Pi Formulas , Part 26

Edgar Valdebenito

abstract

In this note we show some formulas related with the constant Pi

Algunas fórmulas que involucran a la constante Pi

Edgar Valdebenito

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En esta nota mostramos una colección de fórmulas en las que aparece la constante Pi:

$$\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 3.14159265\ldots$$

Fórmulas

(1) $$e^{\pi/6} = \csc\left(\ln\frac{2}{\sqrt{3}}\right)\sum_{n=1}^{\infty}(-1)^n\, a_{2n}3^{-n} + \sqrt{3}\,\sec\left(\ln\frac{2}{\sqrt{3}}\right)\sum_{n=1}^{\infty}(-1)^{n-1}\, a_{2n-1}3^{-n}$$

(2) $$e^{-\pi/6} = \csc\left(\ln\frac{2}{\sqrt{3}}\right)\sum_{n=1}^{\infty}(-1)^n\, a_{2n}3^{-n} - \sqrt{3}\,\sec\left(\ln\frac{2}{\sqrt{3}}\right)\sum_{n=1}^{\infty}(-1)^{n-1}\, a_{2n-1}3^{-n}$$

En las fórmulas (1)-(2) , se tiene:

$$a_{n+2} = -\frac{(n^2+1)a_n + (n+1)(2n+1)a_{n+1}}{(n+1)(n+2)}\ , a_1 = 1, a_2 = -1/2\, , n \in \mathbb{N}$$

poniendo $b_n = n!\, a_n$, se tiene:

$$b_{n+2} = -\left((n^2+1)b_n + (2n+1)b_{n+1}\right), b_1 = 1, b_2 = -1, n \in \mathbb{N}$$

(3) $$e^{\pi/6} = \sec\left(\ln\frac{2}{\sqrt{3}}\right)\sum_{n=0}^{\infty}(-1)^n\, a_{2n}3^{-n} + \frac{1}{\sqrt{3}}\csc\left(\ln\frac{2}{\sqrt{3}}\right)\sum_{n=1}^{\infty}(-1)^{n-1}\, a_{2n+1}3^{-n}$$

(4) $$e^{-\pi/6} = \sec\left(\ln\frac{2}{\sqrt{3}}\right)\sum_{n=0}^{\infty}(-1)^n\, a_{2n}3^{-n} - \frac{1}{\sqrt{3}}\csc\left(\ln\frac{2}{\sqrt{3}}\right)\sum_{n=1}^{\infty}(-1)^{n-1}\, a_{2n+1}3^{-n}$$

En las fórmulas (3)-(4) , se tiene:

$$a_{n+2} = -\frac{(n^2+1)a_n + (n+1)(2n+1)a_{n+1}}{(n+1)(n+2)}\ , a_0 = 1, a_1 = 0, n \in \mathbb{N}\cup\{0\}$$

poniendo $b_n = n!\, a_n$, se tiene:

$$b_{n+2} = -\left((n^2+1)b_n + (2n+1)b_{n+1}\right), b_0 = 1, b_1 = 0, n \in \mathbb{N}\cup\{0\}$$

(5) $$\pi = \frac{48\left(2-\sqrt{3}\right)}{5}\sum_{n=0}^{\infty}\sum_{m=0}^{n} D(m,n-m)\,\frac{(-1)^m}{2m+1}\,5^{-n}\left(\frac{10\left(7-4\sqrt{3}\right)}{3}\right)^{m}$$

(6) $$\pi = \frac{6\left(13+6\sqrt{3}\right)}{61}\sum_{n=0}^{\infty}\sum_{m=0}^{n} D(m,n-m)\,\frac{(-1)^m}{2m+1}\,5^{-2m}\left(\frac{78-25\sqrt{3}}{122}\right)^{n-m}$$

(7) $$\pi = 12\left(2-\sqrt{3}\right)(1-a)\sum_{n=0}^{\infty}\sum_{m=0}^{n} D(m,n-m)\,\frac{(-1)^m}{2m+1}\,a^{n+m}$$

donde

$$a = -\frac{1}{3}+\sqrt[3]{\frac{125-72\sqrt{3}}{27}-\sqrt{\frac{2411-1392\sqrt{3}}{27}}}+\sqrt[3]{\frac{125-72\sqrt{3}}{27}+\sqrt{\frac{2411-1392\sqrt{3}}{27}}}$$

(8) $$\pi = 12\sum_{n=0}^{\infty} D(n,n)\left(\sqrt{2}+1\right)^{2n+2}\int_{0}^{\left(3\sqrt{2}-4\right)\left(\sqrt{3}-1\right)/4} x^{2n}\sqrt{1-x^2}\,dx$$

En las fórmulas (5)-(6)-(7)-(8) , se tiene:

$$D(m,n) = \sum_{k=0}^{n}\binom{n}{k}\binom{m+n-k}{n} = \sum_{k=0}^{n} 2^k\binom{m}{k}\binom{n}{k}$$

$D(m,n)$: Números de Dellanoy

(9) $$\pi = \sum_{n=1}^{\infty}\binom{2n}{n}\frac{(-1)^{n-1}}{2n}\left(24\left(\frac{1}{16}\right)^{n} Im\left(\left(-\frac{1}{4}+2i\right)^{n}\right)+8\left(\frac{1}{57}\right)^{n} Im\left(\left(-\frac{1}{57}+i\right)^{n}\right)\right.$$
$$\left.+4\left(\frac{1}{239}\right)^{n} Im\left(\left(-\frac{1}{239}+i\right)^{n}\right)\right)$$

(10) $$\pi = 6\sum_{n=1}^{\infty}\binom{2n}{n}\frac{(-1)^{n-1}2^{-2n}}{n} Im\left(\left(\left(a+\left(2-\sqrt{3}\right)(1+a)i\right)^{2}-1\right)^{n}\right)$$

$$\left|\left(a+\left(2-\sqrt{3}\right)(1+a)i\right)^{2}-1\right| < 1\ ,\ 0.48\ldots < a < 0.86\ldots$$

(11) $$\pi = 8\sqrt{6}-6\sqrt{3}-6-36\sqrt{3}\sum_{n=0}^{\infty} M(n)\,\frac{1}{n+3}\left(\frac{\sqrt{2}-1}{3}\right)^{n+3}$$

donde

$$M(n) = \sum_{k=0}^{n} \frac{(-1)^k}{n+2-k} \binom{n}{k} \binom{2n+2-2k}{n+1-k}$$

$M(n)$: Números de Motzkin

(12) $$\pi = 12 \sinh^{-1} \left(\sum_{n=1}^{\infty} c_n \left(2-\sqrt{3}\right)^{2n-1} \right)$$

donde

$$c_{n+1} = -\frac{(8n(n-1)+1)c_n + (2n-3)(2n-2)c_{n-1}}{(2n)(2n+1)} \text{ , } c_1 = 1, c_2 = -\frac{1}{6}$$

(13) $$\pi = 12 \cosh^{-1} \left(\sum_{n=0}^{\infty} c_n \left(2-\sqrt{3}\right)^{2n} \right)$$

donde

$$c_{n+1} = -\frac{(8n^2-1)c_n + (2n-2)(2n-1)c_{n-1}}{(2n+1)(2n+2)} \text{ , } c_1 = 1, c_2 = \frac{1}{2}$$

(14) $$\pi = 12 \tanh^{-1} \left(\sum_{n=1}^{\infty} c_n \left(2-\sqrt{3}\right)^{2n-1} \right)$$

donde

$$c_{n+1} = -\frac{n(2n-1)c_n + \sum_{k=0}^{n-1}(2k+1)c_{n-k}c_{k+1}}{n(2n+1)} \text{ , } c_1 = 1$$

(15) $$\pi = 16 \sinh^{-1} \left(\sum_{n=1}^{\infty} c_n 5^{-(2n-1)} \right) - 4 \sinh^{-1} \left(\sum_{n=1}^{\infty} c_n 239^{-(2n-1)} \right)$$

Los números c_n , se definen como en la fórmula (12).

(16) $$\pi = 16 \cosh^{-1} \left(\sum_{n=1}^{\infty} c_n 5^{-2n} \right) - 4 \cosh^{-1} \left(\sum_{n=1}^{\infty} c_n 239^{-2n} \right)$$

Los números c_n , se definen como en la fórmula (13).

(17) $$\pi = 16 \tanh^{-1} \left(\sum_{n=1}^{\infty} c_n 5^{-(2n-1)} \right) - 4 \tanh^{-1} \left(\sum_{n=1}^{\infty} c_n 239^{-(2n-1)} \right)$$

Los números c_n , se definen como en la fórmula (14).

(18) $$\frac{\pi}{24} + \frac{\sqrt{3}}{8} + \frac{\ln\left(2-\sqrt{3}\right)}{4} = \frac{1}{\sqrt{3}} \sum_{n=2}^{\infty} \left[\frac{n}{2}\right] \frac{3^{-n}}{2n+1}$$

(19) $$\frac{\pi}{32}+\frac{1}{8}+\frac{\ln(\sqrt{2}-1)}{4}=\sum_{n=2}^{\infty}\left[\frac{n}{2}\right]\frac{(\sqrt{2}-1)^{2n+1}}{2n+1}$$

(20) $$\frac{\pi}{48}+\frac{\sqrt{3}}{24}-\frac{\ln 3}{8}=\sum_{n=2}^{\infty}\left[\frac{n}{2}\right]\frac{(2-\sqrt{3})^{2n+1}}{2n+1}$$

(21) $$\frac{\pi(\sqrt{2}-1)^2}{12}+\frac{\sqrt{3}\,(\sqrt{2}-1)^2}{8}$$
$$=\frac{(\sqrt{2}-1)\sqrt{9+10\sqrt{2}}}{8}-\frac{1+2\sqrt{2}}{2}\sin^{-1}\left(\frac{(\sqrt{2}-1)^2}{2}\right)$$
$$-2\sum_{n=0}^{\infty}r(n)\,(\sqrt{2}+1)^{2n+2}\int_0^{(\sqrt{2}-1)^2/2}\frac{x^{2n+2}}{\sqrt{1-x^2}}dx$$

donde

$$r(n)=D(n,n)-D(n+1,n-1)\,,n\in\mathbb{N}\,,r(0)=1$$

$D(m,n)$: Números de Dellanoy

$r(n)$: Números de Schroder

(22) $$\pi=4\sqrt{3}-3\ln\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)$$
$$+2\sqrt{3}\sum_{n=2}^{\infty}\frac{3^{-n}}{2n+1}\left(F\left(1,1+\frac{1}{2n};2+\frac{1}{2n};-3^{-n}\right)+(-1)^nF\left(1,1+\frac{1}{2n};2+\frac{1}{2n};3^{-n}\right)\right)$$

(23) $$\pi=16(\sqrt{2}-1)-4\ln(\sqrt{2}+1)$$
$$+8\sum_{n=2}^{\infty}\frac{(\sqrt{2}-1)^{2n+1}}{2n+1}\left(F\left(1,1+\frac{1}{2n};2+\frac{1}{2n};-(\sqrt{2}-1)^{2n}\right)\right.$$
$$\left.+(-1)^nF\left(1,1+\frac{1}{2n};2+\frac{1}{2n};(\sqrt{2}-1)^{2n}\right)\right)$$

(24) $$\pi=24(2-\sqrt{3})-3\ln 3$$
$$+12\sum_{n=2}^{\infty}\frac{(2-\sqrt{3})^{2n+1}}{2n+1}\left(F\left(1,1+\frac{1}{2n};2+\frac{1}{2n};-(2-\sqrt{3})^{2n}\right)\right.$$
$$\left.+(-1)^nF\left(1,1+\frac{1}{2n};2+\frac{1}{2n};(2-\sqrt{3})^{2n}\right)\right)$$

(25) $$\pi = \frac{24(2-3i)}{13} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^{2n+1} e^{(-1)^n (2n+1)\frac{\pi}{3} i}$$

(26) $$\pi = \frac{24(2+3i)}{13} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^{2n+1} e^{-(-1)^n (2n+1)\frac{\pi}{3} i}$$

(27) $$\pi = \frac{24(3-2i)}{13} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^{2n+1} e^{(-1)^n (2n+1)\frac{\pi}{6} i}$$

(28) $$\pi = \frac{24(3+2i)}{13} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^{2n+1} e^{-(-1)^n (2n+1)\frac{\pi}{6} i}$$

(29) $$\pi = 5 \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\sqrt{2}\sqrt{5-2\sqrt{5}}}{1+\sqrt{5-2\sqrt{5}}}\right)^{n} \sin\left(\frac{n\pi}{4}\right)$$

(30) $$\pi = 5 \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{2\sqrt{5-2\sqrt{5}}}{1+\sqrt{3}\sqrt{5-2\sqrt{5}}}\right)^{n} \sin\left(\frac{n\pi}{6}\right)$$

(31) $$\pi = 5 \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{2\sqrt{5-2\sqrt{5}}}{\sqrt{3}+\sqrt{5-2\sqrt{5}}}\right)^{n} \sin\left(\frac{n\pi}{3}\right)$$

(32) $$\pi = 10 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \left(\frac{\sqrt{6-2\sqrt{5}}-1}{\sqrt{5-2\sqrt{5}}}\right)^{2n-1}$$

(33) $$\pi = 10 \sum_{n=1}^{\infty} \frac{1}{2n-1} \left(\frac{\sqrt{11-4\sqrt{5}}-1}{\sqrt{2}\sqrt{5-2\sqrt{5}}}\right)^{2n-1} \sin\left(\frac{(2n-1)\pi}{4}\right)$$

(34) $$\pi = 4\left(\sqrt{2}-1\right) \sum_{n=0}^{\infty} \frac{a_n\, x^{2n+1}}{2n+1} + 4\sqrt{2} \sum_{n=0}^{\infty} \frac{b_n\, e^{-(n+1)x}}{n+1}$$

donde

$$a_0 = 1, a_n = -\left(2-\sqrt{2}\right) \sum_{k=1}^{n} \frac{a_{n-k}}{(2k)!}\ , n \in \mathbb{N}$$

$$b_{n+2} = -\sqrt{2}\, b_{n+1} - b_n\ , b_0 = 1, b_1 = -\sqrt{2}\ , n \in \mathbb{N} \cup \{0\}$$

$$0 < x < \lim_{n\to\infty} \sqrt{\frac{a_n}{a_{n+1}}} = 2.35 \ldots$$

(35) $$\pi = \sqrt{3}\sum_{n=0}^{\infty}\frac{a_n\ x^{2n+1}}{2n+1} + 3\sqrt{3}\ \sum_{n=0}^{\infty}\frac{b_n\ e^{-(n+1)x}}{n+1}$$

donde

$$a_0 = 1, a_n = -\frac{2}{3}\sum_{k=1}^{n}\frac{a_{n-k}}{(2k)!}\ , n \in \mathbb{N}$$

$$b_{n+2} = -\ b_{n+1} - b_n\ , b_0 = 1, b_1 = -1\ , n \in \mathbb{N} \cup \{0\}$$

$$0 < x < \lim_{n\to\infty}\ \sqrt{\frac{a_n}{a_{n+1}}} = 2.09\ ...$$

(36) $$\pi = 6\left(2-\sqrt{3}\right)\sum_{n=0}^{\infty}\frac{a_n\ x^{2n+1}}{2n+1} + 6\ \sum_{n=0}^{\infty}\frac{b_n\ e^{-(n+1)x}}{n+1}$$

donde

$$a_0 = 1, a_n = -2\left(2-\sqrt{3}\right)\sum_{k=1}^{n}\frac{a_{n-k}}{(2k)!}\ , n \in \mathbb{N}$$

$$b_{n+2} = -\sqrt{3}\ b_{n+1} - b_n\ , b_0 = 1, b_1 = -\sqrt{3}\ , n \in \mathbb{N} \cup \{0\}$$

$$0 < x < \lim_{n\to\infty}\ \sqrt{\frac{a_n}{a_{n+1}}} = 2.61\ ...$$

(37) $$\frac{1}{\pi} = \frac{1}{8}\prod_{n=1}^{\infty}\left\{\left(\frac{n}{n+1}\right)\left(1+\frac{1}{4n}\right)^4 x_n\right\}$$

donde

$$x_{n+1} = \frac{2}{1+\sqrt{x_n}}\ , x_1 = 2\ , n \in \mathbb{N}$$

se tiene

$$\frac{1}{\pi} = \frac{1}{8}\left\{\frac{1}{2}\left(\frac{5}{4}\right)^4 2\right\}\left\{\frac{2}{3}\left(\frac{9}{8}\right)^4 \frac{2}{1+\sqrt{2}}\right\}\left\{\frac{3}{4}\left(\frac{13}{12}\right)^4 \frac{2}{1+\sqrt{\frac{2}{1+\sqrt{2}}}}\right\}\ ...$$

(38) $$\pi = 2^{n+2} \sum_{k=0}^{\infty} \tan^{-1}\left(\frac{\left(\tanh\frac{k+1}{2} - \tanh\frac{k}{2}\right) t_n}{1 + t_n^2 \tanh\left(\frac{k+1}{2}\right)\tanh\left(\frac{k}{2}\right)}\right), n \in \mathbb{N}$$

donde

$$t_n = \sqrt{\frac{2 - \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}{2 + \underbrace{\sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}_{n-radicales}}} = \tan\left(\frac{\pi}{2^{n+2}}\right), n \in \mathbb{N}$$

(39) $$\pi = 3 \sum_{n=0}^{\infty} \frac{c_n}{n+1}\left(\ln\left(\sqrt{2} + \sqrt{3}\right)\right)^{2n+2}$$

(40) $$\pi = 5 \sum_{n=0}^{\infty} \frac{(-1)^n c_n}{n+1}\left(\tan^{-1}\sqrt{\frac{2}{\sqrt{5}}}\right)^{2n+2}$$

(41) $$\pi = 3 \sum_{n=0}^{\infty} \frac{(-1)^n c_n}{n+1}\left(\tan^{-1}\sqrt{2}\right)^{2n+2}$$

(42) $$\pi = 4 \sum_{n=0}^{\infty} \frac{c_n}{n+1}\left(\ln\left(-1 + \frac{2}{1 - \sqrt[4]{\frac{2 - \sqrt{2 + \sqrt{2}}}{2 + \sqrt{2 + \sqrt{2}}}}}\right)\right)^{2n+2}$$

En las fórmulas (39)-(40)-(41)-(42) , se tiene:

$$c_0 = 1 , c_n = \frac{1}{(2n+1)!} - \sum_{k=1}^{n} \frac{2^{2k-2}}{(2k)!} c_{n-k} , n \in \mathbb{N}$$

(43) $$\pi = \sum_{n=1}^{\infty} 2^n z^{2n-1}\left(\frac{3\sqrt{2}}{2n-1}\sin\left(\frac{(2n-1)\pi}{4}\right) + \frac{z}{n}\sin\left(\frac{n\pi}{2}\right)\right)$$

(44) $$\pi = \sum_{n=1}^{\infty} (-1)^{n-1} 2^{2n-1} z^{4n-3}\left(\frac{3}{4n-3} + \frac{z}{2n-1} + \frac{6z^2}{4n-1}\right)$$

En las fórmulas (43)-(44) , se tiene:

$$z = \frac{2}{3} - \frac{\left(1 + i\sqrt{3}\right)\left(23 + 3i\sqrt{237}\right)^{1/3}}{6 \cdot 2^{2/3}} - \frac{11\left(1 - i\sqrt{3}\right)}{6\left(2\left(23 + 3i\sqrt{237}\right)\right)^{1/3}} = 0.42681725 \ldots$$

El número z , satisface la ecuación: $2z^3-4z^2-z+1=0$.

(45) $$\pi=\sum_{n=1}^{\infty}2^n\,z^{2n-1}\left(\frac{3\sqrt{2}}{2n-1}\sin\left(\frac{(2n-1)\pi}{4}\right)-\frac{z}{n}\sin\left(\frac{n\pi}{2}\right)\right)$$

(46) $$\pi=\sum_{n=1}^{\infty}(-1)^{n-1}\,2^{2n-1}z^{4n-3}\left(\frac{3}{4n-3}-\frac{z}{2n-1}+\frac{6z^2}{4n-1}\right)$$

En las fórmulas (45)-(46) , se tiene:

$$z=\frac{1}{3}\left(-2+\frac{\left(-23+3i\sqrt{237}\right)^{1/3}}{2^{2/3}}+\frac{11}{\left(2\left(-23+3i\sqrt{237}\right)\right)^{1/3}}\right)=0.55138752\,...$$

El número z , satisface la ecuación: $2z^3+4z^2-z-1=0$.

(47) $$\pi=4\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=0}^{\left[\frac{n-1}{2}\right]}(-1)^k\binom{n}{2k+1}x^{2n-4k-2}(1-x^2)^{2k+1}=4\sum_{n=1}^{\infty}\frac{1}{n}r^n\sin(n\,y)$$

donde x es solución de la ecuación no lineal: $x^2+\sin x-1=0, 0<x<1$,

$$r=\sqrt{x^4+(1-x^2)^2}\ ,y=\tan^{-1}\left(\frac{1}{x^2}-1\right)$$

$$x_{n+1}=x_n-\frac{x_n^2+\sin(x_n)-1}{2x_n+\cos(x_n)}\ ,x_1=\frac{1}{2}\ ,\lim_{n\to\infty}x_n=x=0.636732\,...$$

(48) $$\pi=2\sqrt{3}\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\left(-\frac{1}{3}\right)^k\binom{n}{2k+1}x^{2n-4k-2}(1-x^2)^{2k+1}=6\sum_{n=1}^{\infty}\frac{1}{n}r^n\sin(n\,y)$$

donde x es solución de la ecuación no lineal: $x^2+\sqrt{3}\,\sin x-1=0{,}0<x<1$,

$$r=\sqrt{x^4+\frac{1}{3}(1-x^2)^2}\ ,y=\tan^{-1}\left(\frac{1}{\sqrt{3}}\left(\frac{1}{x^2}-1\right)\right)$$

$$x_{n+1}=x_n-\frac{x_n^2+\sqrt{3}\sin(x_n)-1}{2x_n+\sqrt{3}\cos(x_n)}\ ,x_1=\frac{1}{2}\ ,\lim_{n\to\infty}x_n=x=0.467856\,...$$

(49) $$\pi=8\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=0}^{\left[\frac{n-1}{2}\right]}(-1)^k\binom{n}{2k+1}x^{2n-4k-2}\left(\left(\sqrt{2}-1\right)(1-x^2)\right)^{2k+1}=8\sum_{n=1}^{\infty}\frac{1}{n}r^n\sin(n\,y)$$

donde x es solución de la ecuación no lineal: $x^2+\left(\sqrt{2}+1\right)\sin x-1=0{,}0<x<1$,

$$r=\sqrt{x^4+\left(\sqrt{2}-1\right)^2(1-x^2)^2}\ ,y=\tan^{-1}\left(\left(\sqrt{2}-1\right)\left(\frac{1}{x^2}-1\right)\right)$$

$$x_{n+1}=x_n-\frac{x_n^2+\left(\sqrt{2}+1\right)\sin(x_n)-1}{2x_n+\left(\sqrt{2}+1\right)\cos(x_n)}\ ,x_1=\frac{1}{2}\ ,\lim_{n\to\infty}x_n=x=0.366682\ldots$$

(50) $$\pi=12\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=0}^{\left[\frac{n-1}{2}\right]}(-1)^k\binom{n}{2k+1}x^{2n-4k-2}\left(\left(2-\sqrt{3}\right)(1-x^2)\right)^{2k+1}$$

$$=12\sum_{n=1}^{\infty}\frac{1}{n}r^n\sin(n\,y)$$

donde x es solución de la ecuación no lineal: $x^2+\left(2+\sqrt{3}\right)\sin x-1=0{,}0<x<1$,

$$r=\sqrt{x^4+\left(2-\sqrt{3}\right)^2(1-x^2)^2}\ ,y=\tan^{-1}\left(\left(2-\sqrt{3}\right)\left(\frac{1}{x^2}-1\right)\right)$$

$$x_{n+1}=x_n-\frac{x_n^2+\left(2+\sqrt{3}\right)\sin(x_n)-1}{2x_n+\left(2+\sqrt{3}\right)\cos(x_n)}\ ,x_1=\frac{1}{2}\ ,\lim_{n\to\infty}x_n=x=0.253442\ldots$$

(51) $$\pi=6\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\ln\left(\frac{\sqrt{3}}{\sqrt{3}-3^{-n}}\right)-6\sum_{n=2}^{\infty}\frac{1}{n}\tan^{-1}\left(3^{-n/2}\right)$$

(52) $$\pi=6\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\ln\left(\frac{\sqrt{3}}{\sqrt{3}-3^{-n}}\right)-6\sum_{n=2}^{\infty}(H_n-1)\tan^{-1}\left(\frac{3^{n/2}\left(\sqrt{3}-1\right)}{\sqrt{3}\,3^n+1}\right)$$

$$H_n=\sum_{k=1}^{n}\frac{1}{k}$$

(53) $$\pi=6\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\ln\left(\frac{\sqrt{3}}{\sqrt{3}-3^{-n}}\right)-6\sum_{n=2}^{\infty}\frac{3^{-n/2}}{n(1+3^{-n})}\ F\left(1{,}1;\frac{3}{2};\frac{1}{3^n+1}\right)$$

(54) $$\pi=6\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\ln\left(\frac{\sqrt{3}}{\sqrt{3}-3^{-n}}\right)-6\sum_{n=2}^{\infty}\frac{3^{-n/2}}{n\sqrt{1+3^{-n}}}\ F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};\frac{1}{3^n+1}\right)$$

En las fórmulas (22)-(23)-(24)-(53)-(54) , $F(a,b;c;x)$ es la función hipergeométrica de Gauss.

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