(\Sigma\mathcal{UF})$-convergent to a set function $m_0 : \Sigma \to R$. Then the following are equivalent:*

(i) m_0 *is $\mathcal{U}$-ν-lower semicontinuous at E (resp. globally $\mathcal{U}$-ν-lower semicontinuous on Σ);*

(ii) the family $(m_\lambda)_\lambda$ is weakly $(\mathcal{UF})$-backward exhaustive at E (resp. on Σ);

(iii) there is a family $(U_\pi)_\pi \in \mathcal{U}$, depending on E (resp. independent of E), such that for any $\pi \in \Pi$ there is $F \in \mathcal{F}$ such that for every $\lambda \in F$ there is $\eta > 0$ with $m_\lambda(D) - m_0(D) \in U_\pi$ for each $D \in \Sigma$ with $\rho_\nu(D, E) < \eta$;

(iv) there exists a family $(V_\pi)_\pi \in \mathcal{U}$, depending on E (resp. independent of E), such that for every $\pi \in \Pi$ and $F \in \mathcal{F}$ there are $\lambda_1, \lambda_2, \dots, \lambda_q \in F$ and $\eta > 0$ such that for any $D \in \Sigma$ with $\rho_\nu(D, E) < \eta$ there exists $j \in [1, q]$ with $m_{\lambda_j}(D) - m_0(D) \in V_\pi$.

Moreover, global $\mathcal{U}$-ν-lower semicontinuity of m_0 is equivalent to the following condition:

(v) there is a family $(W_\pi)_\pi \in \mathcal{U}$, such that for every $\pi \in \Pi$ and $F \in \mathcal{F}$ there exist a nonempty set $\Lambda_0 \subset F$ and a finitely uniform cover $\{V_\lambda : \lambda \in \Lambda_0\}$ of Σ with $m_\lambda(D) - m_0(D) \in W_\pi$ for all $\lambda \in \Lambda_0$ and $D \in V_\lambda$.

Remark 4.7. Let $\mathcal{F}$ be an ultrafilter of $\mathbb{N}$ containing $\mathcal{F}_{\text{cofin}}$, ν be as in Remark 4.4 (b) and δ', δ_n, $n \in \mathbb{N}$, be as in (5), (4), respectively. It is not difficult to check that the sequence $(\delta_n)_n$ is weakly $(\mathcal{F})$-ν-backward exhaustive, but not weakly $(\mathcal{F})$-ν-forward exhaustive at $\emptyset$, and that δ' is ν-lower semicontinuous, but not ν-upper semicontinuous at $\emptyset$.