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Binding Number of Some Special Classes of Trees

B.Chaluvaraju[1], H.S.Boregowda[2] and S.Kumbinarsaiah[3]

[1]Department of Mathematics, Bangalore University, Janana Bharathi Campus, Bangalore-560 056, India

[2]Department of Studies and Research in Mathematics, Tumkur University, Tumkur-572 103, India

[3]Department of Mathematics, Karnatak University, Dharwad-580 003, India

E-mail: bchaluvaraju@gmail.com, bgsamarasa@gmail.com, kumbinarasaiah@gmail.com

Abstract: The binding number of a graph $G = (V, E)$ is defined to be the minimum of $|N(X)|/|X|$ taken over all nonempty set $X \subseteq V(G)$ such that $N(X) \neq V(G)$. In this article, we explore the properties and bounds on binding number of some special classes of trees.

Key Words: Graph, tree, realizing set, binding number, Smarandachely binding number.

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§1. Introduction

In this article, we consider finite, undirected, simple and connected graphs $G = (V, E)$ with vertex set V and edge set E. As such $n =| V |$ and $m =| E |$ denote the number of vertices and edges of a graph G, respectively. An edge - induced subgraph is a subset of the edges of a graph G together with any vertices that are their endpoints. In general, we use $\langle X\rangle$ to denote the subgraph induced by the set of edges $X \subseteq E$. A graph G is connected if it has a $u - v$ path whenever $u, v \in V(G)$ (otherwise, G is disconnected). The open neighborhood of a vertex $v \in V(G)$ is $N(v) = \{u \in V : uv \in E(G)\}$ and the closed neighborhood $N[v] = N(v) \cup \{v\}$. The degree of v, denoted by $deg(v)$, is the cardinality of its open neighborhood. A vertex with degree one in a graph G is called pendant or a leaf or an end-vertex, and its neighbor is called its support or cut vertex. An edge incident to a leaf in a graph G is called a pendant edge. A graph with no cycle is acyclic. A tree T is a connected acyclic graph. Unless mentioned otherwise, for terminology and notation the reader may refer Harary [3].

Woodall [7] defined the binding number of G as follows: If $X \subseteq V(G)$, then the open neighborhood of the set X is defined as $N(X) = \bigcup_{x\in X} N(v)$. The binding number of G, denoted $b(G)$, is given by

$$b(G) = min_{x\in F}\frac{|N(X)|}{|X|},$$

where $F = \{X \subseteq V(G) : X \neq \varnothing, N(X) \neq V(G)\}$. We say that $b(G)$ is realized on a set X if $X \in F$ and $b(G) = \frac{|N(X)|}{|X|}$, and the set X is called a realizing set for $b(G)$. Generally, for a given graph H, a *Smarandachely binding number* $b_H(G)$ is the minimum number $b(G)$ on such F with

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$\langle X \rangle_G \ncong H$ for $\forall X \in F$. Clearly, if H is not a spanning subgraph of G, then $b_H(G) = b(G)$.

For complete review and the following existing results on the binding number and its related concepts, we follow [1], [2], [5] and [6].

Theorem 1.1 *For any path P_n with $n \geq 2$ vertices,*

$$b(P_n) = \begin{cases} 1 & \text{if n is even;} \\ \frac{n-1}{n+1} & \text{if n is odd.} \end{cases}$$

Theorem 1.2 *For any spanning subgraph H of a graph G, $b(G) \leq b(H)$.*

In [8], Wayne Goddard established several bounds including ones linking the binding number of a tree to the distribution of its end-vertices $end(G) = \{v \in V(G) : deg(v) = 1\}$. Also, let $\varrho(v) = |N(v) \cap end(G)|$ and $\varrho(G)=$ max $\{\varrho(v) : v \in V(G)\}$. The following result is obviously true if $\varrho(G) = 0$ and if $\varrho(G) = 1$, follows from taking $X = \{N(v) \cap end(G)\}$, where v is a vertex for which $\varrho(v) = \varrho(G)$.

Theorem 1.3 *For any graph G, $\varrho(G).b(G) \leq 1$.*

Theorem 1.4 *For any nontrivial tree T,*

(1) $b(T) \geq 1/\Delta(T)$;
(2) $b(T) \geq 1/\varrho(T) + 1$.

§2. Main Results

Observation 2.1 Let T be a tree with $n \geq 3$ vertices, having $(n-1)$-pendant vertices, which are connected to unique vertex. Then $b(T)$ is the reciprocal of number of vertices connected to unique vertex.

Observation 2.2 Let T be a nontrivial tree. Then $b(T) > 0$.

Observation 2.3 Let T be a tree with $b(T) < 1$. Then every realizing set of T is independent.

Theorem 2.4 *For any Star $K_{1,n-1}$ with $n \geq 2$ vertices,*

$$b(K_{1,n-1}) = \frac{1}{n-1}.$$

Proof Let $K_{1,n-1}$ be a star with $n \geq 2$ vertices. If $K_{1,n-1}$ has $\{v_1, v_2, \cdots, v_n\}$ vertices with $deg(v_1) = n-1$ and $deg(v_2) = deg(v_3) = \cdots = deg(v_n) = 1$. We prove the result by induction on n. For $n = 2$, then $|N(X)| = |X| = 1$ and $b(K_{1,1}) = 1$. For $n = 3$, $|N(X)| < |X| = 2$ and $b(K_{1,2}) = \frac{1}{2}$. Let us assume the result is true for $n = k$ for some k, where k is a positive integer. Hence $b(K_{1,k-1}) = \frac{1}{k-1}$.

Now we shall show that the result is true for $n > k$. Since $(k+1)$- pendant vertices in $K_{1,k+1}$ are connected to the unique vertex v_1. Here newly added vertex v_{k+1} must be adjacent to v_1 only. Otherwise $K_{1,k+1}$ loses its star criteria and v_{k+1} is not adjacent to $\{v_2, v_3, \cdots, v_k\}$, then $K_{1,k+1}$ has k number of pendant vertices connected to vertex v_1. Therefore by Observation 2.1, the desired result follows. □

Theorem 2.5 *Let T_1 and T_2 be two stars with order n_1 and n_2 , respectively. Then $n_1 < n_2$ if and only if $b(T_1) > b(T_2)$.*

Proof By Observation 2.1 and Theorem 2.4, we have $b(T_1) = \frac{1}{n_1}$ and $b(T_2) = \frac{1}{n_2}$. Due to the fact of $n_1 < n_2$ if and only if $\frac{1}{n_1} > \frac{1}{n_2}$. Thus the result follows. □

Definition 2.6 *The double star $K^*_{r,s}$ is a tree with diameter 3 and central vertices of degree r and s respectively, where the diameter of graph is the length of the shortest path between the most distanced vertices.*

Theorem 2.7 *For any double star $K^*_{r,s}$ with $1 \leq r \leq s$ vertices,*

$$b(K^*_{r,s}) = \frac{1}{max\{r, s\} - 1}.$$

Proof Suppose $K^*_{r,s}$ is a double star with $1 \leq r \leq s$ vertices. Then there exist exactly two central vertices x and y for all $x, y \in V(K^*_{r,s})$ such that the degree of x and y are r and s respectively. By definition, the double star $K^*_{r,s}$ is a tree with diameter 3 having only one edge between x and y. Therefore the vertex x is adjacent to $(r-1)$-pendant vertices and the vertex y is adjacent to $(s-1)$-pendant vertices.

Clearly $max\{r-1, s-1\}$ pendant vertices are adjacent to a unique vertex x or y as the case may be. Therefore $b(K^*_{r,s}) = \frac{1}{max\{r-1,s-1\}}$. Hence the result follows. □

Definition 2.8 *A subdivided star, denoted $K^*_{1,n-1}$ is a star $K_{1,n-1}$ whose edges are subdivided once, that is each edge is replaced by a path of length 2 by adding a vertex of degree 2.*

Observation 2.9 Let $K_{1,n-1}$ be a star with $n \geq 2$ vertices. Then cardinality of the vertex set of $K^*_{1,n-1}$ is $p = 2n - 1$.

Theorem 2.10 *For any subdivided star $K^*_{1,n-1}$ with $n \geq 2$ vertices,*

$$b(K^*_{1,n-1}) = \begin{cases} \frac{1}{2} & \text{if n = 2;} \\ \frac{2}{3} & \text{if n = 3;} \\ 1 & \text{otherwise.} \end{cases}$$

Proof By Observation 2.9, the subdivided star $K^*_{1,n-1}$ has $p = 2n - 1$ vertices. Then the following cases arise:

Case 1. If $n = 2$, then by Theorem 1.1, $b(K^*_{1,2-1}) = b(P_3) = \frac{1}{2}$.

Case 2. If $n = 3$, then by Theorem 1.1, $b(K^*_{1,3-1}) = b(P_5) = \frac{2}{3}$.

Case 3. If a vertex $v_1 \in V(K_{1,n-1})$ with $deg(v_1) = n - 1$ and $deg(N(v_1)) = 1$, where $N(v_1) = \{v_2, v_3, \cdots, v_n\}$. Clearly, each edge $\{v_1v_2, v_1v_3, \cdots, v_1v_n\}$ takes one vertex on each edge having degree 2, so that the resulting graph will be subdivided star $K^*_{1,n-1}$, in which $\{v_1\}$ and $\{v_2, v_3, \cdots, v_n\}$ vertices do not lose their properties. But the maximum degree vertex v_1 is a cut vertex of $K^*_{1,n-1}$. Therefore $b(K_{1,n-1}) < b(K^*_{1,n-1})$ for $n \geq 4$ vertices. Since each newly added vertex $\{u_i\}$ is adjacent to exactly one pendent vertex $\{v_j\}$,where $i = j$ and $2 \leq i, j \leq n$, in $K^*_{1,n-1}$. By the definition of binding number $|N(X)| = |X|$. Hence the result follows. □

Definition 2.11 *A $B_{t,k}$ graph is said to be a Banana tree if the graph is obtained by connecting one pendant vertex of each t-copies of an k-star graph with a single root vertex that is distinct from all the stars.*

Theorem 2.12 *For any Banana tree $B_{t,k}$ with $t \geq 2$ copies and $k \geq 3$ number of stars,*

$$b(B_{t,k}) = \frac{1}{k-2}.$$

Proof Let t be the number of distinct k-stars. Then it has $k-1$-pendant vertices and the binding number of each k-stars is $\frac{1}{k-1}$. But in $B_{t,k}$, each t copies of distinct k-stars are joined by single root vertex. Then the resulting graph is connected and each k-star has $k - 2$ number of vertices having degree 1, which are connected to unique vertex. By Observation 2.1, the result follows. □

Definition 2.13 *A caterpillar tree $C^*(T)$ is a tree in which removing all the pendant vertices and incident edges produces a path graph.*

For example, $b(C^*(K_1)) = 0$; $b(C^*(P_2)) = b(C^*(P_4)) = 1$; $b(C^*(P_3)) = \frac{1}{2}$; $b(C^*(P_5)) = \frac{2}{3}$ and $b(C^*(K_{1,n-1})) = \frac{1}{n-1}$.

Theorem 2.14 *For any caterpillar tree $C^*(T)$ with $n \geq 3$ vertices,*

$$b(K_{1,n-1}) \leq b(C^*(T)) \leq b(P_n).$$

Proof By mathematical induction, if $n = 3$, then by Theorem 1.1 and Observation 2.1, we have $b(K_{1,2}) = b(C^*(T)) = b(P_3) = \frac{1}{2}$. Thus the result follows. Assume that the result is true for $n = k$. Now we shall prove the result for $n > k$. Let $C^*(T)$ be a Caterpillar tree with $k + 1$-vertices. Then the following cases arise:

Case 1. If $k + 1$ is odd, then $b(C^*(T)) \leq \frac{k}{k+1}$.

Case 2. If $k + 1$ is even, then $b(C^*(T)) \leq 1$.

By above cases, we have $b(C^*(T)) \leq b(P_n)$. Since, k vertices in $C^*(T)$ exist k-stars, which contributed at least $\frac{1}{k-1}$. Hence the lower bound follows. □

Definition 2.15 *The binary tree B^* is a tree like structure that is rooted and in which each vertex has at least two children and child of a vertex is designated as its left or right child.*

To prove our next result we make use of the following conditions of Binary tree B^*.

C_1**:** If B^* has at least one vertex having two children and that two children has no any child.

C_2**:** If B^* has no vertex having two children which are not having any child.

Theorem 2.16 *Let B^* be a Binary tree with $n \geq 3$ vertices. Then*

$$b(B^*) = \begin{cases} \frac{1}{2} & \text{if } B^* \text{ satisfy } C_1; \\ b(P_n) & \text{if } B^* \text{ satisfy } C_2. \end{cases}$$

Proof Let B^* be a Binary tree with $n \geq 3$ vertices. Then the following cases are arises:

Case 1. Suppose binary tree B^* has only one vertex, say v_1 has two children and that two children has no any child. Then only vertex v_1 has two pendant vertices and no other vertex has more than two pendant vertices. That is maximum at most two pendant vertices are connected to unique vertex. There fore $b(B^*) = \frac{1}{2}$ follows.

Case 2. Suppose binary tree B^* has no vertex having two free child. That is each non-pendant vertex having only one child, then this binary tree gives path. This implies that $b(B^*) = b(P_n)$ with $n \geq 3$ vertices. Thus the result follows. □

Definition 2.17 *The t-centipede C_t^* is the tree on $2t$-vertices obtained by joining the bottoms of t - copies of the path graph P_2 laid in a row with edges.*

Theorem 2.18 *For any t-centipede C_t^* with $2t$-vertices,*

$$b(C_t^*) = 1.$$

Proof If $n = 1$, then tree C_1^* is a 1-centipede with 2-vertices. Thus $b(C_1^*) = 1$. Suppose the result is true for $n > 1$ vertices, say $n = t$ for some t, that is $b(C_t^*) = 1$. Further, we prove $n = t+1$, $b(C_{t+1}^*) = 1$. In a $(t+1)$ - centipede exactly one vertex from each of the $(k+1)$- copies of P_2 are laid on a row with edges. Hence the resulting graph must be connected and each such vertex is connected to exactly one pendant vertex. By the definition of binding number $|N(X)| = |X|$. Hence the result follows. □

Definition 2.19 *The Fire-cracker graph $F_{t,s}$ is a tree obtained by the concatenation of t - copies of s - stars by linking one pendant vertex from each.*

Theorem 2.20 *For any Fire-cracker graph $F_{t,s}$ with $t \geq 2$ and $s \geq 3$.*

$$b(F_{t,s}) = \frac{1}{s-1}.$$

Proof If $s = 2$, then Fire-cracker graph $F_{t,2}$ is a t-centipede and $b(F_{t,2}) = 1$. If $t \geq 2$ and $s \geq 3$, then t - copies of s - stars are connected by adjoining one pendant vertex from each s-stars. This implies that the resulting graph is connected and a Fire-cracker graph $F_{t,s}$. Then this connected graph has $(s-2)$-vertices having degree 1, which are connected to unique vertex. Hence the result follows. □

Theorem 2.21 *For any nontrivial tree T,*

$$\frac{1}{n-1} \leq b(T) \leq 1.$$

Further, the lower bound attains if and only if $T = K_{1,n-1}$ and the upper bound attains if the tree T has 1-factor or there exists a realizing set X such that $X \cap N(X) = \phi$.

Proof The upper bound is proved by Woodall in [7]with the fact of $\delta(T) = 1$. Let $X \in F$ and $\frac{|N(X)|}{|X|} = b(G)$. Then $|N(X)| \geq 1$, since the set X is not empty. Suppose, $|N(X)| \geq n - \delta(T) + 1$. If $\delta(T) = 1$, then any vertex of T is adjacent to atleast one vertex in X. This implies that $N(X) = V(T)$, which is a contradiction. There fore $|X| \leq n-1$ and $b(T) = |N(X)|/|X| \geq 1/(n-1)$. Thus the lower bound follows. □

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