On zeros of some entire functions

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Abstract. Let
$$A_q^{(\alpha)}(a;z)=\sum_{k=0}^{\infty}\frac{(a;q)_k q^{\alpha k^2}z^k}{(q;q)_k},$$
where $\alpha>0,\ 0<q<1$. In a paper of Ruiming Zhang, he asked under what conditions the zeros of the entire function $A_q^{(\alpha)}(a;z)$ are all real and established some results on the zeros of $A_q^{(\alpha)}(a;z)$ which present a partial answer to that question. In the present paper, we will set up some results on certain entire functions which includes that $A_q^{(\alpha)}(q^l;z),\ l\geq 2$ has only infinitely many negative zeros that gives a partial answer to Zhang's question. In addition, we establish some results on zeros of certain entire functions involving the Rogers-Szegő polynomials and the Stieltjes-Wigert polynomials.

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1 Introduction

Recall that entire functions are functions that are holomorphic in the whole complex plane. Given an entire function $f(z)=\sum_{k=0}^{\infty}a_kz^k$, then the order of $f(z)$ can be computed by [5, (2.2.3)]

$$\rho(f)=\limsup_{k\to\infty}\frac{k\log k}{-\log a_k}. \tag{1.1}$$

Following [11], we define the entire function $A_q^{(\alpha)}(a;z)$ by

$$A_q^{(\alpha)}(a;z)=\sum_{k=0}^{\infty}\frac{(a;q)_k q^{\alpha k^2}z^k}{(q;q)_k},$$

where $\alpha > 0,\ 0 < q < 1$ and

$$(a;q)_0 = 1,\ (a;q)_k = \prod_{j=0}^{k-1}(1 - aq^j)\ (k \geq 1).$$

It is easily seen that

$$A_q^{(\frac{1}{2})}(q^{-n};z) = \sum_{k=0}^{\infty} \frac{(q^{-n};q)_k q^{\frac{k^2}{2}} z^k}{(q;q)_k} = (q;q)_n S_n(zq^{\frac{1}{2}-n};q),$$
$$A_q^{(1)}(0;z) = \sum_{k=0}^{\infty} \frac{q^{k^2} z^k}{(q;q)_k} = A_q(-z),$$

where $A_q(z)$ and $S_n(z;q)$ are the Ramanujan entire function and the Stieltjes-Wigert polynomial respectively [10]. So $A_q^{(\alpha)}(a;z)$ generalizes both $A_q(z)$ and $S_n(z;q)$. It is well-known that both of them have only real positive zeros. Therefore, Zhang in [18] asked under what conditions the zeros of the entire function $A_q^{(\alpha)}(a;z)$ are all real. In that paper, Zhang proved that $A_q^{(\alpha)}(-a;z)$ $(a \geq 0, \alpha > 0, 0 < q < 1)$ has only infinitely many negative zeros and $A_q^{(\alpha)}(q^{-n};z)$ $(n \in \mathbb{N}, \alpha \geq 0, 0 < q < 1)$ has only finitely many positive zeros, which gave a partial answer to that question. In addition, Zhang obtained a result on the negativity of zeros of an entire function including many well-known entire functions.

Our motivation for the present work emanates from Zhang's question. In this paper, we will establish the following results which present a partial answer to Zhang's question.

Theorem 1.1. *Let $\alpha > 0$ and $0 < q < 1$. Then*

(i)*if $l \geq 2$ is an integer, then $A_q^{(\alpha)}(q^l;z)$ has only infinitely many real zeros and all of them are negative;*

(ii)*if m and n are nonnegative integers such that at least one of them is positive, $\{l_j\}_{j=1}^m$ are integers not less than 2, $0 < q_j < 1$ $(1 \leq j \leq m)$ and $\nu_r > -1,\ 0 < q_r < 1$ $(1 \leq r \leq n)$, then the function*

$$\sum_{k=0}^{\infty} \prod_{j=1}^{m} \frac{(q_j^{l_j};q_j)_k}{(q_j;q_j)_k} \frac{q^{\alpha k^2}}{\prod_{r=1}^{n}(q_r, q_r^{\nu_r+1};q_r)_k} z^k$$

has only infinitely many real zeros and all of them are negative;

(iii)*if $m \geq 0$ and $n \geq 1$ are integers, $\{l_j\}_{j=1}^m$ are integers not less than 2 and $\nu_r \geq 0$ $(1 \leq r \leq m)$, then the function*

$$\sum_{k=0}^{\infty} \frac{\prod_{j=1}^{m}(l_j)_k}{(k!)^{m+n}\prod_{r=1}^{n}(\nu_r+1)_k} z^k$$

where $(a)_k$ *is defined by* $(a)_0 = 1,\ (a)_k = a(a+1)\cdots(a+k-1)\ (k \geq 1)$, *has only infinitely many real zeros and all of them are negative.*

It should be mentioned that in [13, Theorem 4] Katkova et al. proved that there exists a constant q_∞ (≈ 0.556415) such that the function $A_q^{(\alpha)}(q; z)$ has only real zeros if and only if $q \leq q_\infty$. So the similar result for $A_q^{(\alpha)}(q^l; z)$ does not hold for $l = 1$.

The Gaussian binomial coefficients are q-analogs of the binomial coefficients, which are given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}.$$

We now introduce the definition of the Rogers-Szegő polynomials which were first investigated by Rogers [15] and then by Szegő [16]. The Rogers-Szegő polynomials are defined by

$$h_n(x, y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k}.$$

If q is replaced by q^{-1} in the Rogers-Szegő polynomials, then we obtain the Stieltjes-Wigert polynomials (see [16]):

$$g_n(x, y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)} x^k y^{n-k}.$$

From [18, Theorem 5], we know that $h_n(x|q)$ has only negative zeros for $q \geq 1$ and $g_n(x|, q)$ has only negative zeros for $0 < q \leq 1$, where $h_n(x|q)$ and $g_n(x|, q)$ are defined by

$$h_n(x|q) := h_n(x, 1|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k$$

and

$$g_n(x|q) := g_n(x, 1|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)} x^k.$$

Motivated by Zhang's work, we will establish the following results on zeros of certain entire functions involving the Rogers-Szegő polynomials and the Stieltjes-Wigert polynomials.

Theorem 1.2. *Let* $0 < q < 1$. *If* α *is positive number and* $0 < x, y < 1$, *then*

$$\sum_{n=0}^{\infty} \frac{h_n(x, y|q)}{(q;q)_n} q^{\alpha n^2} z^n$$

has infinitely many real zeros and all of them are negative; if $-1 < x, y < 0$ and $\alpha \geq \frac{1}{2}$, then

$$\sum_{n=0}^{\infty} \frac{g_n(x, y|q)}{(q; q)_n} q^{\alpha n^2} z^n$$

has infinitely many real zeros and all of them are positive.

Remark 1.1. (i) *Applying the method which is used in the proof of Theorem 1.2, we can deduce the following results: let $0 < q < 1$; if α is positive number and $0 < x < 1$, then*

$$\sum_{n=0}^{\infty} \frac{h_n(x|q)}{(q; q)_n} q^{\alpha n^2} z^n$$

has infinitely many real zeros and all of them are negative; if $-1 < x < 0$ and $\alpha \geq \frac{1}{2}$, then

$$\sum_{n=0}^{\infty} \frac{g_n^-(x|q)}{(q; q)_n} q^{\alpha n^2} z^n$$

has infinitely many real zeros and all of them are positive, where $g_n^-(x|q) = g_n(x, -1|q)$. But we need the following results:

$$\left|\frac{h_n(x|q)}{(q; q)_n}\right| \leq \frac{1}{(q, x; q)_\infty}, \quad \left|\frac{g_n^-(x|q)}{(q; q)_n}\right| \leq \frac{1}{(q, |x|; q)_\infty}$$

which can be derived easily.

(ii) *We can establish certain results on the Rogers-Szegő polynomials and the Stieltjes-Wigert polynomials by using similar method. These are analogous to (ii) and (iii) of Theorem 1.1*

We also set up the following result which is analogous to [18, Theorem 7].

Theorem 1.3. *Suppose r and s are two positive integers, $a_j (1 \leq j \leq r)$ and $b_k (1 \leq k \leq s)$ are r+s positive numbers and $\alpha > 0$, $0 < q < 2^{-\frac{1}{\alpha}}$. Then there exists $K_0 \in \mathbb{Z}_{>0}$ such that for all integers $K \geq K_0$, the function*

$$\sum_{k=K}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_r)_k}{(b_1)_k (b_2)_k \cdots (b_s)_k} q^{\alpha k^2} z^k$$

has only infinitely many real zeros and all of them are negative.

In the next section, we will provide some lemmas which are crucial in the proof of Theorems 1.1 and 1.2. Section 3 is devoted to our proof of Theorems 1.1–1.3.

2 Preliminaries

In order to prove Theorems 1.1 and 1.2, we need some auxiliary results. We first recall from [7] that a real entire function $f(z)$ is of Laguerre-Pólya class if

$$f(z) = cz^m e^{-\alpha z^2+\beta z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{z_k}\right) e^{-z/z_k},$$

where $c, \beta, z_k \in \mathbb{R}, \alpha \geq 0, m \in \mathbb{Z}_{\geq 0}$ and $\sum_{k=1}^{\infty} z_k^{-2} < +\infty$.

Let us recall that a real sequence $\{a_n\}_{n=0}^{\infty}$ is called a Pólya frequence (or PF) sequence if the infinite matrix $(a_{j-i})_{i,j=0}^{\infty}$ is totally positive, i.e. all its minors are nonnegative, where we use the notation $a_k = 0$ if $k < 0$. This concept can be extended to finite sequences in the obvious way by completing the sequence with zero terms.

Lemma 2.1. (See [2]) *The sequence $\{a_k\}_{k=0}^{\infty}$ is a PF sequence if and only if the convergent series $\sum_{k=0}^{\infty} a_k z^k$ satisfies*

$$\sum_{k=0}^{\infty} a_k z^k = cz^m e^{\gamma z} \prod_{k=1}^{\infty} \frac{1+\alpha_k z}{1-\beta_k z},$$

where $c \geq 0, \gamma \geq 0, \alpha_k \geq 0, \beta_k \geq 0, m \in \mathbb{Z}^+$ and $\sum_{k=1}^{\infty} (\alpha_k + \beta_k) < +\infty$.

It was proved in [6] that

$$\sum_{k=0}^{\infty} \frac{q^{k^2}}{k!} x^k$$

is a real entire function and in Laguerre-Pólya class for $|q| < 1$ and $x \in \mathbb{R}$. Then by [7, Theorem C], we obtain that $\{\frac{q^{k^2}}{k!}\}_{k=0}^{\infty}$ is a PF sequence for $0 < q < 1$.

Lemma 2.2. (See [8, p. 1047] and [14]) *Let $\{a_k\}_{k=0}^{m}$ and $\{b_k\}_{k=0}^{n}$ be sequences of nonnegative numbers. Then*

(i) *the sequence $\{a_k\}_{k=0}^{m}$ is a a PF sequence if and only if the polynomial $\sum_{k=0}^{m} a_k z^k$ has only nonpositive zeros;*

(ii) *if the sequences $\{a_k\}_{k=0}^{m}$ and $\{b_k\}_{k=0}^{n}$ are PF sequences, then so is the sequence $\{a_k \cdot b_k\}_{k=0}^{\infty}$;*

(iii) *if the sequences $\{a_k\}_{k=0}^{m}$ and $\{b_k\}_{k=0}^{n}$ are PF sequences, then so is the sequence $\{k! \cdot a_k \cdot b_k\}_{k=0}^{\infty}$.*

We also need the following results, namely, Vitali's theorem [17] and Hurwitz's theorm [1, §5, Theorem 2].

Lemma 2.3. (Vitali's theorem) *Let $\{f_n(z)\}$ be a sequence of functions analytic in a domain D and assume that $f_n(z) \to f(z)$ point-wise in D. Then $f_n(z) \to f(z)$ uniformly in any subdomain bounded by a contour C, provided that C is contained in D.*

Lemma 2.4. (Hurwitz's theorm) *If the functions $\{f_n(z)\}$ are nonzero and analytic in a region Ω, and $f_n(z) \to f(z)$ uniformly on every compact subset of Ω, then $f(z)$ either identically zero or never equal to zero in Ω.*

We conclude this section with following result which is very important in the proof of Theorem 1.2.

Lemma 2.5. *Let x, y be two real numbers and $\alpha > 0,\ 0 < q < 1$. Then the functions*

$$\sum_{n=0}^{\infty} \frac{h_n(x,y|q)}{(q;q)_n} q^{\alpha n^2} z^n \ \textit{and} \ \sum_{n=0}^{\infty} \frac{g_n(x,y|q)}{(q;q)_n} q^{\alpha n^2} z^n$$

are all entire functions.

Proof. We first consider the function $\sum_{n=0}^{\infty} \frac{h_n(x,y|q)}{(q;q)_n} q^{\alpha n^2} z^n$. For $|x| \le 1, |y| \le 1$, by [9, (1.3.15)], we have

$$\begin{aligned}
|h_n(x,y|q)| &\le \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q |x|^k |y|^{n-k} \le \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \\
&\le \sum_{k=0}^{n} \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(q;q)_k} q^{k-n} \\
&\le q^{-n} \sum_{k=0}^{n} \frac{q^k}{(q;q)_k} \le q^{-n} \sum_{k=0}^{\infty} \frac{q^k}{(q;q)_k} \\
&= \frac{q^{-n}}{(q;q)_\infty}.
\end{aligned}$$

By the same arguments, we get for $|x| > 1, |y| \le 1$,

$$\begin{aligned}
|h_n(x,y|q)| &\le \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q |y|^k |x|^{n-k} = |x|^n \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q |y|^k |x|^{-k} \\
&\le |x|^n \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \le \frac{(|x|/q)^n}{(q;q)_\infty};
\end{aligned}$$

for $|x| \le 1, |y| > 1$,

$$
\begin{aligned}
|h_n(x, y|q)| &\le |y|^n \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q |x|^k |y|^{-k} \\
&\le |y|^n \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \le \frac{(|y|/q)^n}{(q;q)_\infty};
\end{aligned}
$$

for $|x| > 1, |y| > 1$,

$$
\begin{aligned}
|h_n(x, y|q)| &= |xy|^n \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q |x|^{k-n} |y|^{-k} \\
&\le |xy|^n \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \le \frac{(|xy|/q)^n}{(q;q)_\infty}.
\end{aligned}
$$

In any cases we obtain

$$
|h_n(x, y|q)| \le \frac{a^n}{(q;q)_\infty}
$$

where a is positive number which depends on x, y and q. Then

$$
\left| \frac{h_n(x, y|q)}{(q;q)_n} q^{\alpha n^2} \right| \le \frac{a^n q^{\alpha n^2}}{(q;q)_\infty^2}.
$$

so that

$$
\limsup_{n\to\infty} \left| \frac{h_n(x, y|q)}{(q;q)_n} q^{\alpha n^2} \right|^{\frac{1}{n}} = 0,
$$

which proves that

$$
\sum_{n=0}^{\infty} \frac{h_n(x, y|q)}{(q;q)_n} q^{\alpha n^2} z^n
$$

is an entire function. Similarly, we can deduce that

$$
\sum_{n=0}^{\infty} \frac{g_n(x, y|q)}{(q;q)_n} q^{\alpha n^2} z^n
$$

is also an entire function. This ends the proof of Lemma 2.5. $\square$

3 Proof of Theorems 1.1–1.3

Proof of Theorem 1.1. We first prove (i). According to the q-binomial theorem [3, 9], we obtain that for all complex numbers x and q with $|x| < 1$ and $|q| < 1$, there holds

$$\sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} x^k = \frac{(ax;q)_\infty}{(x;q)_\infty}. \tag{3.1}$$

Setting $a = q^l, x = zq^{-l}$ in (3.1) gives

$$\sum_{k=0}^{\infty} \frac{(q^l;q)_k}{(q;q)_k} q^{-lk} z^k = \frac{(z;q)_\infty}{(zq^{-l};q)_\infty} = \frac{1}{(zq^{-l};q)_l}.$$

Using Lemma 2.1, we get the sequence

$$\left\{ \frac{(q^l;q)_k}{(q;q)_k} q^{-lk} \right\}_{k=0}^{n}$$

is a PF sequence. It follows from (i) of Lemma 2.2 that $\{a_k\}_{k=0}^n$ is a a PF sequence if and only if $\{c^k \cdot a_k\}_{k=0}^n$ is also a PF sequence for any $c > 0$. Hence,

$$\left\{ \frac{(q^l;q)_k}{(q;q)_k} \right\}_{k=0}^{n}$$

is a PF sequence. So from the fact $\left\{ \frac{q^{\alpha k^2}}{k!} \right\}_{k=0}^{n}$ is a PF sequence and (iii) of Lemma 2.2, we arrive at the sequence

$$\left\{ \frac{(q^l;q)_k q^{\alpha k^2}}{(q;q)_k} \right\}_{k=0}^{n}$$

is also a PF sequence, which, by (i) of Lemma 2.2, implies that the polynomial

$$\sum_{k=0}^{n} \frac{(q^l;q)_k q^{\alpha k^2}}{(q;q)_k} z^k$$

has only nonpositive zeros. Here and below, set $\Omega = \mathbb{C} - \{x + yi | x \in (-\infty, 0], y = 0\}$. Then

$$\sum_{k=0}^{n} \frac{(q^l;q)_k q^{\alpha k^2}}{(q;q)_k} z^k \to A_q^\alpha(q^l;z)$$

point-wise in Ω. It is easily seen that for $0 < q < 1,\ \alpha \geq 0$ and each $n \in \mathbb{N},\ z \in \mathbb{C}$,

$$\left| \sum_{k=0}^{n} \frac{(q^l;q)_k q^{\alpha k^2}}{(q;q)_k} z^k \right| \leq \sum_{k=0}^{\infty} \frac{(q^l;q)_k q^{\alpha k^2}}{(q;q)_k} |z|^k < +\infty.$$

We apply Lemma 2.3 to know that

$$\sum_{k=0}^{n} \frac{(q^l;q)_k q^{\alpha k^2}}{(q;q)_k} z^k \to A_q^{\alpha}(q^l;z)$$

uniformly on every compact subset of Ω and then apply Lemma 2.4 to see that $A_q^{\alpha}(q^l;z) \neq 0$ in Ω which means that $A_q^{\alpha}(q^l;z)$ has no zeros outside the set $\{x + yi | x \in (-\infty, 0], y = 0\}$. According to [10, Lemma 14.1.4], we have $A_q^{\alpha}(q^l;z)$ has infinitely many zeros. Therefore, $A_q^{\alpha}(q^l;z)$ has only infinitely many real zeros and all of them are negative, which proves (i).

We next show (ii). According to [18], we know that the sequence

$$\left\{ \frac{1}{(q, q^{\nu+1}; q)_k} \right\}_{k=0}^{N}$$

is a PF sequence for $\nu > -1,\ 0 < q < 1$, which means that

$$\left\{ \frac{1}{(q_r, q_r^{\nu_r+1}; q_r)_k} \right\}_{k=0}^{N}$$

are all PF sequences for $1 \leq r \leq n$.

Since

$$\left\{ \frac{(q_j^{l_j};q_j)_k}{(q_j;q_j)_k} \right\}_{k=0}^{N} \quad (1 \leq j \leq m), \ \left\{ \frac{1}{(q_r, q_r^{\nu_r+1}; q_r)_k} \right\}_{k=0}^{N} \quad (1 \leq r \leq n)$$

and $\left\{ \frac{q^{\alpha k^2}}{k!} \right\}_{k=0}^{n}$ are all PF sequences, we then apply (ii) and (iii) of Lemma 2.2 to find that

$$\left\{ \prod_{j=1}^{m} \frac{(q_j^{l_j};q_j)_k}{(q_j;q_j)_k} \frac{q^{\alpha k^2}}{\prod_{r=1}^{n} (q_r, q_r^{\nu_r+1}; q_r)_k} \right\}_{k=0}^{N}$$

is also a PF sequence, which, by (i) of Lemma 2.2, implies that

$$\sum_{k=0}^{N} \prod_{j=1}^{m} \frac{(q_j^{l_j};q_j)_k}{(q_j;q_j)_k} \frac{q^{\alpha k^2} z^k}{\prod_{r=1}^{n} (q_r, q_r^{\nu_r+1}; q_r)_k}$$

has only negative zeros. For each positive integer N and $z \in \mathbb{C}$, we have

$$\begin{aligned}
&\left|\sum_{k=0}^{N}\prod_{j=1}^{m}\frac{(q_j^{l_j};q_j)_k}{(q_j;q_j)_k}\frac{q^{\alpha k^2}z^k}{\prod_{r=1}^{n}(q_r,q_r^{\nu_r+1};q_r)_k}\right| \\
&\leq \sum_{k=0}^{\infty}\prod_{j=1}^{m}\frac{(q_j^{l_j};q_j)_k}{(q_j;q_j)_k}\frac{q^{\alpha k^2}|z|^k}{\prod_{r=1}^{n}(q_r,q_r^{\nu_r+1};q_r)_k} < +\infty.
\end{aligned}$$

Similarly, we apply Lemmas 2.3 and 2.4 to establish that the function

$$\sum_{k=0}^{\infty}\prod_{j=1}^{m}\frac{(q_j^{l_j};q_j)_k}{(q_j;q_j)_k}\frac{q^{\alpha k^2}z^k}{\prod_{r=1}^{n}(q_r,q_r^{\nu_r+1};q_r)_k}$$

has no zeros outside the set $\{x+yi|x\in(-\infty,0], y=0\}$. In view of [10, Lemma 14.1.4], this function has infinitely many zeros. Then (ii) is proved.

Finally, we give a proof of (iii). Let $q_j = q_r = q$ and $\alpha = n + m/2$. Using the Hôpital's rule, we deduce that

$$\lim_{q\to 1}\frac{(q^{l_j};q)_k}{(q;q)_k} = \frac{(l_j)_k}{k!},\ \lim_{q\to 1}\frac{(1-q)^{2k}}{(q,q^{\nu_r+1};q)_k} = \frac{1}{k!(\nu_r+1)_k}.$$

Then

$$\lim_{q\to 1}\frac{(1-q)^{2kn}\prod_{j=1}^{m}(q^{l_j};q)_k q^{(n+m/2)k^2}\cdot z^k}{(q;q)_k^{m+n}\prod_{r=1}^{n}(q^{\nu_r+1};q)_k} = \frac{\prod_{j=1}^{m}(l_j)_k\cdot z^k}{(k!)^{m+n}\prod_{r=1}^{n}(\nu_r+1)_k}.$$

It is easy to see from

$$lq^{l-1} \leq \frac{1-q^l}{1-q} = 1+q+q^2+\cdots+q^{l-1} \leq l$$

that

$$l!q^{\binom{l}{2}} \leq \frac{(q;q)_l}{(1-q)^l} \leq l!.$$

Then

$$\frac{1}{l!} \leq \frac{(1-q)^l}{(q;q)_l} \leq \frac{q^{-\binom{l}{2}}}{l!} \leq \frac{q^{-l^2/2}}{l!}.$$

Combining this and the fact that $\frac{(1-q)^l}{(q^b;q)_l} \leq \frac{(1-q)^l}{(q;q)_l}$ for $b \geq 1$ gives

$$\left|\frac{(1-q)^{2kn}\prod_{j=1}^{m}(q^{l_j};q)_k q^{\alpha k^2}\cdot z^k}{(q;q)_k^{m+n}\prod_{r=1}^{n}(q^{\nu_r+1};q)_k}\right| \leq \frac{|z|^k}{k!^{m+2n}}$$

This, by Lemma 2.3, shows that

$$\lim_{q\to 1}\sum_{k=0}^{\infty}\frac{(1-q)^{2kn}\prod_{j=1}^{m}(q^{l_j};q)_k q^{(n+m/2)k^2}\cdot z^k}{(q;q)_k^{m+n}\prod_{r=1}^{n}(q^{\nu_r+1};q)_k}=\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{m}(l_j)_k\cdot z^k}{(k!)^{m+n}\prod_{r=1}^{n}(\nu_r+1)_k}.$$

converges uniformly in in any compact subset of $\mathbb{C}$. It follows from Lemma 2.4 that the function

$$\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{m}(l_j)_k\cdot z^k}{(k!)^{m+n}\prod_{r=1}^{n}(\nu_r+1)_k}$$

has no zeros outside the set $\{x+yi|x\in(-\infty,0],y=0\}$.

Set

$$a_k=\frac{\prod_{j=1}^{m}(l_j)_k}{(k!)^{m+n}\prod_{r=1}^{n}(\nu_r+1)_k}.$$

It is easily seen from the Stirling's formula [4] that

$$\lim_{k\to\infty}\frac{-\log a_k}{k\log k}=2n$$

which, by (1.1), means that

$$\rho\left(\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{m}(l_j)_k\cdot z^k}{(k!)^{m+n}\prod_{r=1}^{n}(\nu_r+1)_k}\right)=\frac{1}{2n}\le\frac{1}{2}.$$

Hence, by [10, Theorem 1.2.5], the function

$$\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{m}(l_j)_k\cdot z^k}{(k!)^{m+n}\prod_{r=1}^{n}(\nu_r+1)_k}$$

has infinitely many zeros. Then this function has only infinitely many real zeros and all of them are negative, which proves (iii). This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. We first consider the function $\sum_{n=0}^{\infty}\frac{h_n(x,y|q)}{(q;q)_n}q^{\alpha n^2}z^n$, where $0<x,y<1$ and $\alpha>0$. From [12, Theorem 3.1, (3.1)], we know that

$$\sum_{n=0}^{\infty}\frac{h_n(x,y|q)}{(q;q)_n}t^n=\frac{1}{(xt,yt;q)_\infty}$$

for $\max\{|xt|,|yt|\}<1$. Then, by Lemma 2.1, we have the sequence

$$\left\{\frac{h_n(x,y|q)}{(q;q)_n}\right\}_{n=0}^{\infty}$$

is a PF sequence. It follows from the fact that $\{\frac{q^{k^2}}{k!}\}_{k=0}^{\infty}$ is a PF sequence and (iii) in Lemma 2.2 that

$$\left\{\frac{h_n(x,y|q)}{(q;q)_n}q^{\alpha n^2}\right\}_{n=0}^{N}$$

is also a PF sequence. So, by (i) of Lemma 2.2, we see that

$$\sum_{n=0}^{N}\frac{h_n(x,y|q)}{(q;q)_n}q^{\alpha n^2}z^n$$

has only nonpositive zeros. We know that

$$\sum_{n=0}^{N}\frac{h_n(x,y|q)}{(q;q)_n}q^{\alpha n^2}z^n \to \sum_{n=0}^{\infty}\frac{h_n(x,y|q)}{(q;q)_n}q^{\alpha n^2}z^n$$

point-wise in Ω. It is easy to see that for $0<q<1,\ \alpha>0$ and each $N\in\mathbb{N},\ z\in\mathbb{C}$,

$$\left|\sum_{n=0}^{\infty}\frac{h_n(x,y|q)}{(q;q)_n}q^{\alpha n^2}z^n\right| \le \sum_{n=0}^{\infty}\frac{h_n(x,y|q)}{(q;q)_n}q^{\alpha n^2}|z|^n < +\infty.$$

Applying Lemma 2.3, we find that

$$\sum_{n=0}^{N}\frac{h_n(x,y|q)}{(q;q)_n}q^{\alpha n^2}z^n \to \sum_{n=0}^{\infty}\frac{h_n(x,y|q)}{(q;q)_n}q^{\alpha n^2}z^n$$

uniformly on every compact subset of Ω and then applying Lemma 2.4, we deduce that the function

$$\sum_{n=0}^{\infty}\frac{h_n(x,y|q)}{(q;q)_n}q^{\alpha n^2}z^n \neq 0$$

in Ω which means that

$$\sum_{n=0}^{\infty}\frac{h_n(x,y|q)}{(q;q)_n}q^{\alpha n^2}z^n$$

has no zeros outside the set $\{x+yi|x\in(-\infty,0], y=0\}$.

We use [9, (1.3.15)] to get

$$\begin{aligned}0\le\frac{h_n(x,y|q)}{(q;q)_n} &= \sum_{k=0}^{n}\frac{x^k}{(q;q)_k}\frac{y^{n-k}}{(q;q)_{n-k}}\\ &\le \sum_{k=0}^{\infty}\frac{x^k}{(q;q)_k}\sum_{k=0}^{\infty}\frac{y^k}{(q;q)_k}\\ &= \frac{1}{(x,y;q)_\infty}.\end{aligned}$$

Then, by Lemma 2.5 and [10, Lemma 14.1.4], we attain that the function

$$\sum_{n=0}^{\infty} \frac{h_n(x,y|q)}{(q;q)_n} q^{\alpha n^2} z^n$$

has infinitely many zeros. Therefore, the function

$$\sum_{n=0}^{\infty} \frac{h_n(x,y|q)}{(q;q)_n} q^{\alpha n^2} z^n$$

has infinitely many real zeros and all of them are negative.

We now investigate the function $\sum_{n=0}^{\infty} \frac{g_n(x,y|q)}{(q;q)_n} q^{\alpha n^2} z^n$, where $-1 < x, y < 0$ and $\alpha \geq \frac{1}{2}$. According to [12, Theorem 3.1, (3.2)], we have

$$\sum_{n=0}^{\infty} (-1)^n \frac{g_n(x,y|q) q^{\binom{n}{2}}}{(q;q)_n} t^n = (x,y;q)_\infty,$$

which, by Lemma 2.1, implies that

$$\left\{ (-1)^n \frac{g_n(x,y|q) q^{\binom{n}{2}}}{(q;q)_n} \right\}_{n=0}^{\infty}$$

is a PF sequence, namely,

$$\left\{ (-1)^n \frac{g_n(x,y|q) q^{\frac{n^2}{2}}}{(q;q)_n} \right\}_{n=0}^{\infty}$$

is a PF sequence. So, by the fact that $\{\frac{q^{k^2}}{k!}\}_{k=0}^{\infty}$ is a PF sequence and (iii) in Lemma 2.2,

$$\left\{ (-1)^n \frac{g_n(x,y|q) q^{\alpha n^2}}{(q;q)_n} \right\}_{n=0}^{N}$$

is a PF sequence, which, by (i) of Lemma 2.2, means that

$$\sum_{n=0}^{N} (-1)^n \frac{g_n(x,y|q) q^{\alpha n^2}}{(q;q)_n} z^n$$

has only nonpositive zeros. It is obvious that

$$\sum_{n=0}^{N} (-1)^n \frac{g_n(x,y|q) q^{\alpha n^2}}{(q;q)_n} z^n \to \sum_{n=0}^{\infty} (-1)^n \frac{g_n(x,y|q) q^{\alpha n^2}}{(q;q)_n} z^n$$

point-wise in Ω. For $0<q<1,\ \alpha>0$ and each $N\in\mathbb{N},\ z\in\mathbb{C}$,

$$\left|\sum_{n=0}^{\infty}(-1)^n\frac{g_n(x,y|q)q^{\alpha n^2}}{(q;q)_n}z^n\right|\le\sum_{n=0}^{\infty}(-1)^n\frac{g_n(x,y|q)q^{\alpha n^2}}{(q;q)_n}|z|^n<+\infty.$$

Then, by Lemma 2.3,

$$\sum_{n=0}^{N}(-1)^n\frac{g_n(x,y|q)q^{\alpha n^2}}{(q;q)_n}z^n\to\sum_{n=0}^{\infty}(-1)^n\frac{g_n(x,y|q)q^{\alpha n^2}}{(q;q)_n}z^n$$

uniformly on every compact subset of Ω. We apply Lemma 2.4 to derive that the function

$$\sum_{n=0}^{\infty}(-1)^n\frac{g_n(x,y|q)q^{\alpha n^2}}{(q;q)_n}z^n\neq 0$$

in Ω. This shows that

$$\sum_{n=0}^{\infty}(-1)^n\frac{g_n(x,y|q)q^{\alpha n^2}}{(q;q)_n}z^n$$

has no zeros outside the set $\{x+yi|x\in(-\infty,0],y=0\}$.

It is clear that

$$\begin{aligned}\left|(-1)^n\frac{g_n(x,y|q)}{(q;q)_n}\right|&\le\sum_{k=0}^{n}\frac{|x|^k}{(q;q)_k}\frac{|y|^{n-k}}{(q;q)_{n-k}}\\&\le\sum_{k=0}^{\infty}\frac{|x|^k}{(q;q)_k}\sum_{k=0}^{\infty}\frac{|y|^k}{(q;q)_k}\\&=\frac{1}{(|x|,|y|;q)_\infty}.\end{aligned}$$

By Lemma 2.5 and [10, Lemma 14.1.4], the function

$$\sum_{n=0}^{\infty}(-1)^n\frac{g_n(x,y|q)q^{\alpha n^2}}{(q;q)_n}z^n$$

has infinitely many zeros. Hence,

$$\sum_{n=0}^{\infty}(-1)^n\frac{g_n(x,y|q)q^{\alpha n^2}}{(q;q)_n}z^n$$

has infinitely many real zeros and all of them are negative, namely,

$$\sum_{n=0}^{\infty}\frac{g_n(x,y|q)q^{\alpha n^2}}{(q;q)_n}z^n$$

has infinitely many real zeros and all of them are positive. This finishes the proof of Theorem 1.2. □

Proof of Theorem 1.3. For $0 < q < \frac{1}{2^{\frac{1}{\alpha}}}$. Put

$$A_k = \frac{(a_1)_k(a_2)_k\cdots(a_r)_k}{(b_1)_k(b_2)_k\cdots(b_s)_k}q^{\alpha k^2}.$$

Then

$$\frac{A_{k-1}^2}{A_kA_{k-2}} = \prod_{i=1}^{r}\frac{a_i+k-2}{a_i+k-1}\prod_{j=1}^{s}\frac{b_i+k-1}{b_i+k-2}q^{-2\alpha}$$

which means that

$$\lim_{k\to\infty}\frac{A_{k-1}^2}{A_kA_{k-2}} = q^{-2\alpha} > 4.$$

So there exists a positive integer K_0 such that

$$\frac{A_{k-1}^2}{A_kA_{k-2}} > 4$$

for $k \geq K_0$. It follows from [7, Theorem B] that

$$\sum_{k=K}^{\infty}\frac{(a_1)_k(a_2)_k\cdots(a_r)_k}{(b_1)_k(b_2)_k\cdots(b_s)_k}q^{\alpha k^2}z^k$$

has only negative zeros for any $K \geq K_0$.

On the other hand, by the Stirling's formula and (1.1), we have

$$\rho\left(\sum_{k=K}^{\infty}\frac{(a_1)_k(a_2)_k\cdots(a_r)_k}{(b_1)_k(b_2)_k\cdots(b_s)_k}q^{\alpha k^2}z^k\right) = 0,$$

which, by [10, Theorem 1.2.5], implies that the function

$$\sum_{k=K}^{\infty}\frac{(a_1)_k(a_2)_k\cdots(a_r)_k}{(b_1)_k(b_2)_k\cdots(b_s)_k}q^{\alpha k^2}z^k$$

has infinitely many zeros. Hence, this function has only infinitely many real zeros and all of them are negative. This concludes the proof of Theorem 1.3. □

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