A simple equation for the free-fall time in air

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Abstract

The free-fall time in air was studied using the new dimensionless number G_H (here mentioned as the Galilei-Huygens number to commemorate the achievements of Galileo Galilei and Christian Huygens in physics of free fall), a combination of the falling body mass and effective cross-section area, air density, and air drag coefficient. This number equals zero in vacuum and can be interpreted as the ratio of the air drag resistance force, calculated for the final velocity of the freefall in vacuum from the same height, to the gravity force. The free-fall time in air is shown to be a function of two parameters: the free-fall time in vacuum and dimensionless parameter G_H . In most practical cases ($G_H \leq 15$), this function can be closely approximated as the product of the free-fall time in vacuum and a linear function of the parameter G_H. To illustrate the accuracy and simplicity of the approximate equation, the free-fall time was calculated for various spherical bodies (ping-pong and tennis balls, hailstones, basketball, and track-and-field men's shot) if dropped off the Leaning Tower of Pisa. The results obtained are straightforward and traceable and can be of educational value and interest for physics teachers and students.

I. INTRODUCTION

One of the most popular topics of physics for students is the free fall in vacuum and air. The earliest fundamental studies performed by Galileo and later Huygens are famous (they are compared in particular in essay [1]). The free vertical fall with quadratic air resistance is well studied analytically [2 - 4, etc.] but the direct equation for the free-fall time in air is a relatively cumbersome to apply and interpret without calculator or computer. .A simple and straightforward equation is needed to encourage the interest and participation of students.

II. PRECISE SOLUTION

The mathematical model of this effect can be described by the equation

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$$M\frac{dV}{dt} = Mg - \frac{1}{2}C_d\,\rho_0\,S\,V^2 \tag{1}$$

with the initial conditions $V(0) = 0,\ X(0) = 0$ (at time $t = 0$ when the body starts freely falling down from the altitude H).

Here $V(t) = \dfrac{dX}{dt}$ and $X(t)$ are the instantaneous velocity and displacement of the falling body at time t; $M, S,$ and C_d are respectively the mass, effective cross-section area, and the drag coefficient (a dimensionless number depending on the geometry and velocity of the falling body); ρ_0 is the air density; g is the acceleration of gravity.

To reduce the number of the parameters, introduce the dimensionless variables

$$\xi = \frac{X}{H},\quad \tau = \frac{t}{t_v} \tag{2}$$

where the free-fall time in vacuum

$$t_v = \sqrt{\frac{2H}{g}}\,. \tag{3}$$

Substituting Eqs (2) into Eq. (1), obtain equation

$$\frac{d\upsilon}{dt} = 2 - \frac{1}{2}G_H\,\upsilon^2 \tag{4}$$

for the dimensionless velocity $\upsilon(\tau) = \dfrac{d\xi}{d\tau}$. Here, the dimensionless parameter

$$G_H = \frac{C_d \, \rho_0 \, S H}{M} \tag{5}$$

will be mentioned as the Galilei-Huygens number ((in favor of Galileo Galilei and Christian Huygens for their important inputs in the physics of free fall). The parameter G_H can be interpreted as the ratio of the air drag resistance force $C_d \rho_0 S \, g \, H$ calculated for the final velocity $V = \sqrt{2gH}$ of the freefall in vacuum, to the gravity force Mg.

As known [6], the solution of Eq. (4) with the initial condition $\upsilon(0) = 0$ is expressed in the form

$$\upsilon(\tau) = \frac{d\xi}{d\tau} = \frac{2 \tanh\left(\sqrt{G_H}\,\tau\right)}{\sqrt{G_H}}. \tag{6}$$

if $G_H > 0.$ Integrating Eq. (5) with the initial condition $\xi(0) = 0$, obtain

$$\xi(\tau) = \frac{2}{G_H} \ln\left[\cosh\left(\sqrt{G_H}\,\tau\right)\right] \tag{7}$$

The dimensionless free-fall time in air is calculated from Eq. (6) at the condition $\xi = 1$ (that is, for $X = H$):

$$\tau_a = \frac{t_a}{t_v} = \frac{1}{\sqrt{G_H}} \operatorname{acosh}\left[\exp\left(\frac{G_H}{2}\right)\right]. \tag{8}$$

As shown in the next chapter, the dimensionless time $\tau_a \to 1$ and therefore fits the free-fall time in vacuum if $G_H \to 0.$

III. APPROXIMATE SOLUTION

Now Eq. (7) is rewritten as

$$\cosh\left(\sqrt{G_H}\,\tau_a\right) = \exp\left(G_H/2\right)$$

and simplified to the approximate polynomial equation using the Taylor-Maclaurin series expansions: $\cosh(x) = 1 + x^2/2 + x^4/24 + \ldots.$ and $\exp(x) = 1 + x + x^2/2 + \ldots$. Considering

$G_H << 1$ and neglecting the terms of the second and higher order to the parameter G_H, obtain the biquadratic equation with the unknown variable τ_a:

$$\frac{1}{12}G_H\tau_a^4 + \tau_a^2 - 1 - \frac{G_H}{4} = 0 \tag{9}$$

Consider $\tau_a = 1 + \varepsilon$ where $\varepsilon = \alpha G_H << 1$, so, $\tau_a^4 \approx 1 + 4\varepsilon$ and $\tau_a^2 \approx 1 + 2\varepsilon$. Substituting such approximate relationships into Eq. (8) and neglecting the terms of the second order to the parameter G_H, obtain $\alpha \approx 1/12$ and therefore

$$\tau_a = \frac{t_a}{t_v} = 1 + \frac{G_H}{12}. \tag{10}$$

The precise and linear approximation solutions described by Eqs (7) and (10) are plotted for comparison in FIG. 1. As seen, both plots all but coincide for $G_H \le 10$ and are reasonably similar for $G_H \le 15$ even though the approximate Eq. (10) was derived for $G_H << 1$.

It can be shown that the second term of Eq. (10) generally coincides with the approximate amendment obtained in a more complicated form earlier [3]. But Eq. (10) is more convenient for use and interpretation, the limits of its application are defined, and the mathematical method utilized in this paper is more traceable than in paper [3]. Generally speaking, Eq. (10) is the linear part of the Taylor-Maclaurin series expansion of the function defined by Eq. (8) for $G_H > 0$ but due to the reduction of uncertainties Eq. (10) is true for $G_H = 0$ too. A broad-band approximation can be provided by the empirically deduced equation

$$\tau_a = \frac{t_a}{t_v} = \frac{1 + G_H/9}{\sqrt{1 + G_H/23}} \tag{11}$$

which is also plotted in FIG. 1. This equation is about as accurate as the precise analytical solution given by Eq. (8) but it is not as simple as Eq. (10).

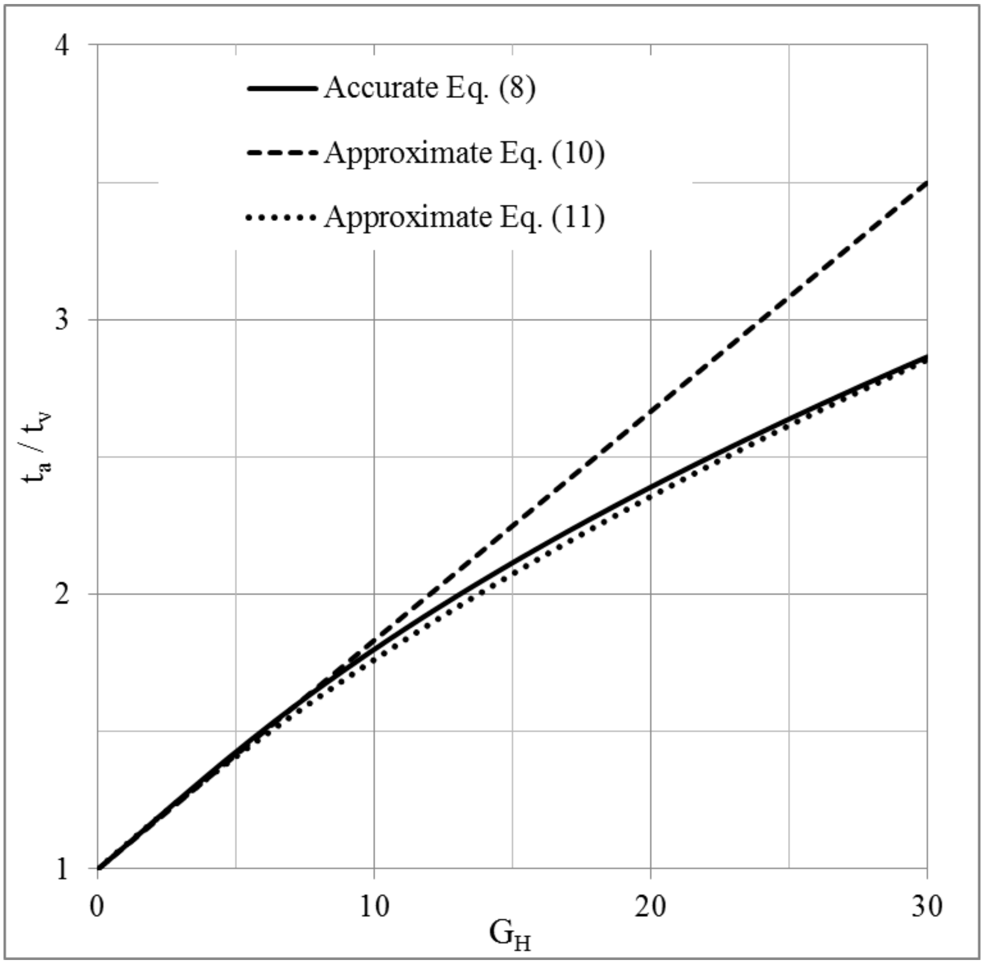

FIG. 1. The free-fall time in air calculated using the accurate equation (8) and two approximate equations (10) and (11).

IV. CALCULATION EXAMPLES

For spherical bodies, the drag coefficient $C_D \approx 0.4$ [1-3], the area $S = \pi R^2$, and the mass $M = 4\pi \rho R^3/3$, so, Eq. (5) can be reduced to the form

$$G_H = 0.4 \frac{\rho_0}{\rho} \frac{H}{R}. \qquad (12)$$

where R is the radius and ρ is the average density of the body, kg/m^3.

The trends described by Eq. (12) are plotted in FIG. 2 for the uniform balls made of steel, aluminum, ice, and wood (here, $\rho = 7800$, 2700, 900, and 500 kg/m^3, respectively; $\rho_a = 1.25$ kg/m^3). In most practical cases parameter $G_H << 10$.

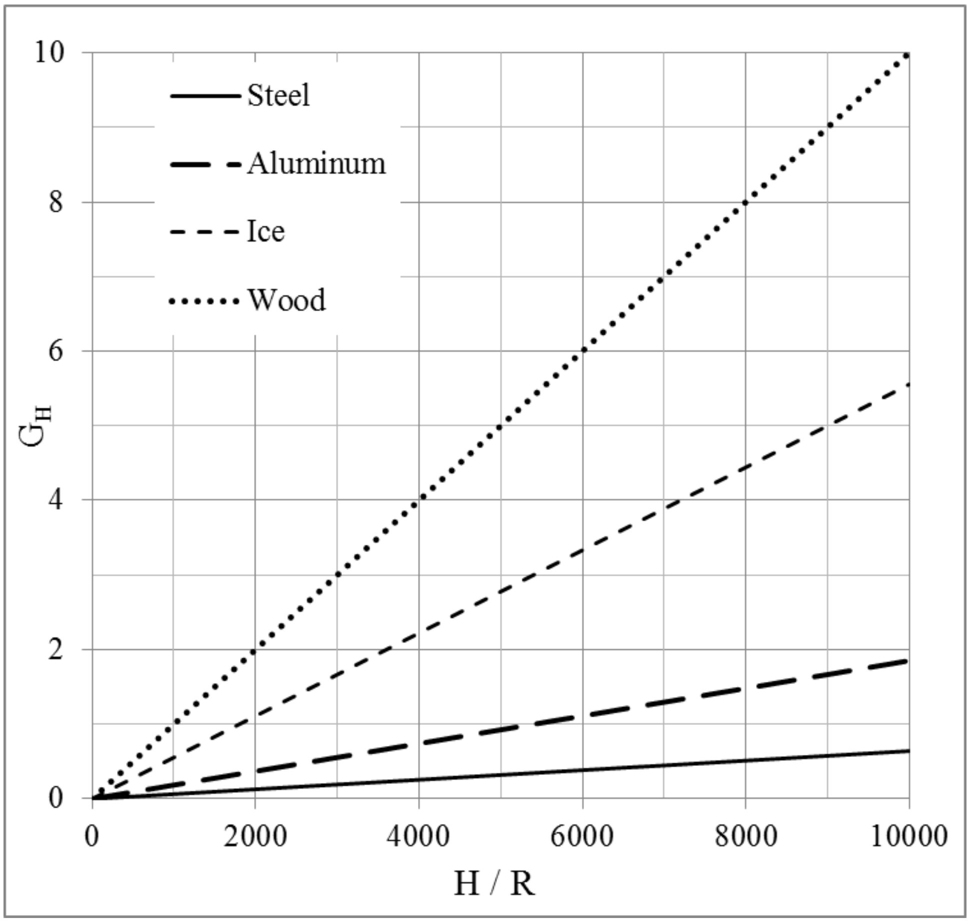

FIG. 2. Dimensionless numbers G_H vs. the dimensionless ratio H/R for uniform spherical bodies made of various materials.

However, for hollow and/or small bodies the average density ρ is low, so, the parameter G_H can be relatively high and the free-fall time in air may notably exceed that in vacuum.

Let's rewrite Eqs. (8) and (10) in a more practical form as

$$t_a = \sqrt{\frac{2H}{g}}\frac{1}{\sqrt{G_H}}\operatorname{acosh}\left[\exp\left(\frac{G_H}{2}\right)\right], \tag{13}$$

$$t_a \approx \sqrt{\frac{2H}{g}}\left(1+\frac{G_H}{12}\right), \tag{14}$$

respectively. Some calculation examples, using both the precise and approximate equations Eq. (13) and Eq. (14) are presented for comparison in Table. 1. In particular, the accuracy of Eq. (10) is quite sufficient to illustrate the effect of the air drag on the free-fall time for various bodies if dropped off the Leaning Tower of Pisa, the alleged place of Galilei's experiments with falling bodies. From the experimental viewpoint, the human visual resolution is good enough to distinguish the free-fall times for men's shot (track and field) and basketball since the estimated difference should be 0.7 s and the mean human reaction time is about 0.2 s, not to mention ping-pong balls or common hailstones.

Table 1. Free-fall times calculated with the Eqs (13) and (14) for various bodies dropped off the Leaning Tower of Pisa (≈50 m high).

Falling body	Diam., m	Mass, kg	Average density, kg/m³	G_H	Free-fall time, s	
					Accurate Eq. (13)	Approximate Eq. (14)
Ping-pong ball	0.038	0.0025	87	15.1	6.8	7.2
Hailstone	0.005	-	900	11.1	6.0	6.1
Hailstone	0.010	-	900	5.6	4.7	4.7
Tennis ball	0.067	0.058	368	2.0	3.7	3.7
Basketball	0.245	0.610	79	2.6	3.9	3.9
Track-and-field men's shot	0.120	7.260	8024	0.1	3.2	3.2
Any free-fall in vacuum	-	-	-	0	3.2	3.2

Notes: The results were rounded to the accuracy of 0.1 s which is considered as the best human reaction time.

V. CONCLUSIONS

A close-form relationship for the free-fall time in air was derived as a function of two factors: the free-fall time in vacuum and dimensionless parameter G_H (mentioned as the Galilei-Huygens number in favor of Galileo Galilei and Christian Huygens for their important inputs in the physics of free fall). This parameter can be interpreted as the ratio of the air drag resistance force, calculated for the final velocity of the freefall in vacuum, to the gravity force. For most practical cases, the relationship is reduced to a quite simple form: the product of the free-fall time in vacuum and a linear function of the parameter G_H. The accuracy and simplicity of the approximate equation are illustrated for various spherical bodies (ping-pong and tennis balls, hailstones, basketball, and track-and-field men's shot) if they were dropped off the Leaning Tower of Pisa. The results are clear and traceable and can be of educational value and interest for physics teachers and students.

1. S. Ducheyne, Steffen "Galileo and Huygens on free fall: Mathematical and methodological differences". *Dynamis : Acta Hispanica ad Medicinae Scientiarumque. Historiam Illustrandam* **28**. pp. 243–274 (2008).
2. J. Lindemuth, "The effect of air resitance on falling balls", Am.J.Phys. **39**, 757 – 759 (1971).
3. P. Timmerman and J.P. van der Weele, "On the rise and fall of a ball with linear or quadratic drag", Am.J.Phys. **67** (6), 538-546 (1998).
4. C.F. Bohren, "Dimensionless analysis, falling bodies, and the fine art of not solving differential equations", Am. J. Phys. 72 (4), 534-537 (2004).

5. H.B. Dwight, *Tables of Integrals and Other Mathematical Data*, Forth Edition (The Macmillan Company, NY, 1961).
6. E. Kamke, *Differentialgleichungen: Losungsmethoden und Losungen, I, Gewohnliche Differentialgleichungen* (B. G. Teubner, Leipzig, 1977).