Solution for Navier-Stokes Equations –
Lagrangian and Eulerian Descriptions

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Abstract – We find an exact solution for the system of Navier-Stokes equations following the Eulerian and Lagrangian descriptions, for spatial dimension n = 3. As we had seen in other previous articles, there are infinite solutions for pressure and velocity, given only the condition of initial velocity.

Keywords – Navier-Stokes equations, velocity, pressure, Eulerian description, Lagrangian description, formulation, classical mechanics, Newtonian mechanics, Newton's law, second law of Newton, equivalent systems, exact solutions, Millennium Problem, existence, smoothness, Bernoulli's law.

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Essentially the Navier-Stokes equations relate to the velocity u and pressure p suffered for a volume element dV at position (x, y, z) and time t. In the formulation or description Eulerian the position (x, y, z) is fixed in time, running different volume elements of fluid in this same position, while the time varies. In the Lagrangian formulation the position (x, y, z) refers to the instantaneous position of a specific volume element $dV = dx\, dy\, dz$ at time t, and this position varies with the movement of this same element dV.

Basically, the deduction of the Navier-Stokes equations is a classical mechanics problem, a problem of Newtonian mechanics, which use the 2nd law of Newton $F = ma$, force is equal to mass multiplied by acceleration. We all know that the force described in Newton's law may have different expressions, varying only in time or also with the position, or with the distance to the source, varying with the body's velocity, etc. Each specific problem must to define how the forces involved in the system are applied and what they mean. I suggest consulting the classic Landau & Lifshitz[1] or Prandtl's book[2] for a more detailed description of the deduction of these equations. Note that the deduction by Landau & Lifshitz [1] contain more parameters than the shown in the references [2] and [3].

In spatial dimension $n = 3$, the Navier-Stokes equations can be put in the form of a system of three nonlinear partial differential equations, as follows:

$$(1) \begin{cases} \dfrac{\partial p}{\partial x} + \dfrac{\partial u_1}{\partial t} + u_1 \dfrac{\partial u_1}{\partial x} + u_2 \dfrac{\partial u_1}{\partial y} + u_3 \dfrac{\partial u_1}{\partial z} = v\nabla^2 u_1 + \dfrac{1}{3}v\nabla_1(\nabla \cdot u) + f_1 \\[2mm] \dfrac{\partial p}{\partial y} + \dfrac{\partial u_2}{\partial t} + u_1 \dfrac{\partial u_2}{\partial x} + u_2 \dfrac{\partial u_2}{\partial y} + u_3 \dfrac{\partial u_2}{\partial z} = v\nabla^2 u_2 + \dfrac{1}{3}v\nabla_2(\nabla \cdot u) + f_2 \\[2mm] \dfrac{\partial p}{\partial z} + \dfrac{\partial u_3}{\partial t} + u_1 \dfrac{\partial u_3}{\partial x} + u_2 \dfrac{\partial u_3}{\partial y} + u_3 \dfrac{\partial u_3}{\partial z} = v\nabla^2 u_3 + \dfrac{1}{3}v\nabla_3(\nabla \cdot u) + f_3 \end{cases}$$

where $u(x, y, z, t) = \big(u_1(x, y, z, t), u_2(x, y, z, t), u_3(x, y, z, t)\big)$, $u : \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}^3$, is the velocity of the fluid, of components u_1, u_2, u_3, p is the pressure, $p : \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}$, and $f(x, y, z, t) = \big(f_1(x, y, z, t), f_2(x, y, z, t), f_3(x, y, z, t)\big)$, $f : \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}^3$, is the density of external force applied in the fluid in point (x, y, z) and at the instant of time t, for example, gravity force per mass unity, with $x, y, z, t \in \mathbb{R}$, $t \geq 0$. The coefficient $v \geq 0$ is the viscosity coefficient, and in the special case that $v = 0$ we have the Euler equations. $\nabla \equiv \left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z}\right)$ is the nabla operator and $\nabla^2 = \nabla \cdot \nabla = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2} \equiv \Delta$ is the Laplacian operator.

The non-linear terms $u_1 \dfrac{\partial u_i}{\partial x} + u_2 \dfrac{\partial u_i}{\partial y} + u_3 \dfrac{\partial u_i}{\partial z}$, $1 \leq i \leq 3$, are a natural consequence of the Eulerian formulation of motion, and corresponds to part of the total derivative of velocity with respect to time of a volume element dV in the fluid, i.e., its acceleration. If $u = (u_1(x, y, z, t), u_2(x, y, z, t), u_3(x, y, z, t))$ and these x, y, z also vary in time, $x = x(t)$, $y = y(t)$, $z = z(t)$, then, by the chain rule,

$$(2) \qquad \frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}.$$

Defining $\dfrac{dx}{dt} = u_1$, $\dfrac{dy}{dt} = u_2$, $\dfrac{dz}{dt} = u_3$, comes

$$(3) \qquad \frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}u_1 + \frac{\partial u}{\partial y}u_2 + \frac{\partial u}{\partial z}u_3,$$

and therefore

$$(4) \qquad \frac{Du_i}{Dt} = \frac{\partial u_i}{\partial t} + u_1 \frac{\partial u_i}{\partial x} + u_2 \frac{\partial u_i}{\partial y} + u_3 \frac{\partial u_i}{\partial z}, \ 1 \leq i \leq 3,$$

which contain the non-linear terms that appear in (1).

Numerically, searching a computational result, i.e., in practical terms, there can be no difference between the Eulerian and Lagrangian formulations for the evaluate of $\dfrac{Du}{Dt}$ (or $\dfrac{du}{dt}$, it is the same physical and mathematical entity). Only conceptually and formally there is difference in the two approaches. I agree, however, that you first consider (x, y, z) variable in time (Lagrangian formulation) and then consider (x, y, z) fixed (Eulerian formulation), seems to be subject to criticism. In our present deduction, starting from Navier-Stokes equations in

Eulerian description, implicitly with a solution (u, p), next the pressure, and its corresponding gradient, they travel with the volume element $dV = dx\, dy\, dz$, i.e., obeys to the Lagrangian description of motion, as well as the external force f, in order to avoid contradictions. The velocity u also will obey to the Lagrangian description, and it is representing the velocity of a generic volume element dV over time, initially at position (x_0, y_0, z_0) and with initial velocity $u^0 = u(0) = const.$, $u = u(t)$. Done the solution in Lagrangian description, the solution for pressure in Eulerian description will be given explicitly and at end a solution in function of the initial data.

Following this definition, the system (1) above can be transformed into

(5)
$$\begin{cases} \frac{\partial p}{\partial x} + \frac{Du_1}{Dt} = \nu\,(\nabla^2 u_1)|_t + \frac{1}{3}\nu(\nabla_1(\nabla \cdot u))|_t + f_1|_t \\ \frac{\partial p}{\partial y} + \frac{Du_2}{Dt} = \nu\,(\nabla^2 u_2)|_t + \frac{1}{3}\nu(\nabla_2(\nabla \cdot u))|_t + f_2|_t \\ \frac{\partial p}{\partial z} + \frac{Du_3}{Dt} = \nu\,(\nabla^2 u_3)|_t + \frac{1}{3}\nu(\nabla_3(\nabla \cdot u))|_t + f_3|_t \end{cases}$$

thus (1) and (5) are equivalent systems, according (4) validity, since that the partial derivatives of the pressure and velocities are correctly transformed to the variable time, using $\partial x = u_1\partial t,\ \partial y = u_2\partial t,\ \partial z = u_3\partial t$. The nabla and Laplacian operators are considered calculated in Lagrangian formulation, i.e., in the variable time. We first choose $u \in C^\infty(\mathbb{R}^3 \times [0, \infty))$ (say, for example, or belonging to Schwartz space, etc.), calculate the operators in variables $x,\ y,\ z,\ t$ in the Eulerian formulation, which is a standard procedure, and then we convert these results to Lagrangian formulation using $x = x(t),\ y = y(t),\ z = z(t)$, the our special method. Likewise for the calculation of $\frac{Du}{Dt}$, following (4), and external force f. In the following the procedure in more details for obtained the pressure in Lagrangian formulation, a time dependent function.

Given $u = u(x, y, z, t) \in C^\infty(\mathbb{R}^3 \times [0, \infty))$ obeying the initial conditions and an integrable vector function f, the system's solution (1) for p, using the condensed notation given by (4), is

(6)
$$p = \int_L S \cdot dl + \theta(t),$$
$$S = \nu(\nabla^2 u) + \frac{1}{3}\nu\nabla(\nabla \cdot u) + f - \frac{Du}{Dt},$$

where L is any continuous path linking a point (x_0, y_0, z_0) to (x, y, z) and $\theta(t)$ is a generic time function, physically and mathematically reasonable, for example with $\theta(0) = 0$. We are supposing that the vector S is a gradient vector function ($\nabla \times S = 0,\ S = \nabla\varphi,\ \varphi$ potential function of S).

In Eulerian description and in special case when the integrand S in (6) is a constant vector or a dependent function only on the time variable, we come to

(7)
$$p = p^0 + S_1(t)(x - x_0) + S_2(t)(y - y_0) + S_3(t)(z - z_0),$$
$$S_i(t) = \nu(\nabla^2 u_i) + \frac{1}{3}\nu\nabla_i(\nabla \cdot u) + f_i - \frac{Du_i}{Dt},$$

where $p^0 = p^0(t)$ is the pressure in the point (x_0, y_0, z_0) at time t.

When the variables x, y, z in (6) as well as f and u are in Lagrangian description, representing a motion over time of a hypothetical volume element dV or particle of fluid, we need eliminate the dependence of the position substituting in (6)

(8)
$$dl = (dx,\ dy,\ dz) = (u_1 dt,\ u_2 dt,\ u_3 dt)$$

and integrating over time. The result is

(9)
$$p(t) = p^0 + \int_0^t \sum_{i=1}^3 S_i(t)\, u_i(t)\, dt,$$

$$p^0 = p(0) = const.$$

This calculation can be more facilitated making $u_i \frac{Du_i}{Dt} dt = u_i du_i$ and $\int_0^t u_i \frac{Du_i}{Dt} dt = \int_{u_i^0}^{u_i} u_i du_i = \frac{1}{2}\left(u_i^2 - u_i^{0\,2}\right)$, so (9) is equal to

(10)
$$p(t) = p^0 - \frac{1}{2}\sum_{i=1}^3\left(u_i^2 - u_i^{0\,2}\right) + \int_0^t \sum_{i=1}^3 R_i(t)\, u_i(t)\, dt,$$
$$R_i(t) = \nu(\nabla^2 u_i)|_t + \frac{1}{3}\nu(\nabla_i(\nabla \cdot u))|_t + f_i,$$

i.e.,

(11)
$$p(t) = p^0 - \frac{1}{2}(u^2 - u^{0\,2}) + \int_0^t R \cdot u\, dt,$$
$$R = \nu(\nabla^2 u)|_t + \frac{1}{3}\nu(\nabla(\nabla \cdot u))|_t + f,$$

$p, p^0 \in \mathbb{R}, u, u^0, f, R \in \mathbb{R}^3, u = (u_1, u_2, u_3)(t),\ u^0 = (u_1^0, u_2^0, u_3^0) = u(0),$ $f = (f_1, f_2, f_3)(t)$, in Lagrangian description. $u^2 = u \cdot u$ and $u^{0\,2} = u^0 \cdot u^0$ are the square modules of the respective vectors u and u^0.

When $f = 0$ and $\nu = 0$ (or most in general $R = 0$) it is simply

(12)
$$p = p^0 - \frac{1}{2}(u^2 - u^{0\,2}),$$

which then can be considered an exact solution for Euler equations in Lagrangian description, and similarly to Bernoulli's law without external force (gravity, in special).

Unfortunately, in Eulerian description, neither

(13) $p(x, y, z, t) = p^0(x, y, z) - \frac{1}{2}(u^2 - u^{0\,2}) + \int_L R \cdot dl,$

$p^0(x, y, z) = p(x, y, z, 0),\ u^0 = u^0(x, y, z) = u(x, y, z, 0),$ nor

(14) $p(x, y, z, t) = p^0(t) - \frac{1}{2}(u^2 - u^{0\,2}) + \int_L R \cdot dl,$

$p^0(t) = p(x_0, y_0, z_0, t),\ u^0 = u^0(t) = u(x_0, y_0, z_0, t),$ solve (1) for all cases of velocities, both formulations supposing $R = \nu\nabla^2 u + \frac{1}{3}\nu\nabla(\nabla \cdot u) + f$ a gradient vector function ($\nabla \times R = 0,\ R = \nabla\phi,\ \phi$ potential function of R) when in Eulerian description.

For example, for $f = 0$ the solution (14) is valid only when

(15) $\frac{\partial p}{\partial x_i} = -\sum_{j=1}^{3} u_j \frac{\partial u_j}{\partial x_i} = -\left(\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j}\right),$

i.e.,

(16) $\frac{\partial u_i}{\partial t} = \sum_{j=1}^{3} u_j \left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right).$

How to return to the Eulerian formulation if only was obtained a complete solution in the Lagrangian formulation? As well as we can choose any convenient velocity $u(t) = (u_1(t),\ u_2(t),\ u_3(t))$ to calculate the pressure (11) that complies with the initial conditions (Lagrangian formulation), we also can choose appropriates $u(x, y, z, t)$ (Eulerian formulation) and $x(t), y(t), z(t)$ to the velocities and positions of the system and taking the corresponding inverse functions in the obtained solution. This choose is not completely free because will be necessary to calculate a system of ordinary differential equations to obtain the correct set of $x(t), y(t), z(t)$, such that

(17) $\begin{cases} \dfrac{dx}{dt} = u_1(x, y, z, t) \\ \dfrac{dy}{dt} = u_2(x, y, z, t) \\ \dfrac{dz}{dt} = u_3(x, y, z, t) \end{cases}$

Nevertheless, this yet can save lots calculation time.

It will be necessary find solutions of (17) such that it is always possible to make any point (x, y, z) of the velocity domain can be achieved for each time t, introducing for this initial positions (x_0, y_0, z_0) conveniently calculated according to (17). Thus we will have velocities and pressures that, in principle, can be calculated for any position and time, independently of one another, not only for a single position for each time. For different values of (x, y, z) and t we will, in the

general case, obtain the velocity and pressure of different volume elements dV, starting from different initial positions (x_0, y_0, z_0).

We can escape the need to solve (17), but admitting its validity and the corresponding existence of solution, previously choosing differentiable functions $x = x(t), y = y(t), z = z(t)$ and then calculating directly the solution for velocity in the Lagrangian formulation,

$$
(18) \qquad
\begin{cases}
u_1(t) = \dfrac{dx}{dt} \\[2mm]
u_2(t) = \dfrac{dy}{dt} \\[2mm]
u_3(t) = \dfrac{dz}{dt}
\end{cases}
$$

hereafter calculating $\dfrac{\partial u_i}{\partial x_j}, \dfrac{\partial^2 u_i}{\partial x_j^2}$ and the differential operations $\nabla \cdot u, \nabla(\nabla \cdot u)$ and $\nabla^2 u$ through of the transformations

$$
(19.1) \qquad \frac{\partial u_i}{\partial x_j} = \frac{\partial u_i/\partial t}{\partial x_j/\partial t} = \frac{1}{u_j}\frac{\partial u_i}{\partial t}
$$

$$
(19.2) \qquad \nabla \cdot u = \sum_{j=1}^{3} \frac{\partial u_j}{\partial x_j} = \sum_{j=1}^{3} \frac{1}{u_j}\frac{\partial u_j}{\partial t}
$$

$$
(19.3) \qquad \nabla_i(\nabla \cdot u) = \frac{\partial/\partial t}{\partial x_i/\partial t}\sum_{j=1}^{3}\frac{1}{u_j}\frac{\partial u_j}{\partial t} =
$$

$$
= \frac{1}{u_i}\sum_{j=1}^{3}\frac{1}{u_j}\left[-\frac{1}{u_j}\left(\frac{\partial u_i}{\partial t}\right)\left(\frac{\partial u_j}{\partial t}\right) + \frac{\partial^2 u_i}{\partial t^2}\right]
$$

and

$$
(20.1) \qquad \frac{\partial^2 u_i}{\partial x_j^2} = \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}\right) = \frac{\partial/\partial t}{\partial x_j/\partial t}\left(\frac{1}{u_j}\frac{\partial u_i}{\partial t}\right) =
$$

$$
= \frac{1}{u_j^2}\left[-\frac{1}{u_j}\left(\frac{\partial u_i}{\partial t}\right)\left(\frac{\partial u_j}{\partial t}\right) + \frac{\partial^2 u_i}{\partial t^2}\right]
$$

$$
(20.2) \qquad \nabla^2 u_i = \sum_{j=1}^{3}\frac{\partial^2 u_i}{\partial x_j^2} = \sum_{j=1}^{3}\frac{1}{u_j^2}\left[-\frac{1}{u_j}\left(\frac{\partial u_i}{\partial t}\right)\left(\frac{\partial u_j}{\partial t}\right) + \frac{\partial^2 u_i}{\partial t^2}\right],
$$

and finally calculating the pressure in (9) or (11), with $\dfrac{Du_i}{Dt} \equiv \dfrac{du_i}{dt}$, supposing finites the limits in equations (19) and (20) when $u_j \to 0$. Note that perhaps the denominators appearing in (19) and (20) explaining the occurrence of *blowup time* reported in the literature[3], when the limits are not finites.

Concluding, answering the question, in the result of pressure in Lagrangian formulation given by (9) or (11), conveniently transforming the initial position

(x_0, y_0, z_0) as function of a generic position (x, y, z) and time t, we will have a correct value of the pressure in Eulerian formulation. The same is valid for the velocity in Lagrangian formulation, if the correspondent Eulerian formulation was not previously obtained.

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Without passing through the Lagrangian formulation, given a velocity $u(x, y, z, t)$ at least two times differentiable and an integrable external force $f(x, y, z, t)$, perhaps a better expression for the solution of the equation (1) is, in fact,

(21)
$$p(x, y, z, t) = \sum_{i=1}^{3} \int_{x_i^0}^{x_i} S_i dx_i + \theta(t),$$

$$S_i = -\left(\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j}\right) + v(\nabla^2 u_i) + \frac{1}{3}v\left(\nabla_i(\nabla \cdot u)\right) + f_i,$$

supposing possible the integrations and that the vector $S = -\left[\frac{\partial u}{\partial t} + (u \cdot \nabla)u\right] + v\nabla^2 u + \frac{1}{3}v\nabla(\nabla \cdot u) + f$ is a gradient function. This is the development of the solution (6) for the specific path L going parallel to axes X, Y and Z from $(x_1^0, x_2^0, x_3^0) \equiv (x_0, y_0, z_0)$ to $(x_1, x_2, x_3) \equiv (x, y, z)$, since that the solution (6) is valid for any L. $\theta(t)$ is a generic time function, physically and mathematically reasonable, for example with $\theta(0) = 0$ or adjustable for any given condition. Again we have seen that the system of Euler equations has no unique solution, only given initial conditions. We can choose different velocities that have the same initial velocity and also result, in general, in different pressures.

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Another way to solve (1) seems to me to be the best of all, for its extreme ease of calculation, also without we need to resort to Lagrangian formulation and its conceptual difficulties. If $u(x, y, z, 0) = u^0(x, y, z)$ is the initial velocity of the system, valid solution in $t = 0$, then $u(x, y, z, t) = u^0(x + t, y + t, z + t)$ is a solution for velocity in $t \geq 0$, a non-unique solution. Similarly, $p(x, y, z, t) = p^0(x + t, y + t, z + t)$ is the correspondent solution for pressure in $t \geq 0$, being $p^0(x, y, z)$ the initial condition for pressure. The velocities $u^0(x + t, y, z)$, $u^0(x, y + t, z)$ and $u^0(x, y, z + t)$ are also solutions, and respectively also the pressures $p^0(x + t, y, z)$, $p^0(x, y + t, z)$ and $p^0(x, y, z + t)$. Other solutions may be searched, for example in the kind $u(x, y, z, t) = u^0(x + T_1(t), y + T_2(t), z + T_3(t))$, $T_i(0) = 0$, and therefore $p(x, y, z, t) = p^0(x + T_1(t), y + T_2(t), z + T_3(t))$.

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I think that this is better than nothing… It is no longer true that the Navier-Stokes and Euler equations do not have a general solution.

Apply these methods to the famous 6th Millenium Problem[3] on existence and smoothness of the Navier-Stokes equations is not so difficult at the same time also it is not absolutely trivial. It takes some time. I hope to do it soon. On the other hand, apply these methods to the case $n = 2$ or $\nu = 0$ (Euler equation) is almost immediate.

> *To Leonard Euler, in memorian,*
> *the greatest mathematician of all time.*
> *309th anniversary of his birth,*
> *April-15-1707-2016.*
>
> *Last update: June-20-2016.*
>
> *Euler, forgive me for my mistakes…*
> *This subject is very difficult!*

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