Some solutions of the equation $4xz = 4 + y^2$

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Abstract

In this note we show some solutions of the equation $4xz = 4 + y^2$, $(x, y, z) \in \mathbb{N}_0^3$,

and *a* relation with the constant Pi : $\pi = 3.141592 \ldots$

Keyword:Ecuación diofántica,Número Pi

I. Introducción

Notación : $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$, $\mathbb{N}_0^3 = \mathbb{N}_0 \times \mathbb{N}_0 \times \mathbb{N}_0$, $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

Sea $(x, y, z) \in \mathbb{N}_0^3$, tal que :

$$4xz = 4 + y^2 \tag{1}$$

entonces es válida la fórmula :

$$\pi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left(x u^2 + y u v + z v^2\right)} \, du \, dv \tag{2}$$

En esta nota mostramos una colección de ternas $(x, y, z) \in \mathbb{N}_0^3$ que satisfacen la ecuación (1).

La prueba de (2) se sigue de : Ref. (53) , pag .297, Prob .73.

La ecuación (1) tiene una solución trivial :

$$S_0 = (1, 0, 1) \tag{3}$$

para $x > 0$, $y > 0$, $z > 0$ observamos que el número y debe ser par, esto es : $y = 2k$, $k \in \mathbb{N}$.

Para justificar esta afirmación supongamos que y es impar : $y = 2k - 1$, $k \in \mathbb{N}$:

$$k \in \mathbb{N}, \; y = 2k - 1 \Rightarrow y^2 = 4k^2 - 4k + 1, \text{ (impar)} \Rightarrow 4 + y^2 = 4k^2 - 4k + 5,$$
$$\text{(impar)}, \text{ pero : } 4xz \text{ es par } (xz > 0), \quad \therefore y = 2k, \; k \in \mathbb{N}.$$

Sea $y = 2k$, $k \in \mathbb{N}$, se tiene :

$$4xz = 4 + 4k^2 \Rightarrow xz = 1 + k^2 \tag{4}$$

Por lo tanto las ternas (x, y, z) son de la forma :

$$S_k = \left\{(x, 2k, z) \in \mathbb{N}_0^3 \; : \; xz = 1 + k^2\right\}, \; k \in \mathbb{N}_0 \tag{5}$$

los conjuntos S_k son disjuntos, esto es :

$$S_n \cap S_m = \emptyset \ , \ n \neq m \tag{6}$$

El conjunto :

$$S = \bigcup_{k=0}^{\infty} S_k \tag{7}$$

contiene todas la soluciones de la ecuación : $4\,x\,z = 4 + y^2$, $(x,\ y,\ z) \in \mathbb{N}_0^3$.

En la ecuación (3) : $x\,z = 1 + k^2$, $k \in \mathbb{N}$, el número k puede ser impar *o* par :

Caso 1 : k impar , $k = 2\,m - 1$, $m \in \mathbb{N}$:

$$x\,z = 1 + (2\,m - 1)^2 = 2\left(2\,m^2 - 2\,m + 1\right) \Rightarrow x\,z \text{ es número par} \tag{8}$$

$$x\,z \text{ par } \Rightarrow (x \text{ par } \wedge \ z \text{ impar}) \vee (x \text{ impar } \wedge \ z \text{ par}) \tag{9}$$

En (7) el número $2\,m^2 - 2\,m + 1$, en algunos casos se puede escribir como producto de números impares : Ej : $m = 7$:

$$2 \times 7^2 - 2 \times 7 + 1 = 85 = 5 \times 17$$

Caso 2 : k par , $k = 2\,m$, $m \in \mathbb{N}$:

$$x\,z = 1 + 4\,m^2 \Rightarrow x\,z \text{ número impar} \Rightarrow x \text{ impar } \wedge \ z \text{ impar} \tag{10}$$

Para encontrar los conjuntos S_k es necesario tener la factorización entera de los números $1 + k^2$, $k \in \mathbb{N}_0$:

$$\left\{1 + k^2 \ : k \in \mathbb{N}_0\right\} = \{1,\ 2,\ 5,\ 10,\ 17,\ 26,\ 37,\ 50,\ 65,\ 82,\ 101,\ 122,\ 145,\ 170,\ 197,\ 226,\ 257,\ 290,\ 325,\ ...\} \tag{11}$$

se tiene :

$$k = 1 \Rightarrow 2 = 1 \times 2 \tag{12}$$

$$k = 2 \Rightarrow 5 = 1 \times 5 \tag{13}$$

$$k = 3 \Rightarrow 10 = 1 \times 2 \times 5 \tag{14}$$

$$k = 4 \Rightarrow 17 = 1 \times 7 \tag{15}$$

$$k = 5 \Rightarrow 26 = 1 \times 2 \times 13 \tag{16}$$

$$k = 6 \Rightarrow 37 = 1 \times 37 \tag{17}$$

$$k = 7 \Rightarrow 50 = 1 \times 2 \times 5^2 \tag{18}$$

....

II. Algunas soluciones de $4\,x\,z = 4 + y^2$

$$S_0 = \{(1,\ 0,\ 1)\} \tag{19}$$

$$S_1 = \{(1,\ 2,\ 2),\ (2,\ 2,\ 1)\} \tag{20}$$

$$S_2 = \{(1,\ 4,\ 5),\ (5,\ 4,\ 1)\} \tag{21}$$

$$S_3 = \{(1,\ 6,\ 10),\ (2,\ 6,\ 5),\ (5,\ 6,\ 2),\ (10,\ 6,\ 1)\} \tag{22}$$

$$S_4 = \{(1,\ 8,\ 17),\ (17,\ 8,\ 1)\} \tag{23}$$

$$S_5 = \{(1,\ 10,\ 26),\ (2,\ 10,\ 13),\ (13,\ 10,\ 2),\ (26,\ 10,\ 1)\} \tag{24}$$

$$S_6 = \{(1,\ 12,\ 37),\ (37,\ 12,\ 1)\} \tag{25}$$

$$S_7 = \{(1,\ 14,\ 50),\ (2,\ 14,\ 25),\ (5,\ 14,\ 10),\ (10,\ 14,\ 5),\ (25,\ 14,\ 2),\ (50,\ 14,\ 1)\} \tag{26}$$

$$S_8 = \{(1,\ 16,\ 65),\ (5,\ 16,\ 13),\ (13,\ 16,\ 5),\ (65,\ 16,\ 1)\} \tag{27}$$

$$S_9 = \{(1,\ 18,\ 82),\ (2,\ 18,\ 41),\ (41,\ 18,\ 2),\ (82,\ 18,\ 1)\} \tag{28}$$

$$S_{10} = \{(1,\ 20,\ 101),\ (101,\ 20,\ 1)\} \tag{29}$$

$$S_{11} = \{(1,\ 22,\ 122),\ (2,\ 22,\ 61),\ (61,\ 22,\ 2),\ (122,\ 22,\ 1)\} \tag{30}$$

$$S_{12} = \{(1,\ 24,\ 145),\ (5,\ 24,\ 29),\ (29,\ 24,\ 5),\ (145,\ 24,\ 1)\} \tag{31}$$

$$S_{13} = \{(1,\ 26,\ 170),\ (2,\ 26,\ 85),\ (5,\ 26,\ 34),\ (10,\ 26,\ 17),\ (17,\ 26,\ 10),\ (34,\ 26,\ 5),\ (85,\ 26,\ 2),\ (170,\ 26,\ 1)\} \tag{32}$$

$$S_{14} = \{(1,\ 28,\ 197),\ (197,\ 28,\ 1)\} \tag{33}$$

$$S_{15} = \{(1,\ 30,\ 226),\ (2,\ 30,\ 113),\ (113,\ 30,\ 2),\ (226,\ 30,\ 1)\} \tag{34}$$

$$S_{16} = \{(1,\ 32,\ 257),\ (257,\ 32,\ 1)\} \tag{35}$$

$$S_{17} = \{(1,\ 34,\ 290),\ (2,\ 34,\ 145),\ (5,\ 34,\ 58),\ (10,\ 34,\ 29),\ (29,\ 34,\ 10),\ (58,\ 34,\ 5),\ (145,\ 34,\ 2),\ (290,\ 34,\ 1)\} \tag{36}$$

$$S_{18} = \{(1,\ 36,\ 225),\ (5,\ 36,\ 65),\ (13,\ 36,\ 25),\ (25,\ 36,\ 13),\ (65,\ 36,\ 5),\ (325,\ 36,\ 1)\} \tag{37}$$

$$S_{19} = \{(1,\ 38,\ 362),\ (2,\ 38,\ 181),\ (181,\ 38,\ 2),\ (362,\ 38,\ 1)\} \tag{38}$$

$$S_{20} = \{(1,\ 40,\ 401),\ (401,\ 40,\ 1)\} \tag{39}$$

III. Otra forma de representar las soluciones de (1)

Sea $k \in \mathbb{N}_0$, $y = 2\,k$, se tiene :

$$4\,x\,z = 4 + 4\,k^2 \Rightarrow x\,z = 1 + k^2 \Rightarrow \begin{cases} x = 1 \ \wedge \ z = 1 + k^2 \\ x = 1 + k^2 \ \wedge \ z = 1 \end{cases} \tag{40}$$

por lo tanto tenemos dos conjuntos de soluciones de (1) :

$$\mathrm{SS}_1 = \left\{\left(1,\ 2\,k,\ 1 + k^2\right) \,:\, k \in \mathbb{N}_0\right\} \tag{41}$$

$$\mathrm{SS}_2 = \left\{\left(1 + k^2,\ 2\,k,\ 1\right) \,:\, k \in \mathbb{N}_0\right\} \tag{42}$$

La fórmula (2) se puede escribir como :

$$\pi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left(u^2 + 2\,k\,u\,v + \left(1 + k^2\right)v\right)}\, du\, dv\ ,\ \ k \in \mathbb{N}_0 \tag{43}$$

Sea $k \in \mathbb{N}$, $y = 4\,k - 2$, se tiene :

$$4\,x\,z = 4 + (4\,k - 2)^2 \Rightarrow x\,z = 2\left(2\,k^2 - 2\,k + 1\right) \Rightarrow \begin{cases} x = 2 \ \wedge z = 2\,k^2 - 2\,k + 1 \\ x = 2\,k^2 - 2\,k + 1 \wedge z = 2 \end{cases} \tag{44}$$

por lo tanto tenemos dos conjuntos de soluciones de (1) :

$$\mathrm{SS}_3 = \left\{\left(2,\ 4\,k - 2,\ 2\,k^2 - 2\,k + 1\right) \,:\, k \in \mathbb{N}\right\} \tag{45}$$

$$\mathrm{SS}_4 = \left\{\left(2\,k^2 - 2\,k + 1,\ 4\,k - 2,\ 2\right) \,:\, k \in \mathbb{N}\right\} \tag{46}$$

Sea $k \in \mathbb{N}$, $y = 10\,k - 4$, se tiene :

$$4\,x\,z = 4 + (10\,k - 4)^2 \Rightarrow x\,z = 5\left(5\,k^2 - 4\,k + 1\right) \Rightarrow \begin{cases} x = 5 \wedge z = 5\,k^2 - 4\,k + 1 \\ x = 5\,k^2 - 4\,k + 1 \wedge z = 5 \end{cases} \tag{47}$$

por lo tanto tenemos dos conjuntos de soluciones de (1) :

$$SS_5 = \left\{\left(5,\ 10\,k - 4,\ 5\,k^2 - 4\,k + 1\right)\ :\ k \in \mathbb{N}\right\} \tag{48}$$

$$SS_6 = \left\{\left(5\,k^2 - 4\,k + 1,\ 10\,k - 4,\ 5\right)\ :\ k \in \mathbb{N}\right\} \tag{49}$$

Sea $k \in \mathbb{N}_0$, $y = 10\,k + 4$, se tiene :

$$4\,x\,z = 4 + (10\,k + 4)^2 \Rightarrow x\,z = 5\left(5\,k^2 + 4\,k + 1\right) \Rightarrow \begin{cases} x = 5 \wedge z = 5\,k^2 + 4\,k + 1 \\ x = 5\,k^2 + 4\,k + 1 \wedge z = 5 \end{cases} \tag{50}$$

por lo tanto tenemos dos conjuntos de soluciones de (1) :

$$SS_7 = \left\{\left(5,\ 10\,k + 4,\ 5\,k^2 + 4\,k + 1\right)\ :\ k \in \mathbb{N}_0\right\} \tag{51}$$

$$SS_8 = \left\{\left(5\,k^2 + 4\,k + 1,\ 10\,k + 4,\ 5\right)\ :\ k \in \mathbb{N}_0\right\} \tag{52}$$

Otros conjuntos de soluciones son :

$$k \in \mathbb{N},\ \ y = 2\,(13\,k - 5)\ \Rightarrow \begin{cases} SS_9 = \{(13,\ 26\,k - 10,\ 13\,k^2 - 10\,k + 2)\ : k \in \mathbb{N}\} \\ SS_{10} = \{(13\,k^2 - 10\,k + 2,\ 26\,k - 10,\ 13)\ : k \in \mathbb{N}\} \end{cases} \tag{53}$$

$$k \in \mathbb{N}_0,\ \ y = 2\,(13\,k + 5)\ \Rightarrow \begin{cases} SS_{11} = \{(13,\ 26\,k + 10,\ 13\,k^2 + 10\,k + 2)\ : k \in \mathbb{N}_0\} \\ SS_{12} = \{(13\,k^2 + 10\,k + 2,\ 26\,k + 10,\ 13)\ : k \in \mathbb{N}_0\} \end{cases} \tag{54}$$

$$k \in \mathbb{N},\ \ y = 2\,(17\,k - 4)\ \Rightarrow \begin{cases} SS_{13} = \{(17,\ 34\,k - 8,\ 17\,k^2 - 8\,k + 1)\ : k \in \mathbb{N}\} \\ SS_{14} = \{(17\,k^2 - 8\,k + 1,\ 34\,k - 8,\ 17)\ : k \in \mathbb{N}\} \end{cases} \tag{55}$$

$$k \in \mathbb{N}_0,\ \ y = 2\,(17\,k + 4)\ \Rightarrow \begin{cases} SS_{15} = \{(17,\ 34\,k + 8,\ 17\,k^2 + 8\,k + 1)\ : k \in \mathbb{N}_0\} \\ SS_{16} = \{(17\,k^2 + 8\,k + 1,\ 34\,k + 8,\ 17)\ : k \in \mathbb{N}_0\} \end{cases} \tag{56}$$

$$k \in \mathbb{N},\ \ y = 2\,(10\,k - 3)\ \Rightarrow \begin{cases} SS_{17} = \{(10,\ 10\,k - 3,\ 10\,k^2 - 6\,k + 1)\ : k \in \mathbb{N}\} \\ SS_{18} = \{(10\,k^2 - 60\,k + 1,\ 10\,k - 3,\ 10)\ : k \in \mathbb{N}\} \end{cases} \tag{57}$$

$$k \in \mathbb{N}_0,\ \ y = 2\,(10\,k + 3)\ \Rightarrow \begin{cases} SS_{19} = \{(10,\ 10\,k + 3,\ 10\,k^2 + 6\,k + 1)\ : k \in \mathbb{N}_0\} \\ SS_{20} = \{(10\,k^2 + 6\,k + 1,\ 10\,k + 3,\ 10)\ : k \in \mathbb{N}_0\} \end{cases} \tag{58}$$

Poniendo $y = 2\,(a\,k \pm b)$ con a , b elegidos de manera conveniente se obtienen más conjuntos de soluciones de (1).

IV. Número Pi

La fórmula (2) se puede escribir como :

$$\pi = \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} e^{-\left(x\,n^2 + y\,n\,m + z\,m^2\right)} f(n,\ m) \tag{59}$$

$$f(n,\ m) = \int_0^1 \int_0^1 e^{-\left(x\,u^2 + 2\,x\,n\,u + y\,m\,u + y\,u\,v + y\,n\,v + 2\,z\,m\,v + z\,v^2\right)}\, du\, dv \tag{60}$$

$$\pi = 2 \int_0^\infty \int_0^\infty \left(e^{-\left(x\,u^2 + y\,u\,v + z\,v^2\right)} + e^{-\left(x\,u^2 - y\,u\,v + z\,v^2\right)}\right) du\, dv \tag{61}$$

$$\pi = 4 \int_0^\infty \int_0^\infty e^{-\left(x\,u^2 + z\,v^2\right)} \cosh(y\,u\,v)\, du\, dv \tag{62}$$

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