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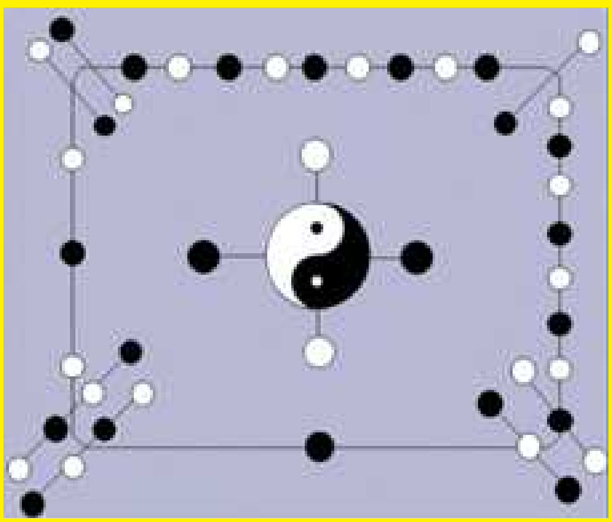
Edited By Linfan MAO



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By Karl Marx, a German revolutionary .

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N^*C^*- Smarandache Curve of Bertrand Curves Pair According to Frenet Frame

Süleyman Şenyurt , Abdussamet Çalışkan and Ünzile Çelik

(Faculty of Arts and Sciences, Department of Mathematics, Ordu University, Ordu, Turkey)

E-mail: senyurtsuleyman@hotmail.com, abdussamet65@gmail.com, unzile.celik@hotmail.com

Abstract: In this paper, let (α, α^*) be Bertrand curve pair, when the unit Darboux vector of the α^* curve are taken as the position vectors, the curvature and the torsion of Smarandache curve are calculated. These values are expressed depending upon the α curve. Besides, we illustrate example of our main results.

Key Words: Bertrand curves pair, Smarandache curves, Frenet invariants, Darboux vector.

AMS(2010): 53A04.

§1. Introduction

It is well known that many studies related to the differential geometry of curves have been made. Especially, by establishing relations between the Frenet Frames in mutual points of two curves several theories have been obtained. The best known of the Bertrand curves discovered by J. Bertrand in 1850 are one of the important and interesting topics of classical special curve theory. A Bertrand curve is defined as a special curve which shares its principal normals with another special curve, called Bertrand mate or Bertrand curve Partner. If $\alpha^* = \alpha + \lambda N$, $\lambda = const.$, then (α, α^*) are called Bertrand curves pair. If α and α^* Bertrand curves pair, then $\langle T, T^* \rangle = \cos\theta = constant$, [9], [10]. The definition of n-dimensional Bertrand curves in Lorentzian space is given by comparing a well-known Bertrand pair of curves in n- dimensional Euclidean space. It shown that the distance between corresponding of Bertrand pair of curves and the angle between the tangent vector fields of these points are constant. Moreover Schell and Mannheim theorems are given in the Lorentzian space, [7]. The Bertrand curves are the Inclined curve pairs. On the other hand, it gave the notion of Bertrand Representation and found that the Bertrand Representation is spherical, [8]. Some characterizations for general helices in space forms were given, [11].

A regular curve in Minkowski space-time, whose position vector is composed by Frenet frame vectors on another regular curve, is called a Smarandache curve [14]. Special Smarandache curves have been studied by some authors. Melih Turgut and Süha Yılmaz studied a special case of such curves and called it Smarandache TB_2 curves in the space $\mathbb{E}_1^4$ ([14]). Ahmad T.Ali

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studied some special Smarandache curves in the Euclidean space. He studied Frenet-Serret invariants of a special case, [1]. Şenyurt and Çalışkan investigated special Smarandache curves in terms of Sabban frame of spherical indicatrix curves and they gave some characterization of Smarandache curves, [4]Özcan Bektaş and Salim Yüce studied some special Smarandache curves according to Darboux Frame in $\mathbb{E}^3$, [3]. Kemal Taşköprü and Murat Tosun studied special Smarandache curves according to Sabban frame on S^2 ([2]). They defined NC-Smarandache curve, then they calculated the curvature and torsion of NB and TNB- Smarandache curves together with NC-Smarandache curve, [12]. It studied that the special Smarandache curve in terms of Sabban frame of Fixed Pole curve and they gave some characterization of Smarandache curves, [12]. When the unit Darboux vector of the partner curve of Mannheim curve were taken as the position vectors, the curvature and the torsion of Smarandache curve were calculated. These values were expressed depending upon the Mannheim curve, [6].

In this paper, special Smarandache curve belonging to α curve such as N^*C^* drawn by Frenet frame are defined and some related results are given.

§2. Preliminaries

The Euclidean 3-space $\mathbb{E}^3$ be inner product given by

$$\langle\ ,\ \rangle = x_1^2 + x_2^3 + x_3^2$$

where $(x_1, x_2, x_3) \in \mathbb{E}^3$. Let $\alpha : I \to \mathbb{E}^3$ be a unit speed curve denote by $\{T, N, B\}$ the moving Frenet frame . For an arbitrary curve $\alpha \in \mathbb{E}^3$, with first and second curvature, κ and τ respectively, the Frenet formulae is given by [9], [10].

$$\begin{cases} T' = \kappa N \\ N' = -\kappa T + \tau B \\ B' = -\tau N. \end{cases} \tag{2.1}$$

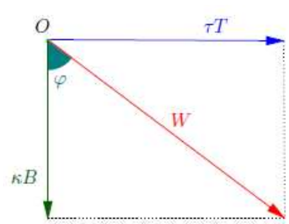

Figure 1 Darboux vector

For any unit speed curve $\alpha : I \to \mathbb{E}^3$, the vector W is called Darboux vector defined by

$$W = \tau T + \kappa B. \tag{2.2}$$

If we consider the normalization of the Darboux, we have

$$\sin\varphi = \frac{\tau}{\|W\|}, \quad \cos\varphi = \frac{\kappa}{\|W\|} \tag{2.3}$$

and

$$C = \sin\varphi T + \cos\varphi B, \tag{2.4}$$

where $\angle(W, B) = \varphi$.

Definition 2.1([9]) *Let $\alpha : I \to \mathbb{E}^3$ and $\alpha^* : I \to \mathbb{E}^3$ be the C^2- class differentiable unit speed two curves and let $\{T(s), N(s), B(s)\}$ and $\{T^*(s), N^*(s), B^*(s)\}$ be the Frenet frames of the curves α and α^*, respectively. If the principal normal vector N of the curve α is linearly dependent on the principal vector N^* of the curve α^*, then the pair (α, α^*) is said to be Bertrand curves pair.*

The relations between the Frenet frames $\{T(s), N(s), B(s)\}$ and $\{T^*(s), N^*(s), B^*(s)\}$ are as follows:

$$\begin{cases} T^* = \cos\theta T + \sin\theta B \\ N^* = N \\ B^* = -\sin\theta T + \cos\theta B. \end{cases} \tag{2.5}$$

where $\angle(T, T^*) = \theta$

Theorem 2.2([9], [10]) *The distance between corresponding points of the Bertrand curves pair in $\mathbb{E}^3$ is constant.*

Theorem 2.3([10]) *Let (α, α^*) be a Bertrand curves pair in $\mathbb{E}^3$. For the curvatures and the torsions of the Bertrand curves pair (α, α^*) we have*

$$\begin{cases} \kappa^* = \dfrac{\lambda\kappa - \sin^2\theta}{\lambda(1-\lambda\kappa)} \ , \lambda = constant \\ \\ \tau^* = \dfrac{\sin^2\theta}{\lambda^2\tau}. \end{cases} \tag{2.6}$$

Theorem 2.4([9]) *Let (α, α^*) be a Bertrand curves pair in $\mathbb{E}^3$. For the curvatures and the torsions of the Bertrand curves pair (α, α^*) we have*

$$\begin{cases} \kappa^* \frac{ds^*}{ds} = \kappa\cos\theta - \tau\sin\theta, \\ \\ \tau^* \frac{ds^*}{ds} = \kappa\sin\theta + \tau\cos\theta. \end{cases} \tag{2.7}$$

By using equation (2.2), we can write Darboux vector belonging to Bertrand mate α^*.

$$W^* = \tau^* T^* + \kappa^* B^*. \tag{2.8}$$

If we consider the normalization of the Darboux vector, we have

$$C^* = \sin\varphi^* T^* + \cos\varphi^* B^*. \tag{2.9}$$

From the equation (2.3) and (2.7), we can write

$$\begin{aligned} \sin\varphi^* &= \frac{\tau^*}{\|W^*\|} = \frac{\kappa\sin\theta + \tau\cos\theta}{\|W\|} = \sin(\varphi+\theta), \\ \cos\varphi^* &= \frac{\kappa^*}{\|W^*\|} = \frac{\kappa\cos\theta - \tau\sin\theta}{\|W\|} = \cos(\varphi+\theta), \end{aligned} \tag{2.10}$$

where $\|W^*\| = \sqrt{\kappa^{*2}+\tau^{*2}} = \|W\|$ and $\angle(W^*, B^*) = \varphi^*$. By the using (2.5) and (2.10), the final version of the equation (2.9) is as follows:

$$C^* = \sin\varphi T + \cos\varphi B. \tag{2.11}$$

§3. N^*C^*- Smarandache Curve of Bertrand Curves Pair According to Frenet Frame

Let (α, α^*) be a Bertrand curves pair in $\mathbb{E}^3$ and $\{T^*, N^*, B^*\}$ be the Frenet frame of the curve α^* at $\alpha^*(s)$. In this case, N^*C^* - Smarandache curve can be defined by

$$\psi(s) = \frac{1}{\sqrt{2}}(N^* + C^*). \tag{3.1}$$

Solving the above equation by substitution of N^* and C^* from (2.5) and (2.11), we obtain

$$\psi(s) = \frac{\sin\varphi T + N + \cos\varphi B}{\sqrt{2}}\cdot \tag{3.2}$$

The derivative of this equation with respect to s is as follows,

$$\psi' = T_\psi \frac{ds_\psi}{ds} = \frac{(-\kappa + \varphi'\cos\varphi)T + (\tau - \varphi'\sin\varphi)B}{\sqrt{2}} \tag{3.3}$$

and by substitution, we get

$$T_\psi = \frac{(-\kappa + \varphi'\cos\varphi)T + (\tau - \varphi'\sin\varphi)B}{\sqrt{\|W\|^2 - 2\varphi'\|W\| + \varphi'^2}}, \tag{3.4}$$

where

$$\frac{ds_\psi}{ds} = \sqrt{\frac{\|W\|^2 - 2\varphi'\|W\| + \varphi'^2}{2}}\cdot \tag{3.5}$$

In order to determine the first curvature and the principal normal of the curve $\psi(s)$, we formalize

$$T'_\psi(s) = \frac{\sqrt{2}\big[(\omega_1 \cos\theta + \omega_3 \sin\theta)T + \omega_2 N + (-\omega_1 \sin\theta + \omega_3 \cos\theta)B\big]}{\big[\|W\|^2 - 2\varphi'\|W\| + \varphi'^2\big]^2}, \tag{3.6}$$

where

$$\begin{cases} \omega_1 = \big(-\kappa\cos\theta + \tau\sin\theta + \varphi'\cos(\varphi+\theta)\big)'\big(\|W\|^2 - 2\varphi'\|W\| + \varphi'^2\big) - \big(-\kappa\cos\theta \\ \qquad +\tau\sin\theta + \varphi'\cos(\varphi+\theta)\big)\big(\|W\|\|W\|' - \varphi''\|W\| - \varphi'\|W\|' + \varphi'\varphi''\big) \\ \omega_2 = \big(-\|W\|^2 + \varphi'\|W\|\big)\big(\|W\|^2 - 2\varphi'\|W\| + \varphi'^2\big) \\ \omega_3 = \big(\kappa\sin\theta + \tau\cos\theta - \varphi'\sin(\varphi+\theta)\big)'\big(\|W\|^2 - 2\varphi'\|W\| + \varphi'^2\big) - \big(\kappa\sin\theta \\ \qquad +\tau\cos\theta - \varphi'\sin(\varphi+\theta)\big)\big(\|W\|\|W\|' - \varphi''\|W\| - \varphi'\|W\|' + \varphi'\varphi''\big) \end{cases}$$

The first curvature is

$$\kappa_\psi = \|T'_\psi\|, \quad \kappa_\psi = \frac{\sqrt{2({\omega_1}^2 + {\omega_2}^2 + {\omega_3}^2)}}{\big[\|W\|^2 - 2\varphi'\|W\| + \varphi'^2\big]^2}.$$

The principal normal vector field and the binormal vector field are respectively given by

$$N_\psi = \frac{\big[(\omega_1\cos\theta + \omega_3\sin\theta)T + \omega_2 N + (-\omega_1\sin\theta + \omega_3\cos\theta)B\big]}{\sqrt{{\omega_1}^2 + {\omega_2}^2 + {\omega_3}^2}}, \tag{3.7}$$

$$B_\psi = \frac{\begin{array}{l} \omega_2\big[-2\kappa\sin\theta\cos\theta + \tau(\sin^2\theta - \cos^2\theta) + \varphi'\sin\varphi\big]T \\ +\omega_1\big[\kappa\sin\theta + \tau\cos\theta - \varphi'\sin(\varphi+\theta)\big]N + \omega_2\big[2\tau\sin\theta\cos\theta \\ +\kappa(\sin^2\theta - \cos^2\theta) + \varphi'\cos\varphi\big]B \end{array}}{\sqrt{\big(\|W\|^2 - 2\varphi'\|W\| + \varphi'^2\big)({\omega_1}^2 + {\omega_2}^2 + {\omega_3}^2)}} \tag{3.8}$$

The torsion is then given by

$$\tau_\psi = \frac{det(\psi', \psi'', \psi''')}{\|\psi' \wedge \psi''\|^2} \ ,$$

$$\tau_\psi = \frac{\sqrt{2}(\vartheta\eta + \varrho\lambda + \mu\rho)}{\vartheta^2 + \varrho^2 + \mu^2}$$

where

$$\begin{cases} \eta = \big(\varphi'\cos(\varphi+\theta) - \kappa\cos\theta + \tau\sin\theta\big)'' + (\kappa\cos\theta - \tau\sin\theta)\|W\|^2 \\ \qquad -(\kappa\cos\theta - \tau\sin\theta)\varphi'\|W\| \\ \lambda = (\kappa\cos\theta - \tau\sin\theta)\big(\varphi'\cos(\varphi+\theta) - \kappa\cos\theta + \tau\sin\theta\big)' + (-\|W\|^2 \\ \qquad +\varphi'\|W\|)' - (\kappa\sin\theta + \tau\cos\theta)\big(\kappa\sin\theta + \tau\cos\theta - \varphi'\sin(\varphi+\theta)\big)' \\ \rho = (-\kappa\sin\theta - \tau\cos\theta)\|W\|^2 + (\kappa\sin\theta + \tau\cos\theta)\varphi'\|W\| + \big(\kappa\sin\theta \\ \qquad +\tau\cos\theta - \varphi'\sin(\varphi+\theta)\big)'', \end{cases}$$

$$\begin{cases}\vartheta = -\big(-\|W\|^2+\varphi'\|W\|\big)\big(\kappa\sin\theta+\tau\cos\theta-\varphi'\sin(\varphi+\theta)\big)\\ \varrho = -\Big[\big(\varphi'\cos(\varphi+\theta)-\kappa\cos\theta+\tau\sin\theta\big)\big(\kappa\sin\theta+\tau\cos\theta-\varphi'\sin(\varphi+\theta)\big)'\\ \qquad +\big(\varphi'\cos(\varphi+\theta)-\kappa\cos\theta+\tau\sin\theta\big)'\big(\kappa\sin\theta+\tau\cos\theta-\varphi'\sin(\varphi+\theta)\big)\Big]\\ \mu = \big(\varphi'\cos(\varphi+\theta)-\kappa\cos\theta+\tau\sin\theta\big)\big(-\|W\|^2+\varphi'\|W\|\big).\end{cases}$$

Example 3.1 Let us consider the unit speed α curve and α^* curve:

$$\alpha(s)=\frac{1}{\sqrt{2}}\big(-\cos s,-\sin s,s\big)\ \ and\ \ \alpha^*(s)=\frac{1}{\sqrt{2}}(\cos s,\sin s,s)\cdot$$

The Frenet invariants of the curve, $\alpha^*(s)$ are given as following:

$$\begin{cases}T^*(s)=\frac{1}{\sqrt{2}}(-\sin s,\cos s,1), N^*(s)=\big(-\cos s,-\sin s,0\big)\\ B^*(s)=\frac{1}{\sqrt{2}}(\sin s,-\cos s,1), C^*(s)=(0,0,1)\\ \kappa^*(s)=\frac{1}{\sqrt{2}}, \tau^*(s)=\frac{1}{\sqrt{2}}\cdot\end{cases}$$

In terms of definitions, we obtain special Smarandache curve, see Figure 1.

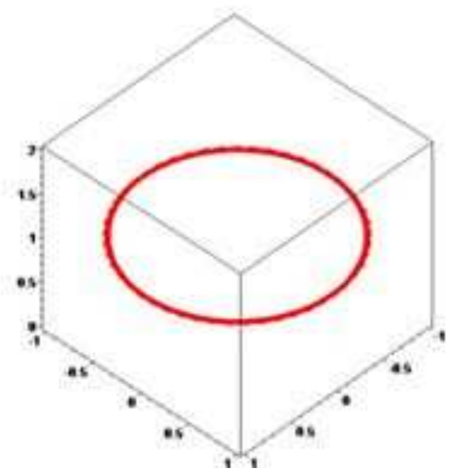

Figure 2 N^*C^*-Smarandache Curve

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On Dual Curves of Constant Breadth According to Dual Bishop Frame in Dual Lorentzian Space $\mathbb{D}_1^3$

Süha Yılmaz[1], Yasin Ünlütürk[2] and Ümit Ziya Savcı[3]

1. Dokuz Eylül University, Buca Educational Faculty, 35150, Buca-Izmir, Turkey
2. Kırklareli University, Department of Mathematics, 39100 Kırklareli, Turkey
3. Celal Bayar University, Department of Mathematics Education, 45900, Manisa-Turkey

E-mail: suha.yilmaz@deu.edu.tr, yasinunluturk@klu.edu.tr, ziyasavci@hotmail.com

Abstract: In this work, dual curves of constant breadth according to Bishop frame are defined, and applications of their differential equations are solved for special cases in dual Lorentzian space $\mathbb{D}_1^3$. Some characterizations of closed dual curves of constant breadth according to Bishop frame are presented in dual Lorentzian space $\mathbb{D}_1^3$. These characterizations are made by obtaining special solutions of differential equations which characterize closed dual curves of constant breadth according to Bishop frame in dual Lorentzian space $\mathbb{D}_1^3$.

Key Words: Dual Lorentzian space, dual curve, dual curves of constant breadth, Bishop frame, differential equations.

AMS(2010): 53A35, 53A40, 53B25.

§1. Introduction

Bishop frame is used in engineering. This special frame has been particulary used in the study of DNA, and tubular surfaces and made in robot. Most of the literature on canal surfaces within the CAGD context has been motivated by the observation that canal surfaces with the rational spine curve and rational radius function are rational, and it is therefore natural to ask for methods which allow one to construct a rational parameterization of canal surface from its spin curve and radius function [8]. The construction of the Bishop frame is due to L. R. Bishop in [2]. That is why he defined this frame that curvature may vanish at some points on the curve. That is, second derivative of the curve may be zero. In this situation, an alternative frame is needed for non continously differentiable curves on which Bishop (parallel transport frame) frame is well defined and constructed in Euclidean and its ambient spaces [4, 18].

Curves of constant breadth have been studied in pure mathematics, optimization, mechanical engineering, physics and related directions. Basic properties of curves of constant breadth can be explained to someone without having any mathematical background knowledge. The existence of non-circular curves of constant breadth in the standard Euclidean plane has been known since the time of Euler; e.g., the Reuleaux triangle was presented by Reuleaux to horn-

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blower, the founder of the compound steam-engine. In recent years, mathematical properties of the Reuleaux triangle have led to some very important applications. Since a curve of constant breadth can be freely rotated in a square always maintaining contact to all four sides of the square, a Reuleaux triangle can be used for drilling holes of maximum area into squares. Another application is given by the basic single-rotor Wankel engine. Its oval-shaped housing surrounds a three-sided rotor similar to a Reuleaux triangle. As the rotor rotates and orbitally revolves, each side of the rotor gets closer and farther from the wall of the housing, as also described above, in view of drilling holes into squares. A Reuleaux triangle is also used in the gear for driving a movie film [12].

In the classical theory of curves in differential geometry, curves of constant breadth have a long history as a research matter [3, 5, 9]. First it was introduced by Euler in [5]. Then Fujivara obtained a problem to determine whether there exist space curves of constant breadth or not, and he defined the concept "breadth" for space curves on a surface of constant breadth [6]. Furthermore, Blaschke defined the curve of constant breadth on the sphere [3]. Reuleaux gave a method to obtain these kinds of curves and applied the results he had by using his method, in kinematics and engineering [14]. Some geometric properties of plane curves of constant breadth were given by Köse in [11]. And, in another work of Köse [10], these properties were studied in the Euclidean 3-space $\mathbb{E}^3$. In Minkowski 3-space as an ambient space, some characterizations of timelike curves of constant breadth were given by Yılmaz and Turgut in [17]. Also, Yılmaz dealt with dual timelike curves of constant breadth in dual Lorentzian space in [16].

Dual numbers were introduced by W. K. Clifford as a tool for his geometrical investigations. Then dual numbers and vectors were used on line geometry and kinematics by Eduard Study. He devoted a special attention to the representation of oriented lines by dual unit vectors and defined the famous mapping: The set of oriented lines in a three-dimensional Euclidean space $\mathbb{E}^3$ is one to one correspondence with the points of a dual space $\mathbb{D}^3$ of triples of dual numbers [7].

In this paper, we study dual curves of constant breadth according to Bishop frame in dual Lorentzian space $\mathbb{D}_1^3$. We give some characterizations of dual curves of constant breadth according to Bishop frame in $\mathbb{D}_1^3$. Then we characterize these kinds of curves by obtaining special solutions of their differential equations in $\mathbb{D}_1^3$.

§2. Preliminaries

Let $\mathbb{E}_1^3$ be the three-dimensional Minkowski space, that is, the three dimensional real vector space $\mathbb{E}^3$ with the metric

$$\langle dx, dx\rangle = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) denotes the canonical coordinates in $\mathbb{E}^3$. An arbitrary vector x of $\mathbb{E}_1^3$ is said to be spacelike if $\langle x, x\rangle > 0$ or $x = 0$, timelike if $\langle x, x\rangle < 0$ and lightlike or null if $\langle x, x\rangle = 0$ and $x \neq 0$. A timelike or light-like vector in $\mathbb{E}_1^3$ is said to be causal. For $x \in \mathbb{E}_1^3$ the norm is defined by $\|x\| = \sqrt{|\langle x, x\rangle|}$, then the vector x is called a spacelike unit vector if $\langle x, x\rangle = 1$ and a timelike unit vector if $\langle x, x\rangle = -1$. Similarly, a regular curve in $\mathbb{E}_1^3$ can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors are spacelike, timelike or null (lightlike), respectively [13].

Dual numbers are given with the set

$$\mathbb{D}=\{\widehat{x}=x+\xi x^{*};x,x^{*}\in\mathbb{E}\},$$

the symbol ξ designates the dual unit with the property $\xi^2=0$ for $\xi\neq 0$. Dual angle is defined as $\widehat{\theta}=\theta+\xi\theta^{*}$, where θ is the projected angle between two spears and θ^{*} is the shortest distance between them. The set $\mathbb{D}$ of dual numbers is commutative ring the the operations $+$ and $\cdot$. The set

$$\mathbb{D}^3=\mathbb{D}\times\mathbb{D}\times\mathbb{D}=\left\{\widehat{\varphi}=\varphi+\xi\varphi^{*};\varphi,\varphi^{*}\in\mathbb{E}^3\right\}$$

is a module over the ring $\mathbb{D}$ [15].

For any $\widehat{a}=a+\xi a^{*}$, $\widehat{b}=b+\xi b^{*}\in\mathbb{D}^3$, if the Lorentzian inner product of $\widehat{a}$ and $\widehat{b}$ is defined by

$$<\widehat{a},\widehat{b}>=<a,b>+\xi(<a^{*},b>+<a,b^{*}>),$$

then the dual space $\mathbb{D}^3$ together with this Lorentzian inner product is called the dual Lorentzian space and denoted by $\mathbb{D}^3_1$ [1]. For $\widehat{\varphi}\neq 0$, the norm $\|\widehat{\varphi}\|$ of $\widehat{\varphi}$ is defined by

$$\|\widehat{\varphi}\|=\sqrt{<\widehat{\varphi},\widehat{\varphi}>}.$$

A dual vector $\widehat{\omega}=\omega+\xi\omega^{*}$ is called dual spacelike vector if $\langle\widehat{\omega},\widehat{\omega}\rangle>0$ or $\widehat{\omega}=0$, dual timelike vector if $\langle\widehat{\omega},\widehat{\omega}\rangle<0$ and dual null (lightlike) vector if $\langle\widehat{\omega},\widehat{\omega}\rangle=0$ for $\widehat{\omega}\neq 0$. Therefore, an arbitrary dual curve which is a differential mapping onto $\mathbb{D}^3_1$, can locally be dual spacelike, dual timelike or dual null if its velocity vector is dual spacelike, dual timelike or dual null, respectively. Also, for the dual vectors $\widehat{a},\widehat{b}\in\mathbb{D}^3_1$, Lorentzian vector product of these dual vectors is defined by

$$\widehat{a}\times\widehat{b}=a\times b+\xi(a^{*}\times b+a\times b^{*})$$

where $a\times b$ is the classical cross product according to the signature $(+,+,-)$ [1].

The dual arc length of the curve $\widehat{\varphi}$ from t_1 to t is defined as

$$s=\int_{t_1}^{t}\|\widehat{\varphi}'(t)\|\,dt=\int_{t_1}^{t}\|\varphi'(t)\|\,dt+\varepsilon\int_{t_1}^{t}\langle t,\varphi'\rangle\,dt=s+\xi s^{*},$$

where t is a unit tangent vector of $\varphi(t)$. From now on we will take the arc-length s of $\varphi(t)$ as the parameter instead of t [9].

Let $\widehat{\varphi}:I\subset\mathbb{E}\to\mathbb{D}^3_1$ be a dual spacelike curve with the arc-length parameter s. The Bishop derivative formula of dual spacelike curve $\widehat{\varphi}$ is expressed as

$$\begin{cases}\widehat{T}'= & \widehat{k}_1\widehat{N}_1-\widehat{k}_2\widehat{N}_2,\\ \widehat{N}_1'= & -\varepsilon\widehat{k}_1\widehat{T},\\ \widehat{N}_2'= & -\varepsilon\widehat{k}_2\widehat{T},\end{cases}\qquad(1)$$

where $\left\langle \widehat{T}, \widehat{T} \right\rangle = 1, \left\langle \widehat{N}_1, \widehat{N}_1 \right\rangle = \varepsilon = \pm 1, \left\langle \widehat{N}_2, \widehat{N}_2 \right\rangle = -\varepsilon$ and $\widehat{k}_1, \widehat{k}_2$ are Bishop curvatures. Here $\widehat{\tau} = \dfrac{d\widehat{\theta}}{ds}$ and $\widehat{\kappa} = \sqrt{\left|\widehat{k}_1^2 - \widehat{k}_2^2\right|}$. Thus, Bishop curvatures are defined by ([1], [2])

$$\widehat{k}_1 = \widehat{\kappa}(s) \cosh \widehat{\theta}(s), \quad \widehat{k}_2 = \widehat{\kappa}(s) \sinh \widehat{\theta}(s) \ .$$

Let $\widehat{\varphi} : I \subset \mathbb{E} \to \mathbb{D}_1^3$ be a dual timelike curve with the arc-length parameter s. The Bishop derivative formula of dual spacelike curve $\widehat{\varphi}$ is expressed as

$$\begin{cases} \widehat{T}' = & \widehat{k}_1 \widehat{N}_1 + \widehat{k}_2 \widehat{N}_2, \\ \widehat{N}_1' = & \widehat{k}_1 \widehat{T}, \\ \widehat{N}_2' = & \widehat{k}_2 \widehat{T}, \end{cases} \tag{2}$$

where $\left\langle \widehat{T}, \widehat{T} \right\rangle = -1, \left\langle \widehat{N}_1, \widehat{N}_1 \right\rangle = 1, \left\langle \widehat{N}_2, \widehat{N}_2 \right\rangle = 1$ and $\widehat{k}_1, \widehat{k}_2$ are Bishop curvatures. Here $\widehat{\tau} = \dfrac{d\widehat{\theta}}{ds}$ and $\widehat{\kappa} = \sqrt{\left|\widehat{k}_1^2 - \widehat{k}_2^2\right|}$. Thus, Bishop curvatures are defined by ([1], [2])

$$\widehat{k}_1 = \widehat{\kappa}(s) \cosh \widehat{\theta}(s), \quad \widehat{k}_2 = \widehat{\kappa}(s) \sinh \widehat{\theta}(s)$$

§3. Main Results

In this section, we give some characterizations of dual spacelike (timelike) curves of constant breadth according to Bishop frame in the dual Lorentzian space $\mathbb{D}_1^3$. First, we give the definition of dual spacelike (timelike) curves of constant breadth in $\mathbb{D}_1^3$. Then we characterize these kinds of curves by obtaining special solutions of their differential equations in $\mathbb{D}_1^3$.

Definition 3.1 *Let (C_1) be a dual spacelike (timelike) curve with position vector $\widehat{\varphi} = \widehat{\varphi}(s)$ in $\mathbb{D}_1^3$. If (C) has parallel tangents in opposite directions at corresponding points $\widehat{\varphi}(s)$ and $\widehat{\alpha}(s_\alpha)$ and the distance between these points is always constant, then (C_1) is called a dual spacelike (timelike) curve of constant breadth. Moreover, a pair of dual curves (C_1) and (C_2) for which the tangents at the corresponding points $\widehat{\varphi}(s)$ and $\widehat{\alpha}(s_\alpha)$, respectively, are parallel and in opposite directions, and the distance between these points is always constant are called a dual (timelike) curve pair of constant breadth.*

3.1 Dual Spacelike Curves of Constant Breadth According to Dual Bishop Frame

Let $\widehat{\varphi} = \widehat{\varphi}(s)$ be a simple closed dual spacelike curve in $\mathbb{D}_1^3$. We consider a dual spacelike curve in the class Γ as in [6] having parallel tangents $\widehat{T}_\varphi$ and $\widehat{T}_\alpha$ in opposite directions at the opposite points $\widehat{\varphi}$ and $\widehat{\alpha}$ of the curve according to Bishop frame. A simple closed dual spacelike curve of constant breadth having parallel tangents in opposite directions at opposite points can be represented with respect to dual Bishop frame by the equation

$$\widehat{\alpha}=\widehat{\varphi}+\widehat{\gamma}\widehat{T}+\widehat{\delta}\widehat{N}_1+\widehat{\lambda}\widehat{N}_2, \tag{3}$$

where $\widehat{\gamma},\widehat{\delta}$ and $\widehat{\lambda}$ are arbitrary functions of s. Differentiating both sides of (4), we get

$$\frac{d\widehat{\alpha}}{ds_\alpha}\frac{ds_\alpha}{ds}=\left(\frac{d\widehat{\gamma}}{ds}-\varepsilon\widehat{\widehat{\delta k_1}}-\varepsilon\widehat{\widehat{\lambda k_2}}+1\right)\widehat{T}+\left(\widehat{\gamma}\widehat{k}_1+\frac{d\widehat{\delta}}{ds}\right)\widehat{N}_1+\left(-\widehat{\gamma}\widehat{k}_2+\frac{d\widehat{\lambda}}{ds}\right)\widehat{N}_2. \tag{4}$$

Considering $\widehat{T}_\alpha=-\widehat{T}_\varphi$ by the definition 3.1, we have the following system of equations

$$\begin{cases}\dfrac{d\widehat{\gamma}}{ds}= & \varepsilon\widehat{\widehat{\delta k_1}}+\varepsilon\widehat{\widehat{\lambda k_2}}-1-\dfrac{ds_\alpha}{ds},\\ \dfrac{d\widehat{\delta}}{ds}= & -\widehat{\gamma}\widehat{k}_1,\\ \dfrac{d\widehat{\lambda}}{ds}= & \widehat{\gamma}\widehat{k}_2.\end{cases} \tag{5}$$

If we call $\widehat{\theta}$ as the angle between the tangent of the curve C at point $\widehat{\varphi}$ with a given direction and taking $\dfrac{d\widehat{\theta}}{ds}=\widehat{\tau},\dfrac{d\widehat{\theta}}{ds_\alpha}=\widehat{\tau}^*$ into account, the equation (5) turns into

$$\begin{cases}\dfrac{d\widehat{\gamma}}{d\widehat{\theta}}= & \varepsilon\widehat{\delta}\dfrac{\widehat{k}_1}{\widehat{\tau}}+\varepsilon\widehat{\lambda}\dfrac{\widehat{k}_2}{\widehat{\tau}}-f(\widehat{\theta}),\\ \dfrac{d\widehat{\delta}}{d\widehat{\theta}}= & -\widehat{\gamma}\dfrac{\widehat{k}_1}{\widehat{\tau}},\\ \dfrac{d\widehat{\lambda}}{d\widehat{\theta}}= & \widehat{\gamma}\dfrac{\widehat{k}_2}{\widehat{\tau}},\end{cases} \tag{6}$$

where $f(\widehat{\theta})=\dfrac{1}{\widehat{\tau}}+\dfrac{1}{\widehat{\tau}^*}$.

Let $\widehat{K}_1=\dfrac{\widehat{k}_1}{\widehat{\tau}},\widehat{K}_2=\dfrac{\widehat{k}_2}{\widehat{\tau}}$ and using the system of ordinary differential equations (6), we have the following dual third order differential equation with respect to $\widehat{\gamma}$ as;

$$\begin{aligned}&\frac{d^3\widehat{\gamma}}{d\widehat{\theta}^3}+\varepsilon(\widehat{K}_1^2-\widehat{K}_2^2)\frac{d\widehat{\gamma}}{d\widehat{\theta}}+3\varepsilon\left(\widehat{K}_1\frac{d\widehat{K}_1}{d\widehat{\theta}}-\widehat{K}_1\frac{d\widehat{K}_1}{d\widehat{\theta}}\right)\widehat{\gamma}\\ &+\varepsilon\left(\int\widehat{\gamma}\widehat{K}_1d\widehat{\theta}\right)\frac{d^2\widehat{K}_1}{d\widehat{\theta}^2}-\varepsilon\left(\int\widehat{\gamma}\widehat{K}_2d\widehat{\theta}\right)\frac{d^2\widehat{K}_2}{d\widehat{\theta}^2}+\frac{d^2f(\widehat{\theta})}{d\widehat{\theta}^2}=0\end{aligned} \tag{7}$$

We can give the following corollary.

Corollary 3.1.1 *The dual differential equation of third order given in (7) is a characterization of the simple closed dual spacelike curve $\widehat{\alpha}$ according to Bishop frame in $\mathbb{D}_1^3$.*

Since position vector of a simple closed dual spacelike curve can be determined by solution of the equation (7), let us investigate solution of the equation (7) in a special case. Let $\widehat{K}_1,\widehat{K}_2$

and $f(\widehat{\theta})$ be constants. Then the equation (7) turns to the following form

$$\frac{d^3\widehat{\gamma}}{d\widehat{\theta}^3}+\varepsilon(\widehat{K}_1^2-\widehat{K}_2^2)\frac{d\widehat{\gamma}}{d\widehat{\theta}}=0. \tag{8}$$

Solution of equation (8) yields the components

$$\begin{cases}\widehat{\gamma}=\widehat{A}+\widehat{B}\cos(\sqrt{\widehat{K}_1^2-\widehat{K}_2^2}\widehat{\theta})+\widehat{C}\sin(\sqrt{\widehat{K}_1^2-\widehat{K}_2^2}\widehat{\theta})\\ \widehat{\delta}=-\int\left\{\widehat{A}+\widehat{B}\cos(\sqrt{\widehat{K}_1^2-\widehat{K}_2^2}\widehat{\theta})+\widehat{C}\sin(\sqrt{\widehat{K}_1^2-\widehat{K}_2^2}\widehat{\theta})\right\}\widehat{K}_1 d\widehat{\theta}\\ \widehat{\lambda}=\int\left\{\widehat{A}+\widehat{B}\cos(\sqrt{\widehat{K}_1^2-\widehat{K}_2^2}\widehat{\theta})+\widehat{C}\sin(\sqrt{\widehat{K}_1^2-\widehat{K}_2^2}\widehat{\theta})\right\}\widehat{K}_2 d\widehat{\theta}.\end{cases} \tag{9}$$

Corollary 3.1.2 *Position vector of a simple dual spacelike closed curve with constant dual curvature and constant dual torsion according to Bishop frame is obtained in terms of the values of $\widehat{\gamma}$, $\widehat{\delta}$ and $\widehat{\lambda}$ as in the equation (9).*

If the distance between opposite points of $\widehat{\varphi}$ and $\widehat{\alpha}$ is constant, then we can write that

$$\|\widehat{\alpha}-\widehat{\varphi}\|=-\widehat{\gamma}^2+\widehat{\delta}^2+\widehat{\lambda}^2=\text{constant}. \tag{10}$$

Differentiating (10) with respect to $\widehat{\theta}$ gives

$$-\widehat{\gamma}\frac{d\widehat{\gamma}}{d\widehat{\theta}}+\widehat{\delta}\frac{d\widehat{\delta}}{d\widehat{\theta}}+\widehat{\lambda}\frac{d\widehat{\lambda}}{d\widehat{\theta}}=0. \tag{11}$$

By virtue of (6), the differential equation (11) yields

$$-\widehat{\delta}\widehat{K}_1(1+\varepsilon)+\widehat{\lambda}\widehat{K}_2(1-\varepsilon)+f(\widehat{\theta})=0, \widehat{\gamma}=0. \tag{12}$$

There are two cases for the equation (12), we study these cases as follows:

Case 1. If $\widehat{K}_1=0$ and $\widehat{K}_2=0$ then we find that the components $\widehat{\delta}$ and $\widehat{\lambda}$ are constants and $f(\widehat{\theta})=0$.

Hence, Dual spacelike curves of constant breadth according to Bishop frame can be written as

$$\widehat{\alpha}=\widehat{\varphi}+\widehat{l}_1\widehat{T}+\widehat{l}_2\widehat{N}_1+\widehat{l}_3\widehat{N}_2, \tag{13}$$

where $\widehat{\gamma}=\widehat{l}_1, \widehat{\delta}=\widehat{l}_2, \widehat{\lambda}=\widehat{l}_3$; $\widehat{l}_1, \widehat{l}_2, \widehat{l}_3$ are constants.

Case 2. If $f(\widehat{\theta})=0$, then we have a relation among radii of curvatures as

$$\frac{1}{\widehat{\tau}}-\frac{1}{\widehat{\tau}^*}=0. \tag{14}$$

For this case, the equation (7) turns into

$$\begin{aligned} &\frac{d^3\widehat{\gamma}}{d\widehat{\theta}^3}+\varepsilon(\widehat{K}_1^2-\widehat{K}_2^2)\frac{d\widehat{\gamma}}{d\widehat{\theta}}+3\varepsilon(\widehat{K}_1\frac{d\widehat{K}_1}{d\widehat{\theta}}-\widehat{K}_1\frac{d\widehat{K}_1}{d\widehat{\theta}})\widehat{\gamma} \\ &+\varepsilon(\textstyle\int\widehat{K}_1d\widehat{\theta})\widehat{\gamma}\frac{d^2\widehat{K}_1}{d\widehat{\theta}^2}-\varepsilon(\textstyle\int\widehat{K}_2d\widehat{\theta})\widehat{\gamma}\frac{d^2\widehat{K}_2}{d\widehat{\theta}^2}=0 \end{aligned} \tag{15}$$

The equation (15) is a characterization for the components. However, its general solution of has not been found. Due to this, we investigate its solutions in special cases.

Let us suppose that $\widehat{K}_1=\widehat{K}_2=0$, then we rewrite the equation (15) as

$$\frac{d^3\widehat{\gamma}}{d\widehat{\theta}^3}=0. \tag{16}$$

By this way, we have the components as follows:

$$\begin{cases} \widehat{\gamma}=\widehat{c}_1+\widehat{c}_2\widehat{\theta}+\widehat{c}_3\widehat{\theta}^2, \\ \widehat{\delta}=\text{constant}, \\ \widehat{\lambda}=\text{constant}. \end{cases} \tag{17}$$

3.2 Dual Timelike Curves of Constant Breadth According to Dual Bishop Frame

Let $\widehat{\varphi}=\widehat{\varphi}(s)$ be a simple closed dual timelike curve in $\mathbb{D}_1^3$. We consider a dual timelike curve in the class Γ as in [6] having parallel tangents $\widehat{T}_\varphi$ and $\widehat{T}_\alpha$ in opposite directions at the opposite points $\widehat{\varphi}$ and $\widehat{\alpha}$ of the curve according to Bishop frame. A simple closed dual timelike curve of constant breadth having parallel tangents in opposite directions at opposite points can be represented with respect to dual Bishop frame by the equation

$$\widehat{\alpha}=\widehat{\varphi}+\widehat{\gamma}\widehat{T}+\widehat{\delta}\widehat{N}_1+\widehat{\lambda}\widehat{N}_2, \tag{18}$$

where $\widehat{\gamma},\widehat{\delta}$ and $\widehat{\lambda}$ are arbitrary functions of s. Differentiating both sides of (18), we get

$$\frac{d\widehat{\alpha}}{ds_\alpha}\frac{ds_\alpha}{ds}=(\frac{d\widehat{\gamma}}{ds}+\widehat{\delta}\widehat{k}_1+\widehat{\lambda}\widehat{k}_2+1)\widehat{T}+(\widehat{\gamma}\widehat{k}_1+\frac{d\widehat{\delta}}{ds})\widehat{N}_1+(\widehat{\gamma}\widehat{k}_2+\frac{d\widehat{\lambda}}{ds})\widehat{N}_2. \tag{19}$$

Considering $\widehat{T}_\alpha=-\widehat{T}_\varphi$ by the Definition 3.1, we have the following system of equations

$$\begin{cases} \frac{d\widehat{\gamma}}{ds}= \frac{ds_\alpha}{ds}-\widehat{\delta}\widehat{k}_1-\widehat{\lambda}\widehat{k}_2-1, \\ \frac{d\widehat{\delta}}{ds}= -\widehat{\gamma}\widehat{k}_1, \\ \frac{d\widehat{\lambda}}{ds}= -\widehat{\gamma}\widehat{k}_2. \end{cases} \tag{20}$$

If we call $\widehat{\theta}$ as the angle between the tangent of the curve C at point $\widehat{\varphi}$ with a given direction and taking $\dfrac{d\widehat{\theta}}{ds}=\widehat{\tau}, \dfrac{d\widehat{\theta}}{ds_\alpha}=\widehat{\tau}^*$ into account, we have (20) as follow;

$$\begin{cases} \dfrac{d\widehat{\gamma}}{d\widehat{\theta}} = & -\widehat{\delta}\dfrac{\widehat{k}_1}{\widehat{\tau}}-\widehat{\lambda}\dfrac{\widehat{k}_2}{\widehat{\tau}}-f(\widehat{\theta}), \\ \dfrac{d\widehat{\delta}}{d\widehat{\theta}} = & -\widehat{\gamma}\dfrac{\widehat{k}_1}{\widehat{\tau}}, \\ \dfrac{d\widehat{\lambda}}{d\widehat{\theta}} = & -\widehat{\gamma}\dfrac{\widehat{k}_2}{\widehat{\tau}}, \end{cases} \tag{21}$$

where $f(\widehat{\theta})=\dfrac{1}{\widehat{\tau}}-\dfrac{1}{\widehat{\tau}^*}$.

Let $\widehat{K}_1=\dfrac{\widehat{k}_1}{\widehat{\tau}}, \widehat{K}_2=\dfrac{\widehat{k}_2}{\widehat{\tau}}$ and using the system of ordinary differential equations (21), we have the following dual third order differential equation with respect to $\widehat{\gamma}$ as;

$$\begin{aligned} &\frac{d^3\widehat{\gamma}}{d\widehat{\theta}^3}-(\widehat{K}_1^2+\widehat{K}_2^2)\frac{d\widehat{\gamma}}{d\widehat{\theta}}-3\varepsilon(\widehat{K}_1\frac{d\widehat{K}_1}{d\widehat{\theta}}+\widehat{K}_1\frac{d\widehat{K}_1}{d\widehat{\theta}})\widehat{\gamma} \\ &-(\textstyle\int \widehat{K}_1 d\widehat{\theta})\widehat{\gamma}\dfrac{d^2\widehat{K}_1}{d\widehat{\theta}^2}-(\textstyle\int \widehat{K}_2 d\widehat{\theta})\widehat{\gamma}\dfrac{d^2\widehat{K}_2}{d\widehat{\theta}^2}-\dfrac{d^2f(\widehat{\theta})}{d\widehat{\theta}^2}=0. \end{aligned} \tag{22}$$

We can give the following corollary.

Corollary 3.2.1 *The dual differential equation of third order given in (22) is a characterization of the simple closed dual timelike curve* $\widehat{\alpha}$ *according to Bishop frame in* $\mathbb{D}_1^3$.

Since position vector of a simple closed dual timelike curve can be determined by solution of (22), let us investigate solution of the equation (22) in a special case. Let $\widehat{K}_1, \widehat{K}_2$ and $f(\widehat{\theta})$ be constants. Then the equation (22) turns into the following form

$$\frac{d^3\widehat{\gamma}}{d\widehat{\theta}^3}-(\widehat{K}_1^2+\widehat{K}_2^2)\frac{d\widehat{\gamma}}{d\widehat{\theta}}=0. \tag{23}$$

Solution of equation (23) yields the components

$$\begin{cases} \widehat{\gamma}=\widehat{A}+\widehat{B}e^{(\widehat{K}_1^2+\widehat{K}_2^2)\widehat{\theta}}+\widehat{C}e^{-(\widehat{K}_1^2+\widehat{K}_2^2)\widehat{\theta}}, \\ \widehat{\delta}=-\int\left\{\widehat{A}+\widehat{B}e^{(\widehat{K}_1^2+\widehat{K}_2^2)\widehat{\theta}}+\widehat{C}e^{-(\widehat{K}_1^2+\widehat{K}_2^2)\widehat{\theta}}\right\}\widehat{K}_1 d\widehat{\theta}, \\ \widehat{\lambda}=-\int\left\{\widehat{A}+\widehat{B}e^{(\widehat{K}_1^2+\widehat{K}_2^2)\widehat{\theta}}+\widehat{C}e^{-(\widehat{K}_1^2+\widehat{K}_2^2)\widehat{\theta}}\right\}\widehat{K}_2 d\widehat{\theta}. \end{cases} \tag{24}$$

Corollary 3.2.3 *Position vector of a simple dual timelike closed curve with constant dual curvature and constant dual torsion according to Bishop frame is obtained in terms of the values of* $\widehat{\gamma}$, $\widehat{\delta}$ *and* $\widehat{\lambda}$ *in the equation (24).*

If the distance between opposite points of $\widehat{\varphi}$ and $\widehat{\alpha}$ is constant, then we can write that

$$\|\widehat{\alpha}-\widehat{\varphi}\| = -\widehat{\gamma}^2+\widehat{\delta}^2+\widehat{\lambda}^2 = \text{constant}. \tag{25}$$

Differentiating (25) with respect to $\widehat{\theta}$ gives

$$-\widehat{\gamma}\frac{d\widehat{\gamma}}{d\widehat{\theta}}+\widehat{\delta}\frac{d\widehat{\delta}}{d\widehat{\theta}}+\widehat{\lambda}\frac{d\widehat{\lambda}}{d\widehat{\theta}}=0. \tag{26}$$

By virtue of (21), the differential equation (26) yields

$$\widehat{\gamma}f(\widehat{\theta})=0. \tag{27}$$

There are two cases for the equation (27), we study these cases as follows:

Case 1. If $\widehat{\gamma}=0$ then we find that the components $\widehat{\delta}$ and $\widehat{\lambda}$ are constants.

Hence, Dual timelike curves of constant breadth according to Bishop frame can be written as

$$\widehat{\alpha}=\widehat{\varphi}+\widehat{l}_1\widehat{T}+\widehat{l}_2\widehat{N}_1+\widehat{l}_3\widehat{N}_2, \tag{28}$$

where $\widehat{\gamma}=\widehat{l}_1, \widehat{\delta}=\widehat{l}_2, \widehat{\lambda}=\widehat{l}_3$; $\widehat{l}_1, \widehat{l}_2, \widehat{l}_3$ are constants.

Case 2. If $f(\widehat{\theta})=0$, then we have a relation among radii of curvatures as

$$\frac{1}{\widehat{\tau}}-\frac{1}{\widehat{\tau}^*}=0. \tag{29}$$

For this case, the equation (22) turns into

$$\begin{aligned}\frac{d^3\widehat{\gamma}}{d\widehat{\theta}^3}-(\widehat{K}_1^2+\widehat{K}_2^2)\frac{d\widehat{\gamma}}{d\widehat{\theta}}-3\varepsilon(\widehat{K}_1\frac{d\widehat{K}_1}{d\widehat{\theta}}+\widehat{K}_1\frac{d\widehat{K}_1}{d\widehat{\theta}})\widehat{\gamma}\\ -(\textstyle\int\widehat{K}_1d\widehat{\theta})\widehat{\gamma}\frac{d^2\widehat{K}_1}{d\widehat{\theta}^2}-(\textstyle\int\widehat{K}_2d\widehat{\theta})\widehat{\gamma}\frac{d^2\widehat{K}_2}{d\widehat{\theta}^2}=0.\end{aligned} \tag{30}$$

The equation (30) is a characterization for the components. However, its general solution has not been found. Due to this, we investigate its solutions in special cases.

Let us suppose that $\widehat{K}_1=\widehat{K}_2=0$, then we rewrite the equation (30) as

$$\frac{d^3\widehat{\gamma}}{d\widehat{\theta}^3}=0. \tag{31}$$

By this way, we have the components as follows:

$$\begin{cases}\widehat{\gamma}=\widehat{c}_1+\widehat{c}_2\widehat{\theta}+\widehat{c}_3\widehat{\theta}^2,\\ \widehat{\delta}=\text{constant},\\ \widehat{\lambda}=\text{constant}.\end{cases} \tag{32}$$

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On (r, m, k)-Regular Fuzzy Graphs

N.R.Santhimaheswari

Department of Mathematics

G.Venkataswamy Naidu College, Kovilpatti-628502, Tamil Nadu, India

C.Sekar

Department of Mathematics

Aditanar College of Arts and Science, Tiruchendur, Tamil Nadu, India

E-mail: nrsmaths@yahoo.com, sekar.acas@gmail.com

Abstract: In this paper, (r, m, k)- regular fuzzy graph and totally (r, m, k)- regular fuzzy graph are defined and compared through various examples. A necessary and sufficient condition under which they are equivalent is provided. Also (r, m, k)-regularity on some fuzzy graphs whose underlying crisp graph is a cycle is studied with some specific membership functions.

Key Words: Degree of a vertex in fuzzy graph, regular fuzzy graph, total degree, totally regular fuzzy graph, d_m- degree of a vertex in graph, semiregular graphs, (m, k)-regular fuzzy graphs, totally (m, k)-regular fuzzy graphs.

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§1. Introduction

Azriel Rosenfeld introduced fuzzy graphs in 1975 [12]. It has been growing fast and has numerous applications in various fields. A.Nagoor Gani and K.Radha [11] introduced regular fuzzy graphs, total degree and totally regular fuzzy graphs. Alison Northup introduced Semiregular graphs that we call it as $(2, k)$-regular graphs and studied some properties on $(2, k)$-regular graphs [2].

N.R.Santhi Maheswari and C. Sekar introduced d_2-degree of a vertex in fuzzy graphs, total d_2-degree of a vertex in fuzzy graphs, $(2, k)$-regular fuzzy graphs and totally $(2, k)$-regular fuzzy graphs [14]. Also they introduced $(r, 2, k)$-regular fuzzy graphs and totally $(r, 2, k)$-regular fuzzy graphs [15].

Also they introduced d_m-degree of a vertex in fuzzy graphs, total d_m-degree of a vertex in fuzzy graphs, m-Neighbourly irregular fuzzy graphs and totally m-Neighbourly irregular fuzzy graphs [16]. Also, they introduced (m, k)-regular fuzzy graphs and totally (m, k)-regular fuzzy graphs [17].

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These motivate us to introduce (r, m, k)-regular fuzzy graphs and totally (r, m, k)-regular fuzzy graphs. We make comparative study between (r, m, k)-regular fuzzy graphs and totally (r, m, k)-regular fuzzy graphs. Then we provide a necessary and sufficient condition under which they are equivalent. Also (r, m, k)-regularity on fuzzy graphs whose underlying crisp graph is a cycle is studied with some specific membership functions.

§2. Preliminaries

We present some known definitions and results for ready reference to go through the work presented in this paper.

Definition 2.1([9]) *A Fuzzy graph denoted by $G : (\sigma, \mu)$ on graph $G^* : (V, E)$ is a pair of functions (σ, μ) where $\sigma : V \to [0, 1]$ is a fuzzy subset of a non empty set V and $\mu : V \times V \to [0, 1]$ is a symmetric fuzzy relation on σ such that for all u, v in V the relation $\mu(u, v) = \mu(uv) \leq \sigma(u) \wedge \sigma(v)$ is satisfied, where σ and μ are called membership function. A fuzzy graph G is complete if $\mu(u, v) = \mu(uv) = \sigma(u) \wedge \sigma(v)$ for all $u, v \in V$, where uv denotes the edge between u and v. $G^* : (V, E)$ is called the underlying crisp graph of the fuzzy graph $G : (\sigma, \mu)$.*

Definition 2.2([10]) *The strength of connectedness between two vertices u and v is $\mu^\infty(u, v) = sup\{\mu^k(u, v)/k = 1, 2, \ldots\}$ where $\mu^k(u, v) = sup\{\mu(uu_1)\wedge\mu(u_1u_2)\wedge\cdots\wedge\mu(u_{k-1}v)/u, u_1, u_2, \cdots, u_{k-1}, v$ is a path connecting u and v of length $k\}$.*

Definition 2.3([11]) *Let $G : (\sigma, \mu)$ be a fuzzy graph. The degree of a vertex u is $d_G(u) = \sum\limits_{u \neq v} \mu(uv)$ for $uv \in E$ and $\mu(uv) = 0$, for uv not in E; this is equivalent to $d_G(u) = \sum\limits_{uv \in E} \mu(uv)$.*

Definition 2.4([11]) *Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^* : (V, E)$. If $d(v) = k$ for all $v \in V$, then G is said to be regular fuzzy graph of degree k.*

Definition 2.5([11]) *Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^* : (V, E)$. The total degree of a vertex u is defined as $td(u) = \sum \mu(u, v) + \sigma(u) = d(u) + \sigma(u)$, $uv \in E$. If each vertex of G has the same total degree k, then G is said to be totally regular fuzzy graph of degree k or k-totally regular fuzzy graph.*

Definition 2.6([14]) *Let $G : (\sigma, \mu)$ be a fuzzy graph. The d_2-degree of a vertex u in G is $d_2(u) = \sum \mu^2(u, v)$, where $\mu^2(uv) = \sup\{\mu(uu_1)\wedge\mu(u_1v) : u, u_1, v$ is the shortest path connecting u and v of length $2\}$. Also, $\mu(uv) = 0$, for uv not in E.*
The minimum d_2-degree of G is $\delta_2(G) = \wedge\{d_2(v) : v \in V\}$.
The maximum d_2-degree of G is $\Delta_2(G) = \vee\{d_2(v) : v \in V\}$.

Definition 2.7([14]) *Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^* : (V, E)$. If $d_2(v) = k$ for all $v \in V$, then G is said to be $(2, k)$-regular fuzzy graph.*

Definition 2.8([14]) *Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^* : (V, E)$. The total d_2-degree of a* *vertex* $u \in V$ *is defined as* $td_2(u) = \sum \mu^2(u,v) + \sigma(u) = d_2(u) + \sigma(u)$.

The minimum td_2*-degree of* G *is* $t\delta_2(G) = \wedge\{td_2(v) : v \in V\}$.

The maximum td_2*-degree of* G *is* $t\Delta_2(G) = \vee\{td_2(v) : v \in V\}$.

Definition 2.9([14]) *If each vertex of* G *has the same total* d_2 *- degree* k, *then* G *is said to be totally* $(2,k)$*-regular fuzzy graph.*

Definition 2.10([15]) *If each vertex of* G *has the same degree* r *and same* d_2*-degree* k, *then* G *is said to be* $(r,2,k)$*-regular fuzzy graph.*

Definition 2.11([15]) *If each vertex of* G *has the same total degree* r *and same total* d_2*-degree* k, *then* G *is said to be totally* $(r,2,k)$*-regular fuzzy graph.*

Definition 2.12([16]) *Let* $G : (\sigma,\mu)$ *be a fuzzy graph on* $G^* : (V,E)$.. *The* d_m*-degree of a vertex* u *in* G *is* $d_m(u) = \sum \mu^m(uv)$, *where* $\mu^m(uv) = sup\{\mu(uu_1) \wedge \mu(u_1u_2) \wedge \ldots, \mu(u_{m-1}v) : u, u_1, u_2, \ldots, u_{m-1}, v$ *is the shortest path connecting* u *and* v *of length* $m\}$. *Also,* $\mu(uv) = 0$, *for* uv *not in* E.

The minimum d_m*-degree of* G *is* $\delta_m(G) = \wedge\{d_m(v) : v \in V\}$.

The maximum d_m*-degree of* G *is* $\Delta_m(G) = \vee\{d_m(v) : v \in V\}$.

Definition 2.13([16]) *Let* $G : (\sigma,\mu)$ *be a fuzzy graph on* $G^* : (V,E)$. *The total* d_m*-degree of a vertex* $u \in V$ *is defined as* $td_m(u) = \sum \mu^m(uv) + \sigma(u) = d_m(u) + \sigma(u)$.

The minimum td_m*-degree of* G *is* $t\delta_m(G) = \wedge\{td_m(v) : v \in V\}$.

The maximum td_m*-degree of* G *is* $t\Delta_m(G) = \vee\{td_m(v) : v \in V\}$.

Definition 2.14([17]) *Let* $G : (\sigma,\mu)$ *be a fuzzy graph on* $G^* : (V,E)$. *If* $d_m(v) = k$ *for all* $v \in V$, *then* G *is said to be* (m,k)*-regular fuzzy graph.*

Definition 2.15([17]) *If each vertex of* G *has the same total* d_m *- degree* k, *then* G *is said to be totally* (m,k)*-regular fuzzy graph.*

§3. (r,m,k)-Regular Fuzzy Graphs

In this section, we define (r,m,k)-Regular Fuzzy Graphs and illustrates this with $(r,3,k)$-regular graph.

Definition 3.1 *Let* $G : (\sigma,\mu)$ *be a fuzzy graph on* $G^* : (V,E)$. *If* $d(v) = r$ *and* $d_m(v) = k$, *for all* $v \in V$, *then* G *is said to be* (r,m,k)*-regular fuzzy graph. That is, if each vertex of* G *has the same degree* r *and same* d_m*-degree* k, *then* G *is said to be* (r,m,k)*-regular fuzzy graph.*

Example 3.2 Consider $G^? : (V,E)$, where $V = \{u_1,u_2,u_3,u_4,u_5,u_6,u_7,u_8,u_9,u_{10}\}$ and $E = \{u_1u_2, u_2u_3, u_3u_4, u_4u_5, u_5u_6, u_6u_7, u_7u_8, u_8u_9, u_9u_{10}, u_{10}u_1\}$. Define $G : (\sigma,\mu)$ by $\sigma(u_1) = 0.3,\ \sigma(u_2) = 0.4,\ \sigma(u_3) = 0.5,\ \sigma(u_4) = 0.6,\ \sigma(u_5) = 0.7,\ \sigma(u_6) = 0.6, \sigma(u_7) = 0.5,\ \sigma(u_8) = 0.4, \sigma(u_9) = 0.3,\ \sigma(u_{10}) = 0.2$ and $\mu(u_1u_2) = 0.3,\ \mu(u_2u_3) = 0.4,\ \mu(u_3u_4) = 0.3,\ \mu(u_4u_5) = 0.4,\ \mu(u_5u_6) = 0.3,\ \sigma(u_6u_7) = 0.4,\ \sigma(u_7u_8) = 0.3,\ \sigma(u_8u_9) = 0.4,\ \sigma(u_9u_{10}) = 0.3, \sigma(u_{10}u_1) =$

0.4.

$$d_3(u_1) = \{0.3 \wedge 0.4 \wedge 0.3\} + \{0.3 \wedge 0.4 \wedge 0.3\} = 0.3 + 0.3 = 0.6.$$
$$d_3(u_2) = \{0.3 \wedge 0.3 \wedge 0.4\} + \{0.4 \wedge 0.3 \wedge 0.4\} = 0.3 + 0.3 = 0.6.$$
$$d_3(u_3) = \{0.4 \wedge 0.3 \wedge 0.3\} + \{0.3 \wedge 0.4 \wedge 0.3\} = 0.3 + 0.3 = 0.6.$$
$$d_3(u_4) = \{0.3 \wedge 0.4 \wedge 0.3\} + \{0.4 \wedge 0.3 \wedge 0.4\} = 0.3 + 0.3 = 0.6.$$
$$d_3(u_5) = \{0.3 \wedge 0.4 \wedge 0.3\} + \{0.4 \wedge 0.3 \wedge 0.4\} = 0.3 + 0.3 = 0.6.$$
$$d_3(u_6) = \{0.4 \wedge 0.3 \wedge 0.4\} + \{0.3 \wedge 0.4 \wedge 0.3\} = 0.3 + 0.3 = 0.6.$$
$$d_3(u_7) = \{0.3 \wedge 0.4 \wedge 0.3\} + \{0.4 \wedge 0.3 \wedge 0.4\} = 0.3 + 0.3 = 0.6.$$
$$d_3(u_8) = \{0.4 \wedge 0.3 \wedge 0.3\} + \{0.3 \wedge 0.4 \wedge 0.3\} = 0.3 + 0.3 = 0.6.$$
$$d_3(u_9) = \{0.3 \wedge 0.3 \wedge 0.4\} + \{0.4 \wedge 0.3 \wedge 0.4\} = 0.3 + 0.3 = 0.6.$$
$$d(u_i) = \{0.3 + 0.4\} = 0.7 \text{ for i = 1,2,3,4,5,6,7,8,9,10.}$$

It is noted that, each vertex has the same d_3-degree 0.6 and each vertex has the same degree 0.7. Hence G is $(0.7, 3, 0.6)$-regular fuzzy graph.

Example 3.3 Consider $G^* : (V, E)$, where $V = \{u, v, w, x, y, z\}$ and
$E = \{uv, vw, wx, xy, yz, zu\}$.

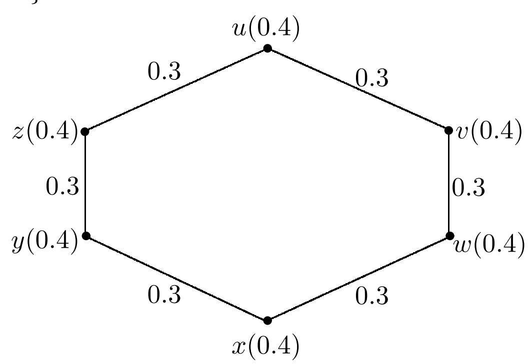

Figure 1

$$d_3(u) = Sup\{0.3 \wedge 0.3 \wedge 0.3, 0.3 \wedge 0.3 \wedge 0.3\} = Sup\{0.3, 0.3\} = 0.3.$$
$$d_3(v) = Sup\{0.3 \wedge 0.3 \wedge 0.3, 0.3 \wedge 0.3 \wedge 0.3\} = Sup\{0.3, 0.3\} = 0.3.$$
$$d_3(w) = Sup\{0.3 \wedge 0.3 \wedge 0.3, 0.3 \wedge 0.3 \wedge 0.3\} = Sup\{0.3, 0.3\} = 0.3.$$
$$d_3(x) = Sup\{0.3 \wedge 0.3 \wedge 0.3, 0.3 \wedge 0.3 \wedge 0.3\} = Sup\{0.3, 0.3\} = 0.3.$$
$$d_3(y) = Sup\{0.3 \wedge 0.3 \wedge 0.3, 0.3 \wedge 0.3 \wedge 0.3\} = Sup\{0.3, 0.3\} = 0.3.$$
$$d_3(z) = Sup\{0.3 \wedge 0.3 \wedge 0.3, 0.3 \wedge 0.3 \wedge 0.3\} = Sup\{0.3, 0.3\} = 0.3.$$

In Figure 1, $d(u) = 0.3 + 0.3 = 0.6$, $d(v) = 0.6$, $d(w) = 0.6, d(x) = 0.6$, $d(y) = 0.6$, $d(z) = 0.6$. Each vertex has the same d_3-degree 0.3 and each vertex has the same degree 0.3. Hence G is a $(0.6, 3, 0.3)$-regular fuzzy graph.

Example 3.4 Non regular fuzzy graphs which is (m, k)-regular

1. Let $G:(\sigma,\mu)$ be a fuzzy graph such that $G^*:(V,E)$, a path on $2m$ vertices. Let all the edges of G have the same membership value c. Then, for $i=1,2,3,4,5,\cdots,m$,

$$\begin{aligned}d_m(v_i) &= \{\mu(e_i)\wedge\mu(e_{i+1})\wedge\{\mu(e_{i+2})\cdots\wedge\mu(e_{m-1+i})\}\\ &= \{c\wedge c\wedge c\cdots\wedge c\}=c.\\ d_m(v_{m+i}) &= \{\mu(e_i)\wedge\mu(e_{i+1})\}+\{\mu(e_{i+2})\cdots\wedge\mu(e_{m-1+i})\}\\ &= \{c\wedge c\wedge c\cdots\wedge c\}=c.\\ d_m(v) &= c,\ \text{for all } v\in V.\end{aligned}$$

Hence $G:(\sigma,\mu)$ is (m,c)-regular fuzzy graph.

For $i=2,3,4,5,\cdots,2m-1$,

$$\begin{aligned}d(v_i) &= \{\mu(e_{i-1})+\mu(e_i)=2c.\\ d(v_1) &= \{\mu(e_1)\}=c.\\ d(v_{2m}) &= \mu(e_{2m-1})=c.\end{aligned}$$

$d(v_1)\neq d(v_i)\neq d(v_{2m})$ for $i=2,4,5,\cdots,2m-1$. Hence G is non- regular fuzzy graph which is (m,c)-regular.

Example 3.5 Let $G:(\sigma,\mu)$ be a fuzzy graph on $G^*:(V,E)$, a cycle of length $\geq 2m+1$. Let

$$\mu(e_i)=\begin{cases} c_1 & \text{if i is odd}\\ \text{membership value } x\geq c_1 & \text{if } i \text{ is even, where } x \text{ is not constant and}\end{cases}$$

$$d_m(v)=min\{c_1,x\}+min\{x,c_1\}=c_1+c_1=2c_1$$

for all $v\in V$.

Case 1. Let $G:(\sigma,\mu)$ be a fuzzy graph on $G^*:(V,E)$ an even cycle of length $\leq 2m+2$. Then $d(v_i)=x+c_1$, for $i=1,2,4,5,\cdots,2m+1$. So, $G:(\sigma,\mu)$ is non-regular (m,k)-regular fuzzy graph, since x is not constant.

Case 2 Let $G:(\sigma,\mu)$ be a fuzzy graph on $G^*:(V,E)$ a odd cycle of length $\leq 2m+1$. Hence $G:(\sigma,\mu)$ is $(m,2c_1)$-regular fuzzy graph and $d(v_1)=2c_1,\ d(v_i)=x+c_1$ for $i=2,4,5,\cdots,2m+1$. So, $G:(\sigma,\mu)$ is non-regular (m,k)-regular fuzzy graph since x is not constant.

§4. Totally (r,m,k)-Regular Fuzzy Graphs

In this section, we introduce totally (r,m,k)-regular fuzzy graph and the necessary and sufficient condition under which (r,m,k)-regular fuzzy graph and totally (r,m,k)-regular fuzzy graph are equivalent is provided.

Definition 4.1 *If each vertex of G has the same total degree r and same total d_m-degree k, then G is said to be totally (r,m,k)-regular fuzzy graph.*

From Figure 1, it is noted that each vertex has the same total d_3-degree 0.7.

$$td_3(u) = d_3(u) + \sigma(u) = 0.3 + 0.4 = 0.7$$
$$td_3(v) = d_3(v) + \sigma(v) = 0.3 + 0.4 = 0.7$$
$$td_3(w) = d_3(w) + \sigma(w) = 0.3 + 0.4 = 0.7$$
$$td_3(x) = d_3(x) + \sigma(x) = 0.3 + 0.4 = 0.7$$
$$td_3(y) = d_3(y) + \sigma(y) = 0.3 + 0.4 = 0.7$$
$$td_3(z) = d_3(z) + \sigma(z) = 0.3 + 0.4 = 0.7$$

$$td(u) = d(u) + \sigma(u) = 0.8 + 0.4 = 1.2$$
$$td(v) = d(v) + \sigma(v) = 0.8 + 0.4 = 1.2$$
$$td(w) = d(w) + \sigma(w) = 0.8 + 0.4 = 1.2$$
$$td(x) = d(x) + \sigma(x) = 0.8 + 0.4 = 1.2$$
$$td(y) = d(y) + \sigma(y) = 0.8 + 0.4 = 1.2$$
$$td(z) = d(z) + \sigma(z) = 0.8 + 0.4 = 1.2$$

In Figure 1, Each vertex has the same total d_3-degree 0.7 and each vertex has the same total degree 1.2. Hence $G : (\sigma, \mu)$ is totally $(1.2, 3, 0.7)$-regular fuzzy graph.

Theorem 4.2 *Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^* : (V, E)$. Then σ is constant function iff the following conditions are equivalent:*

(1) *$G : (\sigma, \mu)$ is (r, m, k)-regular fuzzy graph;*
(2) *$G : (\sigma, \mu)$ is totally (r, m, k)-regular fuzzy graph.*

Proof Suppose that σ is constant function. Let $\sigma(u) = c$, constant for all $u \in V$. Assume that $G : (\sigma, \mu)$ is (r, m, k)-regular fuzzy graph. Then $d(u) = r$ and $d_m(u) = k$, for all $u \in V$. So

$$td(u) = d(u) + \sigma(u) \text{ and } td_m(u) = d_m(u) + \sigma(u) \text{ for all } u \in V.$$
$$\Rightarrow td(u) = r + c \text{ and } td_m(u) = k + c \text{ for all } u \in V.$$

Hence $G : (\sigma, \mu)$ is totally $(r + c, m, k + c)$- regular fuzzy graph. Thus (1) $\Rightarrow$ (2) is proved. Now suppose G is totally (r, m, k)-regular fuzzy graph.

$$\Rightarrow td_m(u) = k \text{ and } td(u) = r \text{ for all } u \in V.$$
$$\Rightarrow d_m(u) + \sigma(u) = k \text{ and } d(u) + \sigma(u) = r \text{ for all } u \in V.$$
$$\Rightarrow d_m(u) + c = k \text{ and } d(u) + \sigma(u) = r \text{ for all } u \in V.$$
$$\Rightarrow d_m(u) = k - c \text{ and } d(u) = r - c \text{ for all } u \in V.$$

Hence $G : (\sigma, \mu)$ is $(r - c, m, k - c)$-regular fuzzy graph and (1) and (2) are equivalent.

Conversely assume that (1) and (2) are equivalent. Suppose σ is not constant function. Then $\sigma(u) \neq \sigma(w)$, for at least one pair $u, w \in V$. Let $G : (\sigma, \mu)$ be a (r, m, k)-regular fuzzy graph. Then, $d_m(u) = d_m(w) = k$ and $d(u) = d(w) = r$. So, $td_m(u) = d_m(u) + \sigma(u) = k + \sigma(u)$ and $td_m(w) = d_m(w) + \sigma(w) = k + \sigma(w)$ and $td(u) = d(u) + \sigma(u) = r + \sigma(u)$ and $td(w) = d(w) + \sigma(w) = r + \sigma(w)$. Since $\sigma(u) \neq \sigma(w) \Rightarrow k + \sigma(u) \neq k + \sigma(w)$ and $r + \sigma(u) \neq r + \sigma(w \Rightarrow td_m(u) \neq td_m(w)$ and $td(u) \neq td(w)$. So $G : (\sigma, \mu)$ is not totally (r, m, k)-regular fuzzy graph which is contradiction to our assumption. Let $G : (\sigma, \mu)$ be a totally (r, m, k)-regular fuzzy graph. Then, $td_m(u) = td_m(w)$ and $td(u) = td(w)$.

$$\begin{aligned}
&\Rightarrow d_m(u) + \sigma(u) = d_m(w) + \sigma(w) \text{ and } d(u) + \sigma(u) = d(w) + \sigma(w)\\
&\Rightarrow d_m(u) - d_m(w) = \sigma(w) - \sigma(u) \neq 0 \text{ and } d(u) - d(w)\\
&\qquad\qquad\qquad = \sigma(w) - \sigma(u) \neq 0\\
&\Rightarrow d_m(u) \neq d_m(w) \text{ and } d(u) \neq d(w).
\end{aligned}$$

So $G : (\sigma, \mu)$ is not (r, m, k)-regular fuzzy graph which is a contradiction to our assumption. Hence σ is constant function. □

Theorem 4.3 *If a fuzzy graph $G : (\sigma, \mu)$ is both (r, m, k)-regular and totally (r, m, k)-regular then σ is constant function.*

Proof Let G be (r_1, m, k_1)-regular and totally (r_2, m, k_2)-regular fuzzy graph. Then $d_m(u) = k_1$ and $td_m(u) = k_2, d(u) = r_1$ and $td(u) = r_2$, for all $u \in V$. Now, $td_m(u) = k_2$ and $td(u) = r_2$, for all $u \in V$.

$$\begin{aligned}
&\Rightarrow d_m(u) + \sigma(u) = k_2 \text{ and } d(u) + \sigma(u) = r_2 \text{ for all } u \in V.\\
&\Rightarrow k_1 + \sigma(u) = k_2 \text{ and } r_1 + \sigma(u) = r_2 \text{ for all } u \in V.\\
&\Rightarrow \sigma(u) = k_2 - k_1 \text{ and } \sigma(u) = r_2 - r_1 \text{ for all } u \in V.
\end{aligned}$$

Hence σ is constant function. □

§5. (r, m, k)- Regular Fuzzy Graph on a Cycle with Some Specific Membership Function.

In this section, (r, m, k)-regularity on a cycle C_{2m}, C_{2m+1} is studied with some specific membership functions.

Theorem 5.1 *For any $m \geq 1$, let $G : (\sigma, \mu)$ be a fuzzy graph on $G^* : (V, E)$, a cycle of length $\geq 2m$. If μ is constant function, then $G : (\sigma, \mu)$ is (r, m, k)-regular fuzzy graph, where $r = 2\mu(uv)$ and $k = \mu(uv)$.*

Proof If μ is constant function say $\mu(uv) = c$, then $d_m(v) = Sup\{(c \wedge c \cdots \wedge c)), (c \wedge c \cdots \wedge c\}$ = c, for all $v \in V$ and $d(v) = c + c = 2c$. Hence G is $(2c, m, c)$-regular fuzzy graph. □

Remark 5.2 Converse of the above Theorem need not be true.

Theorem 5.3 *For any $m \geq 1$, let $G : (\sigma, \mu)$ be a fuzzy graph on $G^* : (V, E)$, a cycle of length*

$\geq 2m+1$. If μ is constant function, then G is (r, m, k)-regular fuzzy graph, where $r = 2\mu(uv)$ and $k = 2\mu(uv)$.

Proof If μ is constant function say $\mu(uv) = c$, then $d_m(v) = \{c \wedge c \cdots \wedge c\} + \{c \wedge c \cdots \wedge c\} = c + c = 2c$, for all $v \in V$ and $d(v) = c + c = 2c$. Hence G is $(2c, m, 2c)$-regular fuzzy graph. □

Remark 5.4 Converse of the above Theorem need not be true.

Theorem 5.5 *For any $m \geq 1$, let $G : (\sigma, \mu)$ be a fuzzy graph on $G^* : (V, E)$, an even cycle of length $\geq 2m+2$. If the alternate edges have the same membership values, then $G : (\sigma, \mu)$ is (r, m, k)-regular fuzzy graph.*

Proof If the alternate edges have the same membership values, then

$$\mu(e_i) = \begin{cases} c_1, & \text{if } i \text{ is odd} \\ c_2, & \text{if } i \text{ is even.} \end{cases}$$

If $c_1 = c_2$, then μ is constant function. So, $G : (\sigma, \mu)$ is $(2c_1, m, 2c_1)$-regular fuzzy graph. If $c_1 < c_2$, then $d_m(v) = \{c_1 \wedge c_2 \ldots c_1 \wedge c_2\} + \{c_1 \wedge c_2 \ldots c_1 \wedge c_2\} = c_1 + c_1 = 2c_1$, for all $v \in V$ and $d(v) = c_1 + c_2$. Hence $G : (\sigma, \mu)$ is $(c_1 + c_2, m, 2c_1)$-regular fuzzy graph.

If $c_1 > c_2$, then $d_m(v) = \{c_1 \wedge c_2 \ldots c_1 \wedge c_2\} + \{c_1 \wedge c_2 \ldots c_1 \wedge c_2\} = c_2 + c_2 = 2c_2$, for all $v \in V$ and $d(v) = c_1 + c_2$. Hence $G : (\sigma, \mu)$ is $(c_1 + c_2, m, 2c_2)$-regular fuzzy graph. □

Remark 5.6 Even if the alternate edges of a fuzzy graph whose underlying graph is an even cycle of length $\geq 2m+2$ have the same membership values, then $G : (\sigma, \mu)$ need not be totally (r, m, k)-regular fuzzy graph, since if σ is not constant function then $G : (\sigma, \mu)$ is not totally (r, m, k)-regular fuzzy graph, for any $m \geq 1$.

Theorem 5.7 *For any $m > 1$, let $G : (\sigma, \mu)$ be a fuzzy graph on $G^* : (V, E)$, a cycle of length $\geq 2m+1$. Let*

$$\mu(e_i) = \begin{cases} c_1, & \textit{if } i \textit{ is odd} \\ c_2 \geq c_1, & \textit{if } i \textit{ is even,} \end{cases}$$

then $G : (\sigma, \mu)$ is a (m, k)-regular fuzzy graph.

Proof Let

$$\mu(e_i) = \begin{cases} c_1, & \text{if } i \text{ is odd} \\ c_2 \geq c_1, & \text{if } i \text{ is even} \end{cases}$$

Case 1. Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^* : (V, E)$ an even cycle of length $\leq 2m+2$. Then by theorem 6.3, G is $(c_1 + c_2, m, 2c_1)$-regular fuzzy graph.

Case 2. Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^* : (V, E)$ an odd cycle of length $\leq 2m+1$. For any $m > 1$, $d_m(v) = 2c_1$, for all $v \in V$. But $d(v_1) = c_1 + c_1 = 2c_1$ and $d(v_i) = c_1 + c_2$, for $i \neq 1$. Hence G is not (r, m, k)-regular fuzzy graph. □

Remark 5.8 Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^* : (V, E)$, an even cycle of length $\geq 2m + 1$. Even if

$$\mu(e_i) = \begin{cases} c_1, & \text{if } i \text{ is odd} \\ c_2 \geq c_1 & \text{if } i \text{ is even,} \end{cases}$$

then G need not be totally (r, m, k)-regular fuzzy graph, since if σ is not constant function then G is not totally (r, m, k)-regular fuzzy graph.

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Super Edge-Antimagic Labeling of Subdivided Star Trees

A.Raheem and A.Q.Baig

Department of Mathematics

COMSATS Institute of information Technology, Islamabad, Pakistan

E-mail: rahimciit7@gmail, makbiyik@yildiz.edu.tr, aqbaig1@gmail.com

Abstract: Let G be a graph with $V(G)$ and $E(G)$ as the vertex set and the edge set respectively. An (a,d)-edge-antimagic total labeling of a graph G is a bijection λ from the set $V(G) \cup E(G) \rightarrow \{1,2,3,\cdots,|V(G|+|E(G)|\}$ such that the set of edge-weights $\{\lambda(x)+\lambda(xy)+\lambda(y) : xy \in E(G)\}$ is equal to $\{a, a+d, a+2d, \cdots, a+(|E(G)|-1)d\}$ where the integers $a > 0$ and $d \geq 0$. An (a,d)-edge-antimagic total labeling of a graph G is called super (a,d)-EAT labeling if the smallest possible labels are assigned to the vertices of the graph G.

Key Words: Labeling, super (a,d)-EAT labeling, subdivision of star trees.

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§1. Introduction

All graphs in this paper are finite, undirected and simple. For a graph G we denote the vertex-set and edge-set by $V(G)$ and $E(G)$, respectively. A (v,e)-graph G is a graph such that $v = |V(G)|$ and $e = |E(G)|$. A general reference for graph-theoretic ideas can be seen in [24]. In the present paper the domain will be the set of all the elements of a graph G and such a labeling is called a total labeling. The more details on antimagic total labeling can be seen in [14, 9]. The subject of edge-magic total labeling of graphs has its origin in the works of Kotzig and Rosa [17, 18] on what they called magic valuations of graphs. The definition of (a,d)-edge-antimagic total labeling was introduced by Simanjuntak, Bertault and Miller in [21] as a natural extension of edge-magic labeling defined by Kotzig and Rosa.

Conjecture 1.1([11]) Every tree admits a super edge-magic total labeling.

In the support of this conjecture, many authors have considered super edge-magic total labeling for many particular classes of trees for example[23, 1, 20, 2, 22, 310, 15, 16, 12, 13, 21]. Lee and Shah [19] verified this conjecture by a computer search for trees with at most 17 vertices. However, this conjecture is still as an open problem.

A star is a particular type of tree graph and many authors have proved the magicness for subdivided stars. Ngurah et. al. [20] proved that $T(m,n,k)$ is also super edge-magic if

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$k = n+3$ or $n+4$. In [23], Salman et. al. found the super edge-magic total labeling of a subdivision of a star S_n^m for $m = 1, 2$. Javaid et. al. [16] proved super edge-magic total labeling on subdivided star $K_{1,4}$ and w-trees.

However, super (a, d)-edge-antimagic total labeling of $G \cong T(n_1, n_2, n_3, \cdots, n_r)$ for different $\{n_i : 1 \leq i \leq r\}$ is still open.

Definition 1.1 *A graph G is called (a, d)-edge-antimagic total ($(a, d) - EAT$) if there exist integers $a > 0$, $d \geq 0$ and a bijection*

$$\lambda : V(G) \cup E(G) \to \{1, 2, 3, \cdots, v+e\}$$

such that $W = \{w(xy) : xy \in E(G)\}$ forms an arithmetic sequence starting from a with the common difference d, where $w(xy) = \lambda(x) + \lambda(y) + \lambda(xy)$ for every $xy \in E(G)$. W is called the set of edge-weights of the graph G.

Definition 1.2 *A (a, d)-edge-antimagic total labeling λ is called super (a, d)-edge-antimagic total labeling if $\lambda(V(G)) = \{1, 2, 3, \cdots, v\}$.*

Definition 1.3 *For $n_i \geq 1$ and $r \geq 3$, let $G \cong T(n_1, n_2, n_3, \cdots, n_r)$ be a graph obtained by inserting $n_i - 1$ vertices to each of the i-th edge of the star $K_{1,r}$ where $1 \leq i \leq r$.*

The notion of a dual labeling has been introduced by Kotzig and Rosa [17]. According to him, if f is an $(a, 0)$-EAT labeling with magic constant a then f_1 is also an $(a, 0)$-EAT labeling with magic constant $a_1 = 3(v+e+1) - a$. The following is defined as $f_1(x) = v+e+1-f(x)$ for all $x \in V(G) \cup E(G)$.

Lemma 1.1[12] *If f is a super edge-magic total labeling of G with the magic constant c, then the function $f_1 : V(G) \cup E(G) \to \{1, 2, 3, \cdots, v+e\}$ defined by*

$$f_1(x) = \begin{cases} v+1-f(x), & \text{for } x \in V(G), \\ 2v+e+1-f(x), & \text{for } x \in E(G). \end{cases}$$

is also a super edge-magic total labeling of G with the magic constant $c_1 = 4v+e+3-c$.

We consider the following proposition which we will use frequently in the main results.

Proposition 1.1([8]) *If a (v, e)-graph G has a (s, d)-EAV labeling then*

(1) *G has a super $(s+v+1, d+1)$-EAT labeling;*
(2) *G has a super $(s+v+e, d-1)$-EAT labeling.*

§2. Super (a, d)-EAT Labeling of Subdivided Stars

In this section we deal with the main results related to the super (a, d)-EAT labelings. on generalized families of subdivided stars for all possible values of d.

Theorem 2.1 *For $n \geq 1$ and $r \geq 4$, $G \cong T(n+1, n+2, 2n+4, n_4, \cdots, n_r)$ admits a super $(a,0)$-EAT labeling with $a = 2v + s - 1$ and a super $(a,2)$-EAT labeling with $a = v + s + 1$, where $v = |V(G)|$ and $s = (n+5) + \sum_{m=4}^{r}[2^{m-4}(n+2)]$ and $n_m = 2^{m-2}(n+2)$ for $4 \leq m \leq r$.*

Proof The vertices and the edges of the graph G are $v = (2n+4) + \sum_{m=4}^{r}[2^{m-3}(n+2)]$ and $e = v - 1$. Define the vertex labeling $\lambda : V(G) \to \{1, 2, \cdots, v\}$ as follows:

Let $\lambda(c) = 1$. For even $1 \leq l_i \leq n_i$, where $i = 1, 2, 3$ and $4 \leq i \leq r$:

$$\lambda(u) = \begin{cases} 1 + \frac{l_1}{2}, & \text{for } u = x_1^{l_1}, \\ (n+3) - \frac{l_2}{2}, & \text{for } u = x_2^{l_2} \\ (2n+5) - \frac{l_3}{2}., & \text{for } u = x_2^{l_3}. \end{cases}$$

$$\lambda(x_i^{l_i}) = (2n+5) + \sum_{m=4}^{i}[2^{m-3}(n+2)] - \frac{l_i}{2} \text{ respectively.}$$

For odd $1 \leq l_i \leq n_i$ and $\alpha = (2n+5) + \sum_{m=4}^{r}[2^{m-3}(n+2)]$, where $i = 1, 2, 3$ and $4 \leq i \leq r$:

$$\lambda(u) = \begin{cases} \alpha + \frac{l_1+1}{2}, & \text{for } u = x_1^{l_1}, \\ (\alpha + n + 3) - \frac{l_2+1}{2}, & \text{for } u = x_2^{l_2}, \\ (\alpha + 2n + 5) - \frac{l_3+1}{2}, & \text{for } u = x_2^{l_2}. \end{cases}$$

and $\lambda(x_i^{l_i}) = (\alpha + 2n + 5) + \sum_{m=4}^{i}[2^{m-3}(n+2)] - \frac{l_i+1}{2}$ respectively.

The set of all edge-sums $\{\lambda(x) + \lambda(y) : xy \in E(G)\}$ generated by the above formulas forms an integer sequence $(\alpha+1)+1, (\alpha+1)+2, \cdots, (\alpha+1)+e$, where $s = \alpha + 2$. Therefore, by Proposition 1.1, λ can be extended to a super $(a,0)$-EAT labeling with $a = 2v - 1 + s = 2v + (n+3) + \sum_{m=4}^{r}[2^{m-3}(n+2)]$ and to a super $(a,2)$-EAT labeling with $a = v + 1 + s = v + (n+4) + \sum_{m=4}^{r}[2^{m-3}(n+2)]$. □

Theorem 2.2 *For $n \geq 1$ and $r \geq 3$, $G \cong T(n+1, n+2, 2n+4, n_4, \cdots, n_r)$ admits a super $(a,1)$-EAT labeling with $a = 2v + s - 1$ and a super $(a,3)$-EAT labeling with $a = v + s + 1$, where $v = |V(G)|$ and $s = 3$ and $n_m = 2^{m-2}(n+2)$ for $4 \leq m \leq r$.*

Proof Let us consider the vertices and edges are defined as in Theorem 2.1. Now, define $\lambda : V(G) \to \{1, 2, \cdots, v\}$ as follows:

$\lambda(c) = 1$. For $1 \leq l_i \leq n_i$, where $i = 1, 2, 3$ and $4 \leq i \leq r$:

$$\lambda(u)=\begin{cases} l_1+1, & \text{for } u=x_1^{l_1},\\ (2n+5)-l_2, & \text{for } u=x_2^{l_2},\\ (4n+9)-l_2, & \text{for } u=x_3^{l_3},\end{cases}$$

and $\lambda(x_i^{l_i})=(4n+9)+\sum\limits_{m=4}^{i}[2^{m-2}(n+2)]-l_i$ respectively.

The set of all edge-sums $\{\lambda(x)+\lambda(y): xy\in E(G)\}$ generated by the above formulas forms an integer sequence $3,3+2,\cdots,3+2(e-1)$, where $s=3$. Therefore, by Proposition 1.1, λ can be extended to a super $(a,1)$-EAT labeling with $a=2v-1+s=2v+2$ and to a super $(a,3)$-EAT labeling with $a=v+1+s=v+4$. □

As a consequence of Lemma 1.1. and the Theorem 2.1., we have the following corollaries:

Corollary 2.3 *For $n\geq 1$ and $r\geq 4$, $G\cong T(n+1,n+2,2n+4,n_4,\cdots,n_r)$ admits a super $(a,0)$-EAT labeling with magic constant $a=(3v-n-1)-\sum\limits_{m=4}^{r}[2^{m-3}(n+2)]$, where $n_m=2^{m-2}(n+2)$ for $4\leq m\leq r$.*

Corollary 2.4 *For $n\geq 1$ and $r\geq 4$, $G\cong T(n+1,n+2,2n+4,n_4,\cdots,n_r)$ admits a super $(a,2)$-EAT labeling with minimum edge weight is $a=(2v-n+1)-\sum\limits_{m=4}^{r}[2^{m-3}(n+2)]$, where $n_m=2^{m-2}(n+2)$ for $4\leq m\leq r$.*

We construct relation between the Super (a,d)-EAT labelings and the (a,d)-EAT labelings deduce from Theorem 2.2. and according to the concept of Kotzig and Rosa related to a dual labeling, we have the following corollary.

Corollary 2.5 *For $n\geq 1$ and $r\geq 4$, $G\cong T(n+1,n+2,2n+4,n_4,\cdots,n_r)$ admits a $(a,1)$-EAT labeling with minimum edge weight is $a=3v$ and $(a,3)$-EAT labeling with minimum edge weight $a=2v+2$, where $n_m=2^{m-2}(n+2)$ for $4\leq m\leq r$.*

Theorem 2.6 *For $n\geq 1$ and $r\geq 4$, $G\cong T(n+1,n+1,n+2,n_4,\cdots,n_r)$ admits a super $(a,0)$-EAT labeling with $a=2v+s-1$ and a super $(a,2)$-EAT labeling with $a=v+s+1$, where $v=|V(G)|$ and*

$$s=1+\lceil\frac{3(n+2)}{2}\rceil+\sum_{m=4}^{r}[2^{m-4}(n+2)]$$

and $n_m=2^{m-3}(n+2)$ for $4\leq m\leq r$.

Proof The vertices and edges of the graph G are $v=(3n+4)+\sum\limits_{m=4}^{r}[2^{m-3}(n+2)]$ and $e=v-1$. Define the vertex labeling $\lambda: V(G)\to\{1,2,\cdots,v\}$ as follows:

$\lambda(c) = \lceil \frac{n+2}{2} \rceil$. For even $1 \leq l_i \leq n_i$, where $i = 1, 2, 3$ and $4 \leq i \leq r$:

$$\lambda(u) = \begin{cases} \frac{n+2}{2} - \frac{l_1}{2}, & \text{for } u = x_1^{l_1}, \\ \frac{n+2}{2} + \frac{l_2}{2}, & \text{for } u = x_2^{l_2}, \\ \lceil \frac{3(n+2)}{2} \rceil - \frac{l_3}{2}, & \text{for } u = x_3^{l_3}. \end{cases}$$

$$\lambda(x_i^{l_i}) = \lceil \frac{3(n+2)}{2} \rceil + \sum_{m=4}^{i} [2^{m-4}(n+2)] - \frac{l_i}{2} \text{ respectively.}$$

For odd $1 \leq l_i \leq n_i$ and $\alpha = \lceil \frac{3(n+2)}{2} \rceil + \sum\limits_{m=4}^{r} [2^{m-4}(n+2)]$, where $i = 1, 2, 3$ and $4 \leq i \leq r$:

$$\lambda(u) = \begin{cases} \alpha + \lceil \frac{n+3}{2} \rceil - \frac{l_1+1}{2}, & \text{for } u = x_1^{l_1}, \\ \alpha + \lceil \frac{n+1}{2} \rceil + \frac{l_2+1}{2}, & \text{for } u = x_2^{l_2}, \\ \alpha + 1 + \lfloor \frac{3(n+2)}{2} \rfloor - \frac{l_3+1}{2}, & \text{for } u = x_3^{l_3}. \end{cases}$$

and

$$\lambda(x_i^{l_i}) = \alpha + 1 + \lfloor \frac{3(n+1)}{2} \rfloor + \sum_{m=4}^{i} [2^{m-4}(n+2)] - \frac{l_i+1}{2} \text{ respectively.}$$

The set of all edge-sums $\{\lambda(x) + \lambda(y) : xy \in E(G)\}$ generated by the above formulas forms a consecutive integer sequence $(\alpha+1)+1, (\alpha+1)+2, \cdots, (\alpha+1)+e$, where $s = \alpha+2$. Therefore, by Proposition 2.1, λ can be extended to a super $(a, 0)$-EAT labeling with

$$a = 2v + s - 1 = 2v + \lceil \frac{3(n+1)}{2} \rceil + \sum_{m=4}^{r} [2^{m-4}(n+2)]$$

and to a super $(a, 2)$-EAT labeling with

$$a = v + 1 + s = v + 2 + \lceil 3n + 72 \rceil + \sum_{m=4}^{r} [2^{m-4}(n+2)]. \quad \square$$

Theorem 2.7 *For $n \geq 1$ and $r \geq 4$, $G \cong T(n+1, n+1, n+2, n_4, \cdots, n_r)$ admits a super $(a, 1)$-EAT labeling with $a = 2v + s - 1$ and a super $(a, 3)$-EAT labeling with $a = v + s + 1$, where $v = |V(G)|$ and $s = 3$ and $n_m = 2^{m-3}(n+2)$ for $4 \leq m \leq r$.*

Proof Let us consider the vertices and edges are defined as in Theorem 2.3. Now, we define $\lambda : V(G) \to \{1, 2, \cdots, v\}$ as follows:

$\lambda(c) = n + 2$. For $1 \leq l_i \leq n_i$, where $i = 1, 2, 3$ and $4 \leq i \leq r$:

$$\lambda(u)=\begin{cases}(n+2)-l_1, & \text{for } u=x_1^{l_1},\\ (n+2)+l_2, & \text{for } u=x_2^{l_2},\\ 3(n+2)-l_3, & \text{for } u=x_3^{l_3},\end{cases}$$

and

$$\lambda(x_i^{l_i})=3(n+2)+\sum_{m=4}^{i}[2^{m-3}(n+2)]-l_i \text{ respectively.}$$

The set of all edge-sums $\{\lambda(x)+\lambda(y) : xy\in E(G)\}$ generated by the above formulas forms an integer sequence $3, 3+2, \cdots, 3+2(e-1)$, where $s=3$. Therefore, by Proposition 2.1, λ can be extended to a super $(a,1)$-EAT labeling with $a=2v-1+s=2v+2$ and to a super $(a,3)$-EAT labeling with $a=v+1+s=v+4$. □

§3. Conclusion

In this paper, we have proved the super edge anti-magicness of subdivided stars for all possible values of d, However the problem of the anti-magicness is still open for different values of magic constant.

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Surface Family with a Common Natural Geodesic Lift

Evren Ergün

Ondokuz Mayıs University

Çarşamba Chamber of Commerce Vocational School, Çarşamba, Samsun, Turkey

Ergin Bayram

Ondokuz Mayıs University

Faculty of Arts and Sciences, Department of Mathematics, Samsun, Turkey

E-mail: eergun@omu.edu.tr, erginbayram@yahoo.com

Abstract: In the present paper, we find a surface family possessing the natural lift of a given curve as a geodesic. We express necessary and sufficient conditions for the given curve such that its natural lift is a geodesic on any member of the surface family. We present a sufficient condition for ruled surfaces with the above property. Finally, we illustrate the method with some examples.

Key Words: Ruled surfaces, curve, geodesic, Frenet frame.

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§1. Introduction

Curves and surfaces play an important role in differential geometry. In recent years, there is an ascending interest on finding surfaces possessing a given curve as a common curve instead of finding and characterizing curves on a given surface. In 2004, Wang et. al. [1] proposed a method to find surfaces having a given curve as a common geodesic. Kasap et. al. [2] generalized the marching-scale functions of Wang and obtained a larger family of surfaces. Li et. al. [3] derived the necessary and sufficient constraint for a line of curvature. Bayram et. al. [4] studied parametric surfaces which interpolate a given curve as a common asymptotic. Ergün et. al. [5] obtained a surface family from a given spacelike or timelike line of curvature in Minkowski 3-space.

Inspired with the above studies, we find a surface family possessing the natural lift of a given curve as a common geodesic. We obtain the sufficient condition for the resulting surface to be a ruled surface.

We start with presenting some background. A parametric curve $\alpha(s)$, $L_1 \leq s \leq L_2$, is a curve on a surface $P(s,t)$ in $\mathbb{R}^3$ that has a constant s or t-parameter value. In this paper, α' denotes the derivative of α with respect to arc length parameter s and we assume that α is a regular curve with $\alpha''(s) \neq 0$, $L_1 \leq s \leq L_2$. For every point of $\alpha(s)$, the set

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$\{T(s), N(s), B(s)\}$ is called the Frenet frame along $\alpha(s)$, where $T(s) = \alpha'(s)$, $N(s) = \frac{\alpha''}{\|\alpha''\|}$ and $B(s) = T(s) \times N(s)$ are the unit tangent, principal normal, and binormal vectors of the curve at the point $\alpha(s)$, respectively. Derivative formulas of the Frenet frame is governed by the relations

$$\frac{d}{ds}\begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix}\begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}, \tag{1}$$

where $\kappa(s) = \|\alpha''(s)\|$ and $\tau(s) = -\langle B'(s), N(s)\rangle$ are called the curvature and torsion of the curve $\alpha(s)$, respectively [6].

Let M be a surface in $\mathbb{R}^3$ and let $\alpha : I \longrightarrow M$ be a parameterized curve. α is called an integral curve of X if

$$\frac{d}{ds}(\alpha(s)) = X(\alpha(s)) \text{ (for all } t \in I),$$

where X is a smooth tangent vector field on M. We have

$$TM = \bigcup_{P \in M} T_P M = \chi(M),$$

where $T_P M$ is the tangent space of M at P and $\chi(M)$ is the space of tangent vector fields on M.

For any parameterized curve $\alpha : I \longrightarrow M$, $\overline{\alpha} : I \longrightarrow TM$ given by ([7])

$$\bar{\alpha}(s) = \left(\alpha(s), \alpha'(s)\right) = \alpha'(s)\,|_{\alpha(s)} \tag{2}$$

is called the *natural lift* of α on TM.

If a rigid body moves along a unit speed curve $\alpha(s)$, then the motion of the body consists of translation along α and rotation about α. The rotation is determined by an angular velocity vector ω which satisfies $T' = \omega \times T$, $N' = \omega \times N$ and $B' = \omega \times B$. The vector ω is called the *Darboux vector*. In terms of Frenet vectors T, N and B, Darboux vector is given by $\omega = \tau T + \kappa B$ [8]. Also, we have $\kappa = \|\omega\| \cos\theta$, $\tau = \|\omega\| \sin\theta$, where θ is the angle between the Darboux vector ω and binormal vector $B(s)$ of α. Observe that $\theta = \arctan\frac{\tau}{\kappa}$ $(Fig.\ 1)$.

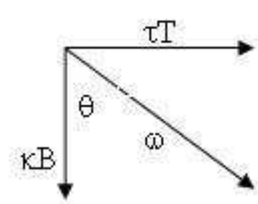

Fig.1 Darboux vector ω, tangent vector T and binormal vector B of α

Let $\alpha(s)$, $L_1 \leq s \leq L_2$, be an arc length curve and $\bar{\alpha}(s)$, $L_1 \leq s \leq L_2$, be the natural lift of α. Then we have

$$\begin{pmatrix} \bar{T}(s) \\ \bar{N}(s) \\ \bar{B}(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\cos\theta & 0 & \sin\theta \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}, \tag{3}$$

where $\{T(s), N(s), B(s)\}$ and $\{\bar{T}(s), \bar{N}(s), \bar{B}(s)\}$ are the Frenet frames of the curves α and $\bar{\alpha}$, respectively, and θ is the angle between the Darboux vector and binormal vector of α.

§2. Surface Family with a Common Natural Geodesic Lift

Suppose we are given a 3-dimensional parametric curve $\alpha(s),\ L_1 \leq s \leq L_2$, in which s is the arc length and $\|\alpha''(s)\| \neq 0,\ L_1 \leq s \leq L_2$. Let $\bar{\alpha}(s),\ L_1 \leq s \leq L_2,$ be the involute of $\alpha(s)$.

Surface family that interpolates $\bar{\alpha}(s)$ as a common curve is given in the parametric form as

$$P(s,t) = \overline{\alpha}(s) + u(s,t)\bar{T}(s) + v(s,t)\bar{N}(s) + w(s,t)\bar{B}(s), \tag{4}$$

$L_1 \leq s \leq L_2,\ T_1 \leq t \leq T_2,$ where $u(s,t),\ v(s,t)$ and $w(s,t)$ are C^1 functions and are called *marching-scale functions* and $\{\bar{T}(s), \bar{N}(s), \bar{B}(s)\}$ is the Frenet frame of the curve $\bar{\alpha}$. Using Eqn. (3) we can express Eqn. (4) in terms of Frenet frame $\{T(s), N(s), B(s)\}$ of the curve α as

$$\begin{aligned} P(s,t) &= \overline{\alpha}(s) + (w(s,t)\sin\theta - v(s,t)\cos\theta)\,T(s) \\ &\quad + u(s,t)\,N(s) + (v(s,t)\sin\theta + w(s,t)\cos\theta)\,B(s), \end{aligned} \tag{5}$$

where $L_1 \leq s \leq L_2,\ T_1 \leq t \leq T_2$.

Remark 1 Observe that choosing different marching-scale functions yields different surfaces possessing $\bar{\alpha}(s)$ as a common curve.

Our goal is to find the necessary and sufficient conditions for which the curve $\overline{\alpha}(s)$ is isoparametric and geodesic on the surface $P(s,t)$. Firstly, as $\bar{\alpha}(s)$ is an isoparametric curve on the surface $P(s,t)$, there exists a parameter $t_0 \in [T_1, T_2]$ such that

$$u(s,t_0) = v(s,t_0) = w(s,t_0) \equiv 0,\ L_1 \leq s \leq L_2,\ T_1 \leq t_0 \leq T_2. \tag{6}$$

Secondly the curve $\bar{\alpha}$ is geodesic on the surface $P(s,t)$ if and only if along the curve the surface normal vector field $n(s,t_0)$ is parallel to the principal normal vector field $\bar{N}$ of the curve $\bar{\alpha}$. The normal vector of $P(s,t)$ can be written as

$$n(s,t) = \frac{\partial P(s,t)}{\partial s} \times \frac{\partial P(s,t)}{\partial t}.$$

By Eqns. (3) and (5), the normal vector along the curve $\overline{\alpha}$ can be expressed as

$$n(s,t_0)=\kappa\left[-\frac{\partial w}{\partial t}(s,t_0)\bar{N}(s)+\frac{\partial v}{\partial t}(s,t_0)\bar{B}(s)\right], \tag{7}$$

where κ is the curvature of the curve α. Since $\kappa(s)\neq 0,\ L_1\leq s\leq L_2$, the curve $\bar{\alpha}$ is a geodesic on the surface $P(s,t)$ if and only if

$$\frac{\partial w}{\partial t}(s,t_0)\neq 0,\ \frac{\partial v}{\partial t}(s,t_0)\equiv 0.$$

So, we can present:

Theorem 2 *Let $\alpha(s),\ L_1\leq s\leq L_2$, be a unit speed curve with nonvanishing curvature and $\bar{\alpha}(s),\ L_1\leq s\leq L_2$, be its natural lift. $\bar{\alpha}(s)$ is a geodesic on the surface* (4) *if and only if*

$$\begin{cases} u(s,t_0)=v(s,t_0)=w(s,t_0)\equiv 0, \\ \frac{\partial w}{\partial t}(s,t_0)\neq 0,\ \frac{\partial v}{\partial t}(s,t_0)\equiv 0, \end{cases} \tag{8}$$

where $L_1\leq s\leq L_2,\ T_1\leq t,\ t_0\leq T_2$ (t_0 fixed).

Corollary 3 *Let $\alpha(s),\ L_1\leq s\leq L_2$, be a unit speed curve with nonvanishing curvature and $\bar{\alpha}(s),\ L_1\leq s\leq L_2$, be its natural lift. If*

$$u(s,t)=w(s,t)=(t-t_0),\ v(s,t)\equiv 0, \tag{9}$$

where $L_1\leq s\leq L_2,\ T_1\leq t,t_0\leq T_2$ (t_0 fixed) then (4) *is a ruled surface and $\bar{\alpha}$ is a geodesic on it.*

§3. **Examples**

Example 1 Let $\alpha(s)=\left(\frac{4}{5}\cos s,1-\sin s,-\frac{3}{5}\cos s\right)$ be a unit speed curve. Then, it is easy to show that

$$\begin{aligned} T(s) &= \left(-\frac{4}{5}\sin s,-\cos s,\frac{3}{5}\sin s\right), \\ N(s) &= \left(-\frac{4}{5}\cos s,\sin s,\frac{3}{5}\cos s\right), \\ B(s) &= \left(-\frac{3}{5},0,-\frac{4}{5}\right), \\ \kappa &= 1,\ \tau=0,\ \theta=0. \end{aligned}$$

We have

$$\bar{\alpha}(s)=\left(-\frac{4}{5}\sin s,-\cos s,\frac{3}{5}\sin s\right)$$

as the natural lift of α with Frenet vectors

$$\begin{aligned}\bar{T}(s) &= \left(-\frac{4}{5}\cos s, \sin s, \frac{3}{5}\cos s\right),\\ \bar{N}(s) &= \left(\frac{4}{5}\sin s, \cos s, -\frac{3}{5}\sin s\right),\\ \bar{B}(s) &= \left(-\frac{3}{5}, 0, -\frac{4}{5}\right).\end{aligned}$$

If we choose $u(s,t) = w(s,t) = t,\ v(s,t) \equiv 0$, then Eqn. (9) is satisfied and we get the ruled surface

$$\begin{aligned}P_1(s,t) &= \bar{\alpha}(s) + t\left[\bar{T}(s) + \bar{B}(s)\right]\\ &= \left(-\frac{4}{5}(\sin s + t\cos s) - \frac{3}{5}t, t\sin s - \cos s,\right.\\ &\quad \left.\frac{3}{5}(\sin s + t\cos s) - \frac{4}{5}t\right),\end{aligned}$$

$-2 < s \leq 2,\ -1 \leq t \leq 1$, possessing $\bar{\alpha}$ as a geodesic such as those shown in Fig.2.

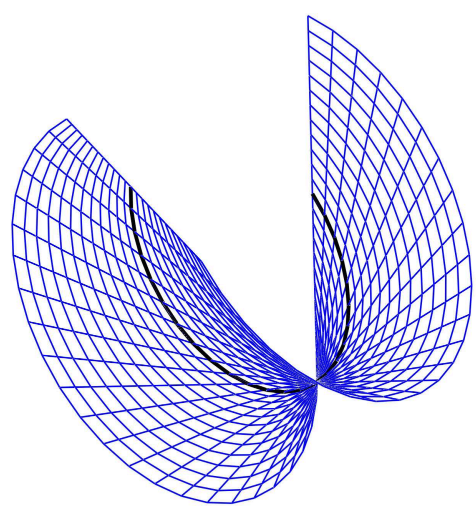

Fig.2 Ruled surface $P_1(s;t)$ as a member of the surface family and its common natural geodesic lift $\overline{\alpha}$

For the same curve, if we choose $u(s,t) = e^{2t} - 1,\ v(s,t) \equiv 0,\ w(s,t) = t$, then Eqn. (8) is satisfied and we obtain the surface

$$\begin{aligned}P_2(s,t) &= \bar{\alpha}(s) + \left(e^{2t} - 1\right)\bar{T}(s) + t\bar{B}(s)\\ &= \left(-\frac{4}{5}\left(\left(e^{2t} - 1\right)\cos s + \sin s\right) - \frac{3}{5}t, \left(e^{2t} - 1\right)\sin s - \cos s,\right.\\ &\quad \left.\frac{3}{5}\left(\left(e^{2t} - 1\right)\cos s + \sin s\right) - \frac{4}{5}t\right),\end{aligned}$$

where $-3 < s \leq 3,\ -1 \leq t \leq 1$ interpolating $\bar{\alpha}$ as the natural geodesic lift $(Fig.\ 3)$.

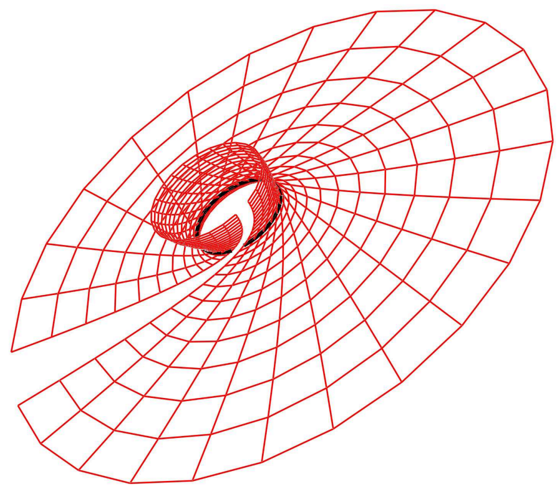

Fig.3 $P_2(s;t)$ as a member of the surface family and its common natural geodesic lift $\overline{\alpha}$

Example 2 Let $\alpha(s) = \left(\frac{\sqrt{3}}{2}\sin s, \frac{s}{2}, \frac{\sqrt{3}}{2}\cos s\right)$ be an arc length helix. One can show that

$$\begin{aligned} T(s) &= \left(\frac{\sqrt{3}}{2}\cos s, \frac{1}{2}, -\frac{\sqrt{3}}{2}\sin s\right), \\ N(s) &= (-\sin s, 0, -\cos s), \\ B(s) &= \left(-\frac{1}{2}\cos s, \frac{\sqrt{3}}{2}, \frac{1}{2}\sin s\right), \\ \kappa &= \frac{\sqrt{3}}{2},\ \tau = \frac{1}{2},\ \theta = \frac{\pi}{6}. \end{aligned}$$

We obtain

$$\bar{\alpha}(s) = \left(\frac{\sqrt{3}}{2}\cos s, \frac{1}{2}, -\frac{\sqrt{3}}{2}\sin s\right)$$

as the natural lift of α with Frenet vectors

$$\begin{aligned} \bar{T}(s) &= (-\sin s, 0, -\cos s), \\ \bar{N}(s) &= (-\cos s, 0, \sin s), \\ \bar{B}(s) &= (0, 1, 0). \end{aligned}$$

Choosing marching scale functions as $u(s,t) = s^2 t,\ v(s,t) \equiv 0,\ w(s,t) = \sin t$ we get the surface

$$\begin{aligned} P_3(s,t) &= \bar{\alpha}(s) + s^2 t\bar{T}(s) + \sin t\bar{B}(s) \\ &= \left(\frac{\sqrt{3}}{2}\cos s - s^2 t\sin s, \frac{1}{2} + \sin t, -\frac{\sqrt{3}}{2}\sin s - s^2 t\cos s\right) \end{aligned}$$

satisfying Eqn. (8) possessing $\bar{\alpha}$ as a common natural geodesic lift $(Fig.\ 4)$.

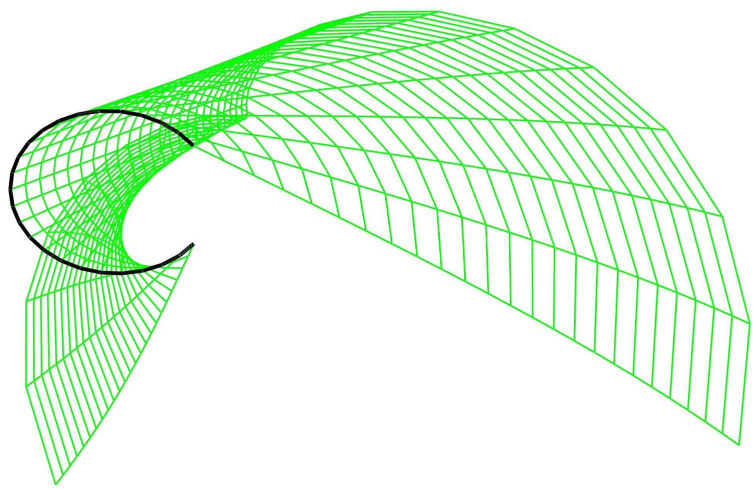

Fig.4 $P_3(s;t)$ as a member of the surface family and its common natural geodesic lift $\overline{\alpha}$

If we let $u(s,t) = s\tan t,\ v(s,t) = (\cos t) - 1,\ w(s,t) = s\sin t$, then Eqn. (8) is satisfied and we have

$$\begin{aligned} P_4(s,t) &= \bar{\alpha}(s) + s\tan t\bar{T}(s) + (\cos t - 1)\bar{N}(s) + s\sin t\bar{B}(s) \\ &= \left(\frac{\sqrt{3}}{2}\cos s - s(\tan t)\sin s + \cos s(1-\cos t), \frac{1}{2} + s\sin t,\right. \\ &\quad \left. -\frac{\sqrt{3}}{2}\sin s - s(\tan t)\cos s + \sin s(\cos t - 1)\right), \end{aligned}$$

$0 < s \leq 3,\ 0 \leq t \leq 1$, as a member of the surface family possessing $\bar{\alpha}$ as a common natural geodesic lift shown in Fig.5.

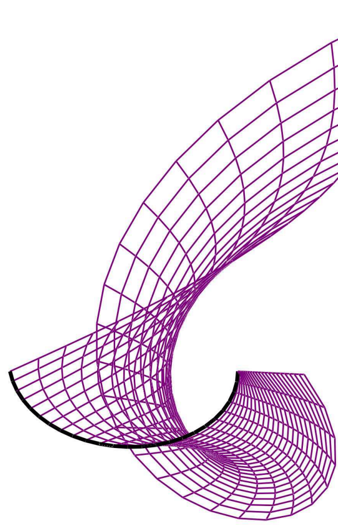

Fig.5 $P_4\,(s,t)$ as a member of the surface family and its common natural geodesic lift $\bar{\alpha}$.

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Some Curvature Properties of LP-Sasakian Manifold with Respect to Quarter-Symmetric Metric Connection

Santu Dey and Arindam Bhattacharyya

(Department of Mathematics, Jadavpur University, Kolkata-700032, India)

E-mail: santu.mathju@gmail.com, bhattachar1968@yahoo.co.in

Abstract: The objective of the present paper is to study the curvature tensor of the quarter-symmetric metric connection with respect to Lorentzian Para-Sasakian manifold (briefly, LP-Sasakian manifold). It is shown that if in the manifold M^n, $\tilde{W}_2 = 0$, then the manifold M^n is locally isomorphic to $S^n(1)$, where $\tilde{W}_2$ is the W_2-curvature tensor of the quarter-symmetric metric connection in a LP-Sasakian manifold. Next we study generalized projective ϕ-Recurrent LP-Sasakian manifold with respect to quarter-symmetric metric connection. After that ϕ-pseudo symmetric LP-Sasakian manifold with respect to quarter-symmetric metric connection is studied and we also discuss LP-Sasakian manifold with respect to quarter-symmetric metric connection when it satisfies the condition $\tilde{P}.\tilde{S} = 0$, where $\tilde{P}$ denotes the projective curvature tensor with respect to quarter-symmetric metric connection. Further, we also study ξ-conharmonically flat LP-Sasakian manifold with respect to quarter-symmetric metric connection. Finally, we give an example of LP-Sasakian manifold with respect to quarter-symmetric metric connection.

Key Words: Quarter-symmetric metric connection, W_2-curvature tensor, generalized projective ϕ-recurrent manifold, ϕ-pseudo symmetric LP-Sasakian manifold, projective curvature tensor, ξ-conharmonically flat LP-Sasakian manifold.

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§1. Introduction

The idea of semi-symmetric linear connection on a differentiable manifold was introduced by Friedmann and Schouten ([1]). Further, Hayden ([3]), introduced the idea of metric connection with torsion on a Riemannian manifold. In ([16]), Yano studied some curvature conditions for semi-symmetric connections in Riemannian manifolds.

The quarter-symmetric connection generalizes the semi-symmetric connection. The semi-symmetric metric connection is important in the geometry of Riemannian manifolds having also physical application; for instance, the displacement on the earth surface following a fixed

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point is metric and semi-symmetric.

In 1975, Golab ([2]) defined and studied quarter-symmetric connection in a differentiable manifold.

A linear connection $\tilde{\nabla}$ on an n-dimensional Riemannian manifold (M^n, g) is said to be a *quarter-symmetric connection* [2] if its torsion tensor $\tilde{T}$ defined by

$$\tilde{T}(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y], \tag{1.1}$$

is of the form

$$\tilde{T}(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y, \tag{1.2}$$

where η is a non-zero 1-form and ϕ is a tensor field of type $(1,1)$. In addition, if a quarter-symmetric linear connection $\tilde{\nabla}$ satisfies the condition

$$(\tilde{\nabla}_X g)(Y,Z) = 0 \tag{1.3}$$

for all $X, Y, Z \in \chi(M)$, where $\chi(M)$ is the set of all differentiable vector fields on M, then $\tilde{\nabla}$ is said to be a quarter-symmetric metric connection. In particular, if $\phi X = X$ and $\phi Y = Y$ for all $X, Y \in \chi(M)$, then the quarter-symmetric connection reduces to a semi-symmetric connection [1].

On the other hand Matsumoto ([5]) introduced the notion of LP-Sasakian manifold. Then Mihai and Rosoca([9]) introduced the same notion independently and obtained several results on this manifold. LP-Sasakian manifolds are also studied by Mihai([9]), Singh([15]) and others.

Definition 1.1 *A LP-Sasakian manifold is said to be generalized projective ϕ-recurrent if its curvature tensor R satisfies the condition*

$$\phi^2((\nabla_W P)(X,Y)Z) = A(W)P(X,Y)Z + B(W)[g(Y,Z)X - g(Y,Z)X], \tag{1.4}$$

where A and B are 1-forms, β is non-zero and these are defined by

$$A(W) = g(W, \rho_1), B(W) = g(W, \rho_2),$$

and where ρ_1 and ρ_2 are vector fields associated with 1-forms A and B respectively and P is the projective curvature tensor for an n-dimensional Riemannian manifold M, given by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y], \tag{1.5}$$

where R and S are the curvature tensor and Ricci tensor of the manifold.

Definition 1.2 *A LP-Sasakian manifold $(M^n, \phi, \xi, \eta, g)(n > 2)$ is said to be ϕ-pseudosymmetric*

([4]) if the curvature tensor R satisfies

$$\begin{aligned}\phi^2((\nabla_W R)(X,Y)Z) &= 2A(W)R(X,Y)Z + A(X)R(W,Y)Z \\ &+ A(Y)R(X,W)Z + A(Z)R(X,Y)W \\ &+ g(R(X,Y)Z,W)\rho \end{aligned} \tag{1.6}$$

for any vector field X, Y, Z and W, where ρ is the vector field associated to the 1-form A such that $A(X) = g(X,\rho)$. In particular, if $A = 0$ then the manifold is said to be ϕ-symmetric.

After Golab([2]), Rastogi ([13], [14]) continued the systematic study of quarter-symmetric metric connection. In 1980, Mishra and Pandey ([8]) studied quarter-symmetric metric connection in a Riemannian, Kaehlerian and Sasakian manifold. In 1982, Yano and Imai([17]) studied quarter-symmetric metric connection in Hermition and Kaehlerian manifolds. In 1991, Mukhopadhyay et al.([10]) studied quarter-symmetric metric connection on a Riemannian manifold with an almost complex structure ϕ. However these manifolds have been studied by many geometers like K. Matsumoto ([6]), K. Matsumoto and I. Mihai ([8]), I. Mihai and R. Rosca([5]) and they obtained many results on this manifold.

In 1970, Pokhariyal and Mishra ([11]) have introduced new tensor fields, called W_2 and E-tensor fields in a Riemannian manifold and studied their properties. Again, Pokhariyal ([12]) have studied some properties of these tensor fields in a Sasakian manifolds. Recently, Matsumoto, Ianus and Mihai ([6]) have studied P-Sasakian manifolds admitting W_2 and E-tensor fields. The W_2-curvature tensor is defined by

$$W_2(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}\{g(X,Z)QY - g(Y,Z)QX\}, \tag{1.7}$$

where R and Q are the curvature tensor and Ricci operator and for all $X, Y, Z \in \chi(M)$.

The conharmonic curvature tensor of LP-Sasakian Manifold M^n is given by

$$\begin{aligned}C(X,Y)Z &= R(X,Y)Z - \frac{1}{n-2}[g(Y,Z)QX - g(X,Z)QY \\ &+ S(Y,Z)X - S(X,Z)Y], \end{aligned} \tag{1.8}$$

where R and S are the curvature tensor and Ricci tensor of the manifold.

Motivated by the above studies, in the present paper, we consider the W_2-curvature tensor of a quarter-symmetric metric connection and study some curvature conditions. Section 2 is devoted to preliminaries. In third section, we find expression for the curvature tensor, Ricci tensor and scalar curvature of LP-Sasakian manifold with respect to quarter-symmetric metric connection and investigate relations between curvature tensor (resp. Ricci tensor) with respect to the semi-symmetric metric connection and curvature tensor (resp. Ricci tensor) with respect to Levi-Civita connection. In section four, W_2 curvature tensor with respect to quarter-symmetric metric connection is studied. In this section, it is seen that if $\tilde{W}_2 = 0$ in M^n, then M^n is locally isomorphic to $S^n(1)$, where $\tilde{W}_2$ is curvature tensor with respect to quarter-symmetric metric connection $\tilde{\nabla}$. Next we have obtained some expression of Ricci tensor when $(\tilde{W}_2(\xi,Z).\tilde{S})(X,Y) = 0$ in LP-Sasakian manifold with respect to quarter-symmetric

metric connection. In section five deals with generalized projective ϕ-Recurrent LP-Sasakian manifold with respect to quarter-symmetric metric connection. In section six, ϕ-pseudo symmetric LP-Sasakian manifold with respect to quarter-symmetric metric connection is studied. In next section, we cultivate LP-Sasakian manifold with respect to quarter-symmetric metric connection satisfying when it satisfies the condition $\tilde{P}.\tilde{S} = 0$, where $\tilde{P}$ denotes the projective curvature tensor with respect to quarter-symmetric metric connection. Finally, We study ξ-conharmonically flat LP-Sasakian manifold with respect to quarter-symmetric metric connection.

§2. Preliminaries

A n-dimensional, $(n = 2m + 1)$, differentiable manifold M^n is called Lorentzian para-Sasakian (briefly, LP-Sasakian) manifold ([5], [7]) if it admits a $(1,1)$-tensor field ϕ, a contravariant vector field ξ, a 1-form η and a Lorentzian metric g which satisfy

$$\eta(\xi) = -1, \tag{2.1}$$

$$\phi^2 X = X + \eta(X)\xi, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.3}$$

$$g(X,\xi) = \eta(X), \tag{2.4}$$

$$\nabla_X \xi = \phi X, \tag{2.5}$$

$$(\nabla_X \phi)(Y) = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.6}$$

where, ∇ denotes the covariant differentiation with respect to Lorentzian metric g. It can be easily seen that in an LP-Sasakian manifold the following relations hold:

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \tag{2.7}$$

$$rank(\phi) = n - 1. \tag{2.8}$$

If we put

$$\Phi(X,Y) = g(X, \phi Y), \tag{2.9}$$

for any vector field X and Y, then the tensor field $\Phi(X,Y)$ is a symmetric $(0,2)$-tensor field ([5]). Also since the 1-form η is closed in an LP-Sasakian manifold, we have ([5])

$$(\nabla_X \eta)(Y) = \Phi(X,Y), \Phi(X,\xi) = 0 \tag{2.10}$$

for all $X, Y \in \chi(M)$.

Also in an LP-Sasakian manifold, the following relations hold ([7]):

$$g(R(X,Y)Z,\xi) = \eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{2.11}$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.12}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.13}$$

$$R(\xi, X)\xi = X + \eta(X)\xi, \tag{2.14}$$

$$S(X, \xi) = (n-1)\eta(X), \tag{2.15}$$

$$QX = (n-1)X, r = n(n-1), \tag{2.16}$$

where Q is the Ricci operator, i.e.

$$g(QX, Y) = S(X, Y) \tag{2.17}$$

and r is the scalar curvature of the connection ∇. Also

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y), \tag{2.18}$$

for any vector field X, Y and Z, where R and S are the Riemannian curvature tensor and Ricci tensor of the manifold respectively.

§3. Curvature tensor of LP-Sasakian Manifold with Respect to Quarter-Symmetric Metric Connection

In this section we express $\tilde{R}(X, Y)Z$ the curvature tensor with respect to quarter-symmetric metric connection in terms of $R(X, Y)Z$ the curvature tensor with respect to Riemannian connection.

Let $\tilde{\nabla}$ be the linear connection and ∇ be Riemannian connection of an almost contact metric manifold such that

$$\tilde{\nabla}_X Y = \nabla_X Y + L(X, Y), \tag{3.1}$$

where L is the tensor field of type $(1, 1)$. For $\tilde{\nabla}$ to be a quarter-symmetric metric connection in M^n, we have ([2])

$$L(X, Y) = \frac{1}{2}[\tilde{T}(X, Y) + \tilde{T}'(X, Y) + \tilde{T}'(Y, X)], \tag{3.2}$$

and

$$g(\tilde{T}'(X, Y), Z) = g(\tilde{T}(X, Y), Z). \tag{3.3}$$

From the equation (1.2) and (3.3), we get

$$\tilde{T}'(X, Y) = \eta(X)\phi Y + g(\phi X, Y)\xi. \tag{3.4}$$

Now putting the equations (1.2) and (3.4) in (3.2), we obtain

$$L(X, Y) = \eta(Y)\phi X + g(\phi X, Y)\xi. \tag{3.5}$$

So, a quarter-symmetric metric connection $\tilde{\nabla}$ in an LP-Sasakian manifold is given by

$$\tilde{\nabla}_X Y = \nabla_Y X + \eta(Y)\phi X + g(\phi X, Y)\xi. \tag{3.6}$$

Thus the above equation gives us the relation between quarter-symmetric metric connection and the Levi-Civita connection.

The curvature tensor $\tilde{R}$ of M^n with respect to quarter-symmetric metric connection $\tilde{\nabla}$ is defined by

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z. \tag{3.7}$$

A relation between the curvature tensor of M with respect to the quarter-symmetric metric connection $\tilde{\nabla}$ and the Riemannian connection ∇ is given by

$$\begin{aligned}\tilde{R}(X,Y)Z &= R(X,Y)Z + g(\phi X, Z)\phi Y - g(\phi Y, Z)\phi X \\ &+ \eta(Z)\{\eta(Y)X - \eta(X)Y\} + \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi,\end{aligned} \tag{3.8}$$

where $\tilde{R}$ and R are the Riemannian curvature tensor with respect to $\tilde{\nabla}$ and ∇ respectively.

From the equation (3.8), we get

$$\tilde{S}(Y,Z) = S(Y,Z) + (n-1)\eta(Y)\eta(Z), \tag{3.9}$$

where $\tilde{S}$ and S are the Ricci tensor with respect to $\tilde{\nabla}$ and ∇ respectively. This gives

$$\tilde{Q}Y = QY + (n-1)\eta(Y)\xi. \tag{3.10}$$

Contracting (3.9), we obtain,

$$\tilde{r} = r - (n-1), \tag{3.11}$$

where $\tilde{r}$ and r are the scalar curvature tensor with respect to $\tilde{\nabla}$ and ∇ respectively. Also we have

$$\tilde{R}(X,Y)\xi = 0, \tag{3.12}$$

which gives

$$\eta(\tilde{R}(X,Y)\xi) = 0, \tag{3.13}$$

and

$$\tilde{R}(\xi,Y)Z = 0, \tag{3.14}$$

which gives

$$\eta(\tilde{R}(\xi,Y)Z) = 0. \tag{3.15}$$

§4. W_2-Curvature Tensor of LP-Sasakian Manifold with Respect to Quarter-Symmetric Metric Connection

The W_2-curvature tensor of LP-Sasakian manifold M^n with respect to quarter-symmetric metric connection $\tilde{\nabla}$ is given by

$$\tilde{W}_2(X,Y)Z = \tilde{R}(X,Y)Z + \frac{1}{n-1}\{g(X,Z)\tilde{Q}Y - g(Y,Z)\tilde{Q}X\}. \tag{4.1}$$

Using the equations (3.8) and (3.10) in (4.1), we get

$$\begin{aligned}\tilde{W}_2(X,Y)Z = &= R(X,Y)Z + g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X \\ &+ \eta(Z)\{\eta(Y)X - \eta(X)Y\} \\ &+ \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi \\ &+ \frac{1}{n-1}[g(X,Z)\{QY + (n-1)\eta(Y)\xi\} \\ &- g(Y,Z)\{QX + (n-1)\eta(X)\xi\}]. \end{aligned}\tag{4.2}$$

Now using the equation (1.7) in (4.2), we obtain

$$\begin{aligned}\tilde{W}_2(X,Y)Z = &= W_2(X,Y)Z + g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X \\ &+ \eta(Z)\{\eta(Y)X - \eta(X)Y\} \\ &+ \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi \\ &+ \frac{1}{n-1}[g(X,Z)(n-1)\eta(Y)\xi \\ &- g(Y,Z)(n-1)\eta(X)\xi]. \end{aligned}\tag{4.3}$$

Putting $Z = \xi$ in (4.3) and using the equations (2.1), (2.4), (2.7) and (1.7), we get

$$\tilde{W}_2(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{4.4}$$

which gives

$$\eta(\tilde{W}_2(X,Y)\xi) = 0. \tag{4.5}$$

Again putting $X = \xi$ in (4.3) and using the equations (2.1), (2.4), (2.7), (2.12) and (1.7), we get

$$\tilde{W}_2(\xi,Y)Z = \eta(Z)Y + \eta(Y)\eta(Z)\xi. \tag{4.6}$$

This gives

$$\eta(\tilde{W}_2(\xi,Y)Z) = 0. \tag{4.7}$$

Theorem 4.1 *In LP-Sasakian Manifold M^n, if the W_2-Curvature tensor of with respect to quarter-symmetric metric connection vanishes, then it is locally isomorphic to $S^n(1)$.*

Proof Let $\tilde{W}_2 = 0$. From the equation (4.2), we have

$$\begin{aligned}R(X,Y)Z &= g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y + \eta(Z)\{\eta(X)Y - \eta(Y)X\} \\ &+ \{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}\xi - \frac{1}{n-1}[g(X,Z)\{QY + (n-1)\eta(Y)\xi\} \\ &- g(Y,Z)\{QX + (n-1)\eta(X)\xi\}]. \end{aligned}\tag{4.8}$$

Taking the inner product of the above equation and using (2.1), (2.4), (2.7), we get

$$\eta(R(X,Y)Z) = \{g(Y,Z)X - g(X,Z)Y\}, \tag{4.9}$$

which gives

$$R(X,Y,Z,U) = \{g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\}. \tag{4.10}$$

This shows that M^n is a space of constant curvature is 1, that is, it is locally isomorphic to $S^n(1)$. □

Suppose let $(\tilde{W}_2(\xi,Z).\tilde{S})(X,Y) = 0$. This gives

$$\tilde{S}(\tilde{W}_2(\xi,Z)X,Y) + \tilde{S}(X,\tilde{W}_2(\xi,Z)Y) = 0. \tag{4.11}$$

Now using the equation (3.9) in (4.11), we get

$$\begin{aligned} &S(\tilde{W}_2(\xi,Z)X,Y) + (n-1)\eta(\tilde{W}_2(\xi,Z)X)\eta(Y) \\ &\qquad S(X,\tilde{W}_2(\xi,Z)Y) + (n-1)\eta(\tilde{W}_2(\xi,Z)Y)\eta(X) = 0. \end{aligned} \tag{4.12}$$

Using the equation (2.15), (4.6) and (4.7) in (4.12), we obtain

$$\begin{aligned} \eta(X)S(Y,Z) &+ (n-1)\eta(X)\eta(Y)\eta(Z) + \eta(Y)S(X,Z) \\ &+ (n-1)\eta(X)\eta(Y)\eta(Z) = 0. \end{aligned} \tag{4.13}$$

Putting $X = \xi$ and using the equation (2.1) and (2.4) in (4.13), we get

$$S(Y,Z) = (1-n)\eta(Y)\eta(Z). \tag{4.14}$$

So, we have the following theorem.

Theorem 4.2 *A LP-Sasakian manifold M^n with respect to quarter-symmetric metric connection $\tilde{\nabla}$ satisfying $(\tilde{W}_2(\xi,Z).\tilde{S})(X,Y) = 0$ is the product of two 1-forms.*

§5. Generalized Projective ϕ-Recurrent LP-Sasakian Manifold with Respect to Quarter-Symmetric Metric Connection

The projective curvature tensor for an n-dimensional Riemannian manifold M with respect to quarter-symmetric metric connection is given by

$$\tilde{P}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{1}{n-1}[\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y], \tag{5.1}$$

where R and S are the curvature tensor and Ricci tensor of the manifold.

Let us consider generalized projective ϕ-recurrent LP-Sasakian manifold with respect to quarter-symmetric metric connection. By virtue of (1.4) and (2.2), we get

$$\begin{aligned}(\tilde{\nabla}_W\tilde{P})(X,Y)Z &+ \eta((\tilde{\nabla}_W\tilde{P})(X,Y)Z)\xi = A(W)\tilde{P}(X,Y)Z \\ &+ B(W)[g(Y,Z)X-g(X,Z)Y], \end{aligned} \tag{5.2}$$

from which it follows that

$$\begin{aligned}g((\tilde{\nabla}_W\tilde{P})(X,Y)Z,U) &+ \eta((\tilde{\nabla}_W\tilde{P})(X,Y)Z)\eta(U) = A(W)g(\tilde{P}(X,Y)Z,U) \\ &+ B(W)[g(Y,Z)g(X,U)-g(X,Z)g(Y,U)]. \end{aligned} \tag{5.3}$$

Let $\{e_i\}$, $i=1,2,\cdots,n$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X=U=e_i$ in (5.3) and taking summation over i, $1\leq i\leq n$, we get

$$\begin{aligned}(\tilde{\nabla}_W\tilde{S})(X,U) &- \frac{\tilde{\nabla}_W\tilde{r}}{n-1}g(X,U)+\frac{(\tilde{\nabla}_W\tilde{S})(X,U)}{n-1}-(\tilde{\nabla}_W\tilde{S})(X,\xi)\eta(U) \\ &+ \frac{\tilde{\nabla}_W\tilde{r}}{n-1}\eta(X)\eta(U)-\frac{(\tilde{\nabla}_W\tilde{S})(X,U)}{n-1}\eta(U) \\ &= A(W)[\frac{n}{n-1}\tilde{S}(X,U)-\frac{\tilde{r}}{n-1}g(X,U)] \\ &+ 2nB(W)g(X,U). \end{aligned} \tag{5.4}$$

Putting $U=\xi$ in (5.4) and using the equation (3.6), (3.9) an (3.11), we obtain

$$A(W)[1-\frac{r}{n-1}]\eta(X)+(n-1)B(W)\eta(X)=0. \tag{5.5}$$

Putting $X=\xi$ in (5.5), we get

$$B(W)=[\frac{r-n+1}{(n-1)^2}]A(W). \tag{5.6}$$

Thus we can state the following theorem.

Theorem 5.1 *In a generalized projective ϕ-ecurrent LP-Sasakian manifold M^n $(n>2)$, the 1-forms A and B are related as* (5.6).

§6. ϕ-Pseudo Symmetric LP-Sasakian Manifold with Respect to Quarter-Symmetric Metric Connection

Definition 6.1 A LP-Sasakian manifold $(M^n,\phi,\xi,\eta,g)(n>2)$ is said to be ϕ-pseudosymmetric with respect to quarter symmetric metric connection if the curvature tensor $\tilde{R}$ satisfies

$$\begin{aligned}\phi^2((\tilde{\nabla}_W\tilde{R})(X,Y)Z) &= 2A(W)\tilde{R}(X,Y)Z+A(X)\tilde{R}(W,Y)Z \\ &+ A(Y)\tilde{R}(X,W)Z+A(Z)\tilde{R}(X,Y)W+g(\tilde{R}(X,Y)Z,W)\rho \end{aligned} \tag{6.1}$$

for any vector field X, Y, Z and W, where ρ is the vector field associated to the 1-form A such that $A(X) = g(X, \rho)$. Now using (2.2) in (6.1), we have

$$\begin{aligned}(\tilde{\nabla}_W \tilde{R})(X,Y)Z &+ \eta((\tilde{\nabla}_W \tilde{R})(X,Y)Z)\xi = 2A(W)\tilde{R}(X,Y)Z \\ &+ A(X)\tilde{R}(W,Y)Z + A(Y)\tilde{R}(X,W)Z \\ &+ A(Z)\tilde{R}(X,Y)W + g(\tilde{R}(X,Y)Z,W)\rho. \end{aligned} \tag{6.2}$$

From which it follows that

$$\begin{aligned}g((\tilde{\nabla}_W \tilde{R})(X,Y)Z,U) &+ \eta((\tilde{\nabla}_W \tilde{R})(X,Y)Z)\eta(U) = 2A(W)g(\tilde{R}(X,Y)Z,U) \\ &+ A(X)g(\tilde{R}(W,Y)Z,U) + A(Y)g(\tilde{R}(X,W)Z,U) \\ &+ A(Z)g(\tilde{R}(X,Y)W,U) + g(\tilde{R}(X,Y)Z,W)A(U). \end{aligned} \tag{6.3}$$

Let $\{e_i : i = 1, 2, \cdots, n\}$ be an orthonormal basis of the tangent space at any point of the manifold. Setting $X = U = e_i$ in (6.3) and taking summation over i, $1 \leq i \leq n$, and then using (2.1), (2.4) and (2.7) in (6.3), we obtain

$$\begin{aligned}(\tilde{\nabla}_W \tilde{S})(Y,Z) &+ g((\tilde{\nabla}_W \tilde{R})(\xi,Y)Z,\xi) = 2A(W)\tilde{S}(Y,Z) \\ &+ A(Y)\tilde{S}(W,Z) + A(Z)\tilde{S}(Y,W) \\ &+ A(\tilde{R}(W,Y)Z) + A(\tilde{R}(W,Z)Y). \end{aligned} \tag{6.4}$$

By virtue of (3.14) it follows from (6.4) that

$$\begin{aligned}(\tilde{\nabla}_W \tilde{S})(Y,Z) &= 2A(W)\tilde{S}(Y,Z) + A(Y)\tilde{S}(W,Z) + A(Z)\tilde{S}(Y,W) \\ &+ A(\tilde{R}(W,Y)Z) + A(\tilde{R}(W,Z)Y). \end{aligned} \tag{6.5}$$

So, we have the following theorem:

Theorem 6.1 *A ϕ-pseudo symmetric LP-Sasakian manifold with respect to quarter-symmetric metric connection is pseudo Ricci symmetric with respect to quarter sym- metric non-metric connection if and only if*

$$A(\tilde{R}(W,Y)Z) + A(\tilde{R}(W,Z)Y) = 0.$$

§7. *LP*-Sasakian Manifold with Respect to Quarter-Symmetric Metric Connection Satisfying $\tilde{P}.\tilde{S} = 0$.

A *LP*-Sasakian manifold with respect to the quarter-symmetric metric connection satisfying

$$(\tilde{P}(X,Y).\tilde{S})(Z,U) = 0, \tag{7.1}$$

where $\tilde{S}$ is the Ricci tensor with respect to a quarter-symmetric metric connection. Then, we have

$$\tilde{S}(\tilde{P}(X,Y)Z,U)+\tilde{S}(Z,\tilde{P}(X,Y)U)=0. \tag{7.2}$$

Putting $X=\xi$ in the equation (7.2), we have

$$\tilde{S}(\tilde{P}(\xi,Y)Z,U)+\tilde{S}(Z,\tilde{P}(\xi,Y)U)=0. \tag{7.3}$$

In view of the equation (5.1), we have

$$\tilde{P}(\xi,Y)Z=\tilde{R}(\xi,Y)Z-\frac{1}{n-1}[\tilde{S}(Y,Z)\xi-\tilde{S}(\xi,Z)Y] \tag{7.4}$$

for $X,Y,Z\in\chi(M)$.

Using equations (3.9) and (3.14) in the equation (7.4), we get

$$\tilde{P}(\xi,Y)Z=-\frac{1}{n-1}[S(Y,Z)\xi+(n-1)\eta(Y)\eta(Z)\xi]. \tag{7.5}$$

Now using the equation (7.5) and putting $U=\xi$ in the equation (7.3) and using the equations (2.2), (2.15) and (3.9) we get

$$S(Y,Z)+(n-1)\eta(Y)\eta(Z)=0. \tag{7.6}$$

i.e.,

$$S(Y,Z)=-(n-1)\eta(Y)\eta(Z). \tag{7.7}$$

In view of above discussions we can state the following theorem:

Theorem 7.1 *A n-dimensional LP-Sasakian manifold with a quarter-symmetric metric connection satisfying $\tilde{P}.\tilde{S}=0$ is the product of two 1-forms.*

§8. ξ-Conharmonically Flat LP-Sasakian Manifold with Respect to Quarter-Symmetric Metric Connection

The conharmonic curvature tensor of LP-Sasakian manifold M^n with respect to quarter-symmetric metric connection $\tilde{\nabla}$ is given by

$$\begin{aligned}\tilde{C}(X,Y)Z &= \tilde{R}(X,Y)Z-\frac{1}{n-2}[g(Y,Z)\tilde{Q}X-g(X,Z)\tilde{Q}Y\\ &+ \tilde{S}(Y,Z)X-\tilde{S}(X,Z)Y],\end{aligned} \tag{8.1}$$

where $\tilde{R}$ and $\tilde{S}$ are the curvature tensor and Ricci tensor with respect to quarter-symmetric metric connection.

Using (3.8), (3.9) and (3.10) in (8.1), we get

$$\begin{aligned}\tilde{C}(X,Y)Z &= R(X,Y)Z + g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X\\ &+ \eta(Z)\{\eta(Y)X - \eta(X)Y\}\\ &+ \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi\\ &- \frac{1}{n-1}[g(X,Z)\{QY + (n-1)\eta(Y)\xi\}\\ &- g(Y,Z)\{QX + (n-1)\eta(X)\xi\}]\\ &- \frac{1}{n-2}[g(Y,Z)\{QX + (n-1)\eta(X)\xi\}\\ &- g(X,Z)\{QY + (n-1)\eta(Y)\xi\} + S(Y,Z)X\\ &+ (n-1)\eta(Y)\eta(Z)X - S(X,Z)Y\\ &- (n-1)\eta(X)\eta(Z)Y]. \end{aligned} \tag{8.2}$$

$$\begin{aligned}\tilde{C}(X,Y)Z &= C(X,Y)Z + g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X\\ &+ \eta(Z)\{\eta(Y)X - \eta(X)Y\} + \{g(Y,Z)\eta(X)\\ &- g(X,Z)\eta(Y)\}\xi - \frac{n-1}{n-2}[g(Y,Z)\eta(X)\xi\\ &- g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X\\ &- \eta(X)\eta(Z)Y], \end{aligned} \tag{8.3}$$

where C is given in (1.8). Putting $Z = \xi$ in (8.3) and using (2.1), (2.4) and (2.7), we obtain

$$\begin{aligned}\tilde{C}(X,Y)\xi &= C(X,Y)\xi - \{\eta(Y)X - \eta(X)Y\}\\ &- \frac{n-1}{n-2}[\eta(X)Y - \eta(Y)X]. \end{aligned} \tag{8.4}$$

Suppose X and Y are orthogonal to ξ, then from (8.4), we obtain

$$\tilde{C}(X,Y)\xi = C(X,Y)\xi. \tag{8.5}$$

So, by the above discussion we can state the following theorem:

Theorem 8.1 *An n-dimensional LP-Sasakian manifold is ξ-conharmonically flat with respect to the quarter-symmetric metric connection if and only if the manifold is also ξ-conharmonically flat with respect to the Levi-Civita connection provided the vector fields X and Y are orthogonal to the associated vector field ξ.*

§9. Example 3-Dimensional LP-Sasakian Manifold with Respect to Quarter-Symmetric Metric Connection

We consider a 3-dimensional manifold $M = \{(x, y, u) \in R^3\}$, where (x, y, u) are the standard coordinates of R^3. Let e_1, e_2, e_3 be the vector fields on M^3 given by

$$e_1 = -e^u \frac{\partial}{\partial x},\ e_2 = -e^{u-x} \frac{\partial}{\partial y},\ e_3 = -\frac{\partial}{\partial u}.$$

Clearly, $\{e_1, e_2, e_3\}$ is a set of linearly independent vectors for each point of M and hence a basis of $\chi(M)$. The Lorentzian metric g is defined by

$$g(e_1, e_2) = g(e_2, e_3) = g(e_1, e_3) = 0,$$
$$g(e_1, e_1) = 1,\ g(e_2, e_2) = 1,\ g(e_3, e_3) = -1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$ and the $(1,1)$ tensor field ϕ is defined by

$$\phi e_1 = -e_1,\ \phi e_2 = -e_2,\ \phi e_3 = 0.$$

From the linearity of ϕ and g, we have

$$\eta(e_3) = -1,$$
$$\phi^2 X = X + \eta(X) e_3$$

and

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y)$$

for any $X \in \chi(M)$. Then for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines a Lorentzian paracontact structure on M. Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g. Then we have

$$[e_1, e_2] = -e^u e_2,\ [e_1, e_3] = -e_1,\ [e_2, e_3] = -e_2.$$

Koszul's formula is defined by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad -g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned}$$

Then from above formula we can calculate followings:

$$\nabla_{e_1} e_1 = e_3,\ \nabla_{e_1} e_2 = 0,\ \nabla_{e_1} e_3 = -e_2,$$
$$\nabla_{e_2} e_1 = -e^u e_2,\ \nabla_{e_2} e_2 = -e_3 - e^u e_1,\ \nabla_{e_2} e_3 = -e_2,$$
$$\nabla_{e_3} e_1 = 0,\ \nabla_{e_3} e_2 = 0,\ \nabla_{e_3} e_3 = 0.$$

From the above calculations, we see that the manifold under consideration satisfies $\eta(\xi) = -1$ and $\nabla_X \xi = \phi X$. Hence the structure (ϕ, ξ, η, g) is a LP-Sasakian manifold.

Using (3.6), we find $\tilde{\nabla}$, the quarter-symmetric metric connection on M following:

$$\tilde{\nabla}_{e_1} e_1 = 0,\ \tilde{\nabla}_{e_1} e_2 = 0,\ \tilde{\nabla}_{e_1} e_3 = 0,$$
$$\tilde{\nabla}_{e_2} e_1 = -e^u e_2,\ \tilde{\nabla}_{e_2} e_2 = -e^u e_1,\ \tilde{\nabla}_{e_2} e_3 = 0$$

and

$$\tilde{\nabla}_{e_3}e_1 = 0,\ \tilde{\nabla}_{e_3}e_2 = 0,\ \tilde{\nabla}_{e_3}e_3 = 0.$$

Using (1.2), the torson tensor T, with respect to quarter-symmetric metric connection $\tilde{\nabla}$ as follows:

$$\tilde{T}(e_i, e_i) = 0,\ \forall i = 1, 2, 3,$$
$$\tilde{T}(e_1, e_2) = 0,\ \tilde{T}(e_1, e_3) = e_3,\ \tilde{T}(e_2, e_3) = e_2.$$

Also,

$$(\tilde{\nabla}_{e_1}g)(e_2, e_3) = 0,\ (\tilde{\nabla}_{e_2}g)(e_3, e_1) = 0,\ (\tilde{\nabla}_{e_3}g)(e_1, e_2) = 0.$$

Thus M is LP-Sasakian manifold with quarter-symmetric metric connection $\tilde{\nabla}$.

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On Net-Regular Signed Graphs

Nutan G.Nayak

Department of Mathematics and Statistics

S. S. Dempo College of Commerce and Economics, Goa, India

E-mail: nayaknutan@yahoo.com

Abstract: A signed graph is an ordered pair $\Sigma = (G, \sigma)$, where $G = (V, E)$ is the underlying graph of Σ and $\sigma : E \to \{+1, -1\}$, called signing function from the edge set $E(G)$ of G into the set $\{+1, -1\}$. It is said to be homogeneous if its edges are all positive or negative otherwise it is heterogeneous. Signed graph is balanced if all of its cycles are balanced otherwise unbalanced. It is said to be net-regular of degree k if all its vertices have same net-degree k i.e. $k = d^{\pm}_{\Sigma}(v) = d^{+}_{\Sigma}(v) - d^{-}_{\Sigma}(v)$, where $d^{+}_{\Sigma}(v)(d^{-}_{\Sigma}(v))$ is the number of positive(negative) edges incident with a vertex v. In this paper, we obtained the characterization of net-regular signed graphs and also established the spectrum for one class of heterogeneous unbalanced net-regular signed complete graphs.

Key Words: Smarandachely k-signed graph, net-regular signed graph,co-regular signed graphs, signed complete graphs.

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§1. Introduction

We consider graph G is a simple undirected graph without loops and multiple edges with n vertices and m edges. A Smarandachely k-signed graph is defined as an ordered pair $\Sigma = (G, \sigma)$, where $G = (V, E)$ is an underlying graph of Σ and $\sigma : E \to \{\overline{e_1}, \overline{e_2}, \overline{e_3}, \cdots, \overline{e_k}\}$ is a function, where $\overline{e_i} \in \{+, -\}$. A Smarandachely 2-signed graph is known as signed graph. It is said to be homogeneous if its edges are all positive or negative otherwise it is heterogeneous. We denote positive and negative homogeneous signed graphs as $+G$ and $-G$ respectively.

The adjacency matrix of a signed graph is the square matrix $A(\Sigma) = (a_{ij})$ where (i, j) entry is $+1$ if $\sigma(v_i v_j) = +1$ and -1 if $\sigma(v_i v_j) = -1$, 0 otherwise. The characteristic polynomial of the signed graph Σ is defined as $\Phi(\Sigma : \lambda) = \det(\lambda I - A(\Sigma))$, where I is an identity matrix of order n. The roots of the characteristic equation $\Phi(\Sigma : \lambda) = 0$, denoted by $\lambda_1, \lambda_2, \cdots, \lambda_n$ are called the eigenvalues of signed graph Σ. If the distinct eigenvalues of $A(\Sigma)$ are $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and their multiplicities are $m_1, m_2, \ldots, m_n$, then the spectrum of Σ is $Sp(\Sigma) = \{\lambda_1^{(m_1)}, \lambda_2^{(m_2)}, \cdots, \lambda_n^{(m_n)}\}$.

Two signed graphs are cospectral if they have the same spectrum. The spectral criterion

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for balance in signed graph is given by B.D.Acharya as follows:

Theorem 1.1([1]) *A signed graph is balanced if and only if it is cospectral with the underlying graph. i.e.* $Sp(\Sigma) = Sp(G)$.

The sign of a cycle in a signed graph is the product of the signs of its edges. Thus a cycle is positive if and only if it contains an even number of negative edges. A signed graph is said to be balanced (or cycle balanced) if all of its cycles are positive otherwise unbalanced. The negation of a signed graph $\Sigma = (G, \sigma)$, denoted by $\eta(\Sigma) = (G, \sigma)$ is the same graph with all signs reversed. The adjacency matrices are related by $A(-\Sigma) = -A(\Sigma)$.

Theorem 1.2([12]) *Two signed graphs* $\Sigma_1 = (G, \sigma_1)$ *and* $\Sigma_2 = (G, \sigma_2)$ *on the same underlying graph are switching equivalent if and only if they are cycle isomorphic.*

In signed graph Σ, the degree of a vertex v is defined as $sdeg(v) = d(v) = d^+_\Sigma(v) + d^-_\Sigma(v)$, where $d^+_\Sigma(v)(d^-_\Sigma(v))$ is the number of positive(negative) edges incident with v. The net degree of a vertex v of a signed graph Σ is $d^\pm_\Sigma(v) = d^+_\Sigma(v) - d^-_\Sigma(v)$. It is said to be net-regular of degree k if all its vertices have same net-degree equal to k. Hence net-regularity of a signed graph can be either positive, negative or zero. We denote net-regular signed graphs as Σ^k_n. We know [13] that if Σ is a k net-regular signed graph, then k is an eigenvalue of Σ with j as an eigenvector with all 1's.

K.S.Hameed and K.A.Germina [6] defined co-regularity pair of signed graphs as follows:

Definition 1.3([6]) *A signed graph* $\Sigma = (G, \sigma)$ *is said to be co-regular, if the underlying graph* G *is regular for some positive integer* r *and* Σ *is net-regular with net-degree* k *for some integer* k *and the co-regularity pair is an ordered pair of* (r, k).

The following results give the spectra of signed paths and signed cycles respectively.

Lemma 1.4([3]) *The signed paths* $P^{(r)}_n$, *where* r *is the number of negative edges and* $0 \le r \le n-1$, *have the eigenvalues(independent of* r*) given by*

$$\lambda_j = 2\cos\frac{\pi j}{n+1}, j = 1, 2, \cdots, n.$$

Lemma 1.5([9]) *The eigenvalues* λ_j *of signed cycles* $C^{(r)}_n$ *and* $0 \le r \le n$ *are given by*

$$\lambda_j = 2\cos\frac{(2j-[r])\pi}{n}, j = 1, 2, \cdots, n$$

where r *is the number of negative edges and* $[r] = 0$ *if* r *is even,* $[r] = 1$ *if* r *is odd.*

Spectra of graphs is well documented in [2] and signed graphs is discussed in [3, 4, 5, 9]. For standard terminology and notations in graph theory we follow D.B.West [10] and for signed graphs T. Zaslavsky [14].

The main aim of this paper is to characterize net-regular signed graphs and also to prove that there exists a net-regular signed graph on every regular graph but the converse does not hold good. Further, we construct a family of connected net-regular signed graphs whose underlying graphs are not regular. We established the spectrum for one class of heterogeneous unbalanced net-regular signed complete graphs.

§2. Main Results

Spectral properties of regular graphs are well known in graph theory.

Theorem 2.1([2]) *If G is an r regular graph, then its maximum adjacency eigenvalue is equal to r and $r = \frac{2m}{n}$.*

Here we generalize Theorem 2.1 to signed graphs as graph is considered as one case in signed graph theory. We denote total number of positive and negative edges of Σ as m^+ and m^- respectively. The following lemma gives the structural characterization of signed graph Σ so that Σ is net-regular.

Lemma 2.2 *If $\Sigma = (G, \sigma)$ is a connected net-regular signed graph with net degree k then $k = \frac{2M}{n}$, where $M = (m^+ - m^-)$, m^+ is the total number of positive edges and m^- is the total number of negative edges in Σ.*

Proof Let $\Sigma = (G, \sigma)$ be a net-regular signed graph with net degree k. Then by definition, $d^{\pm}_{\Sigma}(v) = d^+_{\Sigma}(v) - d^-_{\Sigma}(v)$. Hence,

$$\sum_{i=1}^{n} d^{\pm}_{\Sigma}(v) = \sum_{i=1}^{n} d^+_{\Sigma}(v) - \sum_{i=1}^{n} d^-_{\Sigma}(v).$$

Thus,

$$nk = \sum_{i=1}^{n} d^+_{\Sigma}(v) - \sum_{i=1}^{n} d^-_{\Sigma}(v).$$

Whence,

$$\begin{aligned} k &= \frac{1}{n}\left[\sum_{i=1}^{n} d^+_{\Sigma}(v) - \sum_{i=1}^{n} d^-_{\Sigma}(v)\right] = \frac{1}{n}\left[2m^+ - 2m^-\right] \\ &= \left[\frac{2(m^+ - m^-)}{n}\right] = \frac{2M}{n}. \end{aligned}$$

□

Corollary 2.3 *If $\Sigma = (G, \sigma)$ is a signed graph with co-regularity pair (r, k) then $r \geq k$.*

Proof Let Σ be a k net-regular signed graph then by Lemma 2.2, $k = \frac{2M}{n}$, where $M = (m^+ - m^-)$. Since G is its underlying graph with regularity r on n vertices then $r = \frac{2m}{n}$,where $m = m^+ + m^-$. It is clear that $\frac{2m}{n} \geq \frac{2(m^+ - m^-)}{n}$. Hence $r \geq k$. □

Remark 2.4 By Corollary 2.3, if $\Sigma = (G, \sigma)$ is a signed graph with co-regularity pair (r, k) on

n vertices then $-r \leq k \leq r$.

Now the question arises whether all regular graphs can be net-regular and vice-versa. From Lemma 2.2, it is evident that at least two net-regular signed graphs exist on every regular graph when $m^+ = 0$ or $m^- = 0$. We feel the converse also holds good. But contrary to the intuition, the answer is negative. Next result proves that underlying graph of all net-regular signed graphs need not be regular.

Theorem 2.5 *Let Σ be a net-regular signed graph then its underlying graph is not necessarily a regular graph.*

Proof Let Σ be a net-regular signed graph with net degree k. Then by Lemma 2.2, $k = \left[\frac{2(m^+ - m^-)}{n}\right]$. By changing negative edges into positive edges we get $k = \frac{2m}{n}$ where $m = m^+ + m^-$. If $k = \frac{2m}{n}$ is a positive integer then underlying graph is of order $k = r$. If $k = \frac{2m}{n}$ is not a positive integer then $k \neq r$. Hence the underlying graph of a net-regular signed graph need not be a regular graph. □

Shahul Hameed et.al. [7] gave an example of a connected signed graph on $n = 5$ whose underlying is not a regular graph. Here we construct an infinite family of net-regular signed graphs whose underlying graphs are not regular.

Example 2.6 Here is an infinite family of net-regular signed graphs with the property that whose underlying graphs are not regular. Take two copies of C_n, join at one vertex and assign positive and negative signs so that degree of the vertex common to both cycles will have net degree 0 and also assign positive and negative signs to other edges in order to get net-degree 0. The resultant signed graph is a net-regular signed graph with net-degree 0 whose underlying graph is not regular. We denote it as $\Sigma^{(0)}_{(2n-1)}$ for each C_n and illustration is shown in Fig.1, 2 and 3. In chemistry, underlying graphs of these signed graphs are known as spiro compounds.

In the following figures, solid lines represent positive edges and dotted lines represent negative edges respectively.

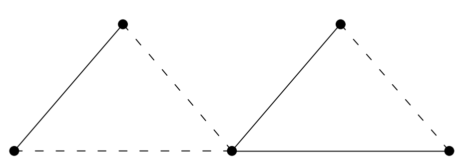

Fig.1 Net-regular signed graph Σ_5^0 for C_3

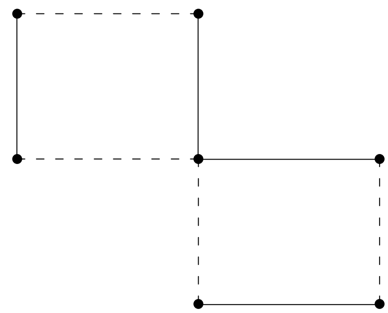

Fig.2 Net-regular signed graph Σ_7^0 for C_4

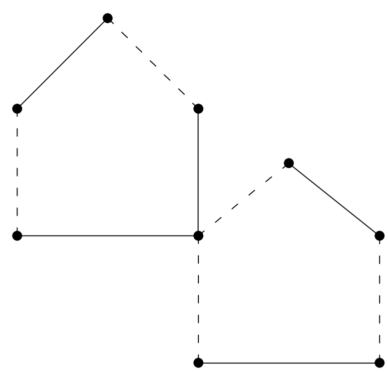

Fig.3 Net-regular signed graph Σ_9^0 for C_5

From Figures 1, 2 and 3, we can see that Σ_7^0 is a bipartite signed graph, but Σ_5^0 and Σ_9^0 are non-bipartite signed graphs. The spectrum of these net -regular signed graphs are

Sp(Σ_5^0)= $\{\pm 2.2361, \pm 1, 0\}$,

Sp(Σ_7^0)= $\{\pm 2.4495, \pm 1.4142, (0)^3\}$,

Sp(Σ_9^0)=$\{\pm 2.3028, \pm 1.6180, \pm 1.3028, \pm 0.6180, 0\}$.

Remark 2.7 Spectrum of this family of connected signed graphs $\Sigma_{(2n-1)}^{(0)}$ satisfy the pairing property i.e. spectrum is symmetric about the origin and also these are non-bipartite when cycle C_n is odd.

Heterogeneous signed complete graphs which are cycle isomorphic to the underlying graph $+K_n$ will have the spectrum $\{(n-1), (-1)^{(n-1)}\}$ and which are cycle isomorphic to $-K_n$ will have the spectrum $\{(1-n), (1)^{(n-1)}\}$. Here we established the spectrum for one class of heterogeneous unbalanced net-regular signed complete graphs.

Let C_n be a cycle on n vertices and $\overline{C}_n$ be its complement where $n \geq 4$. Define σ :

$E(K_n) \to \{1, -1\}$ by

$$\sigma(e) = \begin{cases} 1, & \text{if e} \in \mathrm{C_n} \\ -1, & \text{if e} \in \overline{\mathrm{C_n}} \end{cases}$$

Then $\Sigma = (K_n, \sigma)$ is an unbalanced net-regular signed complete graph and we denote it as K_n^{net} where $n \geq 4$.

The following spectrum for K_n^{net} is given by the author in [10].

Lemma 2.8([10]) *Let K_n^{net} be a heterogeneous unbalanced net-regular signed complete graph then*

$$Sp(K_n^{net}) = \begin{pmatrix} 5-n & 1+4\cos(\frac{2\pi j}{n}) \\ 1 & 1 \end{pmatrix} : j = 1, ..., n-1,$$

where $(5-n)$ gives the net-regularity of K_n^{net}.

Lemma 2.9 $\omega^r + \omega^{n-r} = 2\cos\frac{2r\pi j}{n}$ *for $1 \leq j \leq n$ and $1 \leq r \leq n$, where ω is the n^{th} root of unity.*

Proof Let $1 \leq j \leq n$ and $1 \leq r \leq n$.

$$\begin{aligned} \omega^r + \omega^{n-r} &= e^{\frac{2r\pi ij}{n}} + e^{2r\pi ij} e^{\frac{-2r\pi ij}{n}} = e^{\frac{2r\pi ij}{n}} + e^{\frac{-2r\pi ij}{n}} \\ &= \cos\frac{2r\pi j}{n} + i\sin\frac{2r\pi j}{n} + \cos\frac{2r\pi j}{n} - i\sin\frac{2r\pi j}{n} \\ &= 2\cos\frac{2r\pi j}{n}. \end{aligned}$$

□

By using the properties of the permutation matrix [8] and from Lemma 2.9, we give a new spectrum for K_n^{net}.

Theorem 2.10 *Let K_n^{net} be a heterogeneous signed complete graph as defined above. If n is odd then*

$$Sp(K_n^{net}) = \{2\cos\frac{2\pi j}{n} - \sum_{r=2}^{\frac{n-1}{2}} 2\cos\frac{2r\pi j}{n} : 1 \leq j \leq n\}$$

and if n is even then

$$Sp(K_n^{net}) = \{2\cos\frac{2\pi j}{n} - \cos\pi j - \sum_{r=2}^{\frac{n-2}{2}} 2\cos\frac{2r\pi j}{n} : 1 \leq j \leq n\}.$$

Proof Label the vertices of a circulant graph as $0, 1, \cdots, (n-1)$. Then the adjacency matrix A is

$$A = \begin{pmatrix} 0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & 0 & c_1 & \cdots & c_{n-2} \\ c_{n-2} & c_{n-1} & 0 & \cdots & c_{n-3} \\ \vdots & & & \vdots & \\ c_1 & c_2 & c_3 & \cdots & 0 \end{pmatrix},$$

where $c_i = c_{n-i} = 0$ if vertices i and $n-i$ are not adjacent and $c_i = c_{n-i} = 1$ if vertices i and $n-i$ are adjacent.

Hence

$$\begin{aligned} A &= c_1 P^1 + c_2 P^2 + \cdots + c_{n-1} P^{n-1} \\ &= \sum_{r=1}^{n-1} c_r P^r, \end{aligned}$$

where P is a permutation matrix.

Let K_n^{net} be the heterogeneous signed complete graph and $A(K_n^{net})$ be its adjacency matrix. $A(K_n^{net})$ is a circulant matrix with first row $[0, 1, -1, -1, \cdots, -1, 1]$. Here $c_1 = 1, c_2 = -1, c_3 = -1, \cdots, c_{n-1} = 1$. Hence $A(K_n^{net})$ can be written as a linear combination of permutation matrix P. $A(K_n^{net}) = P^1 - P^2 - P^3 \cdots - P^{n-2} + P^{n-1}$.

Case 1. If n is odd then

$$A(K_n^{net}) = \Big\{(P^1 + P^{n-1}) - (P^2 + P^{n-2}) - \cdots - (P^{\frac{n-1}{2}} + P^{\frac{n+1}{2}})\Big\}$$

and $\omega \in Sp(P)$. Hence

$$\begin{aligned} Sp(K_n^{net}) &= \Big\{(\omega^1 + \omega^{n-1}) - (\omega^2 + \omega^{n-2}) - \cdots - (\omega^{\frac{n-1}{2}} + \omega^{\frac{n+1}{2}})\Big\} \\ &= \left\{2\cos\frac{2\pi j}{n} - \cdots - 2\cos\frac{2(\frac{n-1}{2})\pi j}{n}\right\} \\ Sp(K_n^{net}) &= \left\{2\cos\frac{2\pi j}{n} - \sum_{r=2}^{\frac{n-1}{2}} 2\cos\frac{2r\pi j}{n} : 1 \le j \le n\right\} \end{aligned}$$

Case 2. If n is even then

$$A(K_n^{net}) = \Big\{(P^1 + P^{n-1}) - (P^2 + P^{n-2}) - \cdots - (P^{\frac{n-1}{2}} + P^{\frac{n+1}{2}}) - (P^{\frac{n}{2}})\Big\}$$

and $\omega \in Sp(P)$. Hence

$$\begin{aligned} Sp(K_n^{net}) &= \Big\{(\omega^1 + \omega^{n-1}) - (\omega^2 + \omega^{n-2}) - \cdots - (\omega^{\frac{n-1}{2}} + \omega^{\frac{n+1}{2}}) - (\omega^{\frac{n}{2}})\Big\} \\ &= \left\{2\cos\frac{2\pi j}{n} - \cdots - 2\cos\frac{2(\frac{n-1}{2})\pi j}{n} - \cos\frac{2(\frac{n}{2})\pi j}{n} : 1 \le j \le n\right\}. \end{aligned}$$

So

$$Sp(K_n^{net}) = \left\{ 2\cos\frac{2\pi j}{n} - \cos\pi j - \sum_{r=2}^{\frac{n-2}{2}} 2\cos\frac{2r\pi j}{n} : 1 \leq j \leq n \right\}. \qquad \square$$

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On Common Fixed Point Theorems With Rational Expressions in Cone b-Metric Spaces

G.S.Saluja

Department of Mathematics, Govt. Nagarjuna P.G. College of Science

Raipur - 492010 (C.G.), India

E-mail: saluja1963@gmail.com

Abstract: In this paper, we establish some common fixed point theorems for rational contraction in the setting of cone b-metric spaces with normal solid cone. Also, as an application of our result, we obtain some results of integral type for such mappings. Our results extend and generalize several known results from the existing literature.

Key Words: Common fixed point, rational expression, cone b- metric space, normal cone.

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§1. Introduction and Preliminaries

Fixed point theory plays a very significant role in the development of nonlinear analysis. In this area, the first important result was proved by Banach in 1922 for contraction mapping in complete metric space, known as the Banach contraction principle [2].

In 1989, Bakhtin [3] introduced b-metric spaces as a generalization of metric spaces. He proved the contraction mapping principle in b-metric spaces that generalized the famous contraction principle in metric spaces. Czerwik used the concept of b-metric space and generalized the renowned Banach fixed point theorem in b-metric spaces (see, [5, 6]). In 2007, Huang and Zhang [9] introduced the concept of cone metric spaces as a generalization of metric spaces and establish some fixed point theorems for contractive mappings in normal cone metric spaces. In 2008, Rezapour and Hamlbarani [14] omitted the assumption of normality in cone metric space, which is a milestone in developing fixed point theory in cone metric space.

In 2011, Hussain and Shah [10] introduced the concept of cone b-metric space as a generalization of b-metric space and cone metric spaces. They established some topological properties in such spaces and improved some recent results about KKM mappings in the setting of a cone b-metric space.

In this note, we establish some common fixed point theorems satisfying rational inequality in the framework of cone b-metric spaces with normal solid cone.

Definition 1.1([9]) *Let E be a real Banach space. A subset P of E is called a cone whenever*

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the following conditions hold:

(C1) P *is closed, nonempty and* $P \neq \{0\}$*;*

(C2) $a, b \in R$*,* $a, b \geq 0$ *and* $x, y \in P$ *imply* $ax + by \in P$*;*

(C3) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$*, we define a partial ordering* $\leq$ *with respect to* P *by* $x \leq y$ *if and only if* $y - x \in P$*. We shall write* $x < y$ *to indicate that* $x \leq y$ *but* $x \neq y$*, while* $x \ll y$ *will stand for* $y - x \in P^0$*, where* P^0 *stands for the interior of* P*. If* $P^0 \neq \emptyset$ *then* P *is called a solid cone (see [15]).*

There exist two kinds of cones- normal (with the normal constant K) and non-normal ones following ([7]):

Let E be a real Banach space, $P \subset E$ a cone and $\leq$ partial ordering defined by P. Then P is called normal if there is a number $K > 0$ such that for all $x, y \in P$,

$$0 \leq x \leq y \quad \text{imply} \quad \|x\| \leq K \|y\|, \tag{1.1}$$

or equivalently, if $(\forall n)$ $x_n \leq y_n \leq z_n$ and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = x \quad \text{imply} \quad \lim_{n\to\infty} y_n = x. \tag{1.2}$$

The least positive number K satisfying (1.1) is called the normal constant of P.

Example 1.2 ([15]) Let $E = C^1_{\mathbb{R}}[0,1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ on $P = \{x \in E : x(t) \geq 0\}$. This cone is not normal. Consider, for example, $x_n(t) = \frac{t^n}{n}$ and $y_n(t) = \frac{1}{n}$. Then $0 \leq x_n \leq y_n$, and $\lim_{n\to\infty} y_n = 0$, but $\|x_n\| = \max_{t\in[0,1]} |\frac{t^n}{n}| + \max_{t\in[0,1]} |t^{n-1}| = \frac{1}{n} + 1 > 1$; hence x_n does not converge to zero. It follows by (1.2) that P is a non-normal cone.

Definition 1.3([9, 16]) *Let* X *be a nonempty set. Suppose that the mapping* $d\colon X \times X \to E$ *satisfies:*

(CM1) $0 \leq d(x,y)$ *for all* $x, y \in X$ *with* $x \neq y$ *and* $d(x,y) = 0 \Leftrightarrow x = y$*;*

(CM2) $d(x,y) = d(y,x)$ *for all* $x, y \in X$*;*

(CM3) $d(x,y) \leq d(x,z) + d(z,y)$ $x, y, z \in X$.

Then d *is called a cone metric on* X *and* (X,d) *is called a cone metric space (CMS).*

The concept of a cone metric space is more general than that of a metric space, because each metric space is a cone metric space where $E = \mathbb{R}$ and $P = [0, +\infty)$.

Example 1.4 ([9]) Let $E = \mathbb{R}^2$, $P = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$, $X = \mathbb{R}$ and $d\colon X \times X \to E$ defined by $d(x,y) = (|x-y|, \alpha|x-y|)$, where $\alpha \geq 0$ is a constant. Then (X,d) is a cone metric space with normal cone P where $K = 1$.

Example 1.5 ([13]) Let $E = \ell^2$, $P = \{\{x_n\}_{n\geq 1} \in E : x_n \geq 0, \text{for all } n\}$, (X,ρ) a metric space, and $d\colon X \times X \to E$ defined by $d(x,y) = \{\rho(x,y)/2^n\}_{n\geq 1}$. Then (X,d) is a cone metric space.

Clearly, the above examples show that class of cone metric spaces contains the class of metric spaces.

Definition 1.6([10]) *Let X be a nonempty set and $s \geq 1$ be a given real number. A mapping $d\colon X \times X \to E$ is said to be cone b-metric if and only if, for all x, y, $z \in X$, the following conditions are satisfied:*

(CbM1) $0 \leq d(x,y)$ *with* $x \neq y$ *and* $d(x,y) = 0 \Leftrightarrow x = y$*;*
(CbM2) $d(x,y) = d(y,x)$*;*
(CbM3) $d(x,y) \leq s[d(x,z) + d(z,y)]$.

The pair (X,d) is called a cone b-metric space (CbMS).

Remark 1.7 The class of cone b-metric spaces is larger than the class of cone metric space since any cone metric space must be a cone b-metric space. Therefore, it is obvious that cone b-metric spaces generalize b-metric spaces and cone metric spaces.

We give some examples, which show that introducing a cone b-metric space instead of a cone metric space is meaningful since there exist cone b-metric spaces which are not cone metric spaces.

Example 1.8 ([8]) Let $E = \mathbb{R}^2$, $P = \{(x,y) \in E : x \geq 0, y \geq 0\} \subset E$, $X = \mathbb{R}$ and $d\colon X \times X \to E$ defined by $d(x,y) = (|x-y|^p, \alpha|x-y|^p)$, where $\alpha \geq 0$ and $p > 1$ are two constants. Then (X,d) is a cone b-metric space with the coefficient $s = 2^p > 1$, but not a cone metric space.

Example 1.9 ([8]) Let $X = \ell^p$ with $0 < p < 1$, where $\ell^p = \{\{x_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$. Let $d\colon X \times X \to \mathbb{R}_+$ defined by

$$d(x,y) = \Big(\sum_{n=1}^{\infty} |x_n - y_n|^p\Big)^{\frac{1}{p}},$$

where $x = \{x_n\}$, $y = \{y_n\} \in \ell^p$. Then (X,d) is a cone b-metric space with the coefficient $s = 2^p > 1$, but not a cone metric space.

Example 1.10 ([8]) Let $X = \{1,2,3,4\}$, $E = \mathbb{R}^2$, $P = \{(x,y) \in E : x \geq 0, y \geq 0\}$. Define $d\colon X \times X \to E$ by

$$d(x,y) = \begin{cases} (|x-y|^{-1}, |x-y|^{-1}) & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

Then (X,d) is a cone b-metric space with the coefficient $s = \frac{6}{5} > 1$. But it is not a cone metric space since the triangle inequality is not satisfied,

$$d(1,2) > d(1,4) + d(4,2), \quad d(3,4) > d(3,1) + d(1,4).$$

Definition 1.11([10]) *Let (X,d) be a cone b-metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. Then*

• $\{x_n\}$ *is a Cauchy sequence whenever, if for every* $c \in E$ *with* $0 \ll c$*, then there is a natural number* N *such that for all* $n, m \geq N$*,* $d(x_n, x_m) \ll c$*;*

• $\{x_n\}$ *converges to* x *whenever, for every* $c \in E$ *with* $0 \ll c$*, then there is a natural number* N *such that for all* $n \geq N$*,* $d(x_n, x) \ll c$*. We denote this by* $\lim_{n\to\infty} x_n = x$ *or* $x_n \to x$ *as* $n \to \infty$*.*

• (X, d) *is a complete cone* b*-metric space if every Cauchy sequence is convergent.*

In the following (X, d) will stands for a cone b-metric space with respect to a cone P with $P^0 \neq \emptyset$ in a real Banach space E and $\leq$ is partial ordering in E with respect to P.

§2. Main Results

In this section we shall prove some common fixed point theorems for rational contraction in the framework of cone b-metric spaces with normal solid cone.

Theorem 2.1 *Let* (X, d) *be a complete cone* b*-metric space (CCbMS) with the coefficient* $s \geq 1$ *and* P *be a normal cone with normal constant* K*. Suppose that the mappings* $S, T\colon X \to X$ *satisfy the rational contraction:*

$$d(Sx, Ty) \leq \alpha\Big[\frac{d(x, Sx)\, d(x, Ty) + [d(x, y)]^2 + d(x, Sx)d(x, y)}{d(x, Sx) + d(x, y) + d(x, Ty)}\Big] \tag{2.1}$$

for all $x, y \in X$*,* $\alpha \in [0, 1)$ *with* $s\alpha < 1$ *and* $d(x, Sx) + d(x, y) + d(x, Ty) \neq 0$*. Then* S *and* T *have a common fixed point in* X*. Further if* $d(x, Sx) + d(x, y) + d(x, Ty) = 0$ *implies that* $d(Sx, Ty) = 0$*, then* S *and* T *have a unique common fixed point in* X*.*

Proof Choose $x_0 \in X$. Let $x_1 = S(x_0)$ and $x_2 = T(x_1)$ such that $x_{2n+1} = S(x_{2n})$ and $x_{2n+2} = T(x_{2n+1})$ for all $n \geq 0$. Let $d(x, Sx) + d(x, y) + d(x, Ty) \neq 0$. From (2.1), we have

$$\begin{aligned}
d(x_{2n+1}, x_{2n+2}) &= d(Sx_{2n}, Tx_{2n+1}) \\
&\leq \alpha\Big[\Big(d(x_{2n}, Sx_{2n})\, d(x_{2n}, Tx_{2n+1}) + [d(x_{2n}, x_{2n+1})]^2 \\
&\quad + d(x_{2n}, Sx_{2n})\, d(x_{2n}, x_{2n+1})\Big) \\
&\quad \times\Big(d(x_{2n}, Sx_{2n}) + d(x_{2n}, x_{2n+1}) + d(x_{2n}, Tx_{2n+1})\Big)^{-1}\Big] \\
&= \alpha\Big[\Big(d(x_{2n}, x_{2n+1})\, d(x_{2n}, x_{2n+2}) + [d(x_{2n}, x_{2n+1})]^2 \\
&\quad + d(x_{2n}, x_{2n+1})\, d(x_{2n}, x_{2n+1})\Big) \\
&\quad \times\Big(d(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n+2})\Big)^{-1}\Big] \\
&= \alpha\, d(x_{2n}, x_{2n+1}) \\
&\quad \times\Big[\frac{d(x_{2n}, x_{2n+2}) + d(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n+1})}{d(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n+2})}\Big] \\
&= \alpha\, d(x_{2n}, x_{2n+1}).
\end{aligned} \tag{2.2}$$

Similarly, we have

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &= d(Sx_{2n}, Tx_{2n-1}) \\ &\leq \alpha\Big[\Big(d(x_{2n}, Sx_{2n})\, d(x_{2n}, Tx_{2n-1}) + [d(x_{2n}, x_{2n-1})]^2 \\ &\quad + d(x_{2n}, Sx_{2n})\, d(x_{2n}, x_{2n-1})\Big) \\ &\quad \times\Big(d(x_{2n}, Sx_{2n}) + d(x_{2n}, x_{2n-1}) + d(x_{2n}, Tx_{2n-1})\Big)^{-1}\Big] \\ &= \alpha\Big[\Big(d(x_{2n}, x_{2n+1})\, d(x_{2n}, x_{2n}) + [d(x_{2n}, x_{2n-1})]^2 \\ &\quad + d(x_{2n}, x_{2n+1})\, d(x_{2n}, x_{2n-1})\Big) \\ &\quad \times\Big(d(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n-1}) + d(x_{2n}, x_{2n})\Big)^{-1}\Big] \\ &= \alpha\, d(x_{2n}, x_{2n-1}) \\ &\quad \times\Big[\frac{d(x_{2n}, x_{2n-1}) + d(x_{2n}, x_{2n+1})}{d(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n-1})}\Big] \\ &= \alpha\, d(x_{2n}, x_{2n-1}). \end{aligned} \tag{2.3}$$

By induction, we have

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \alpha\, d(x_{n-1}, x_n) \leq \alpha^2\, d(x_{n-2}, x_{n-1}) \leq \dots \\ &\leq \alpha^n\, d(x_0, x_1). \end{aligned} \tag{2.4}$$

Let $m, n \geq 1$ and $m > n$, we have

$$\begin{aligned} d(x_n, x_m) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)] \\ &= sd(x_n, x_{n+1}) + sd(x_{n+1}, x_m) \\ &\leq sd(x_n, x_{n+1}) + s^2[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)] \\ &= sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^2 d(x_{n+2}, x_m) \\ &\leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^3 d(x_{n+2}, x_{n+3}) \\ &\quad + \cdots + s^{n+m-1} d(x_{n+m-1}, x_m) \\ &\leq s\alpha^n d(x_1, x_0) + s^2\alpha^{n+1} d(x_1, x_0) + s^3\alpha^{n+2} d(x_1, x_0) \\ &\quad + \cdots + s^m \alpha^{n+m-1} d(x_1, x_0) \\ &= s\alpha^n[1 + s\alpha + s^2\alpha^2 + s^3\alpha^3 + \cdots + (s\alpha)^{m-1}] d(x_1, x_0) \\ &\leq \Big[\frac{s\alpha^n}{1 - s\alpha}\Big] d(x_1, x_0). \end{aligned}$$

Since P is a normal cone with normal constant K, so we get

$$\|d(x_n, x_m)\| \leq K \frac{s\alpha^n}{1 - s\alpha} \|d(x_1, x_0)\|.$$

This implies $\|d(x_n, x_m)\| \to 0$ as $n, m \to \infty$ since $0 < s\alpha < 1$. Hence $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete cone b-metric space, there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$.

Now, since

$$\begin{aligned} d(z,Tz) &\leq s[d(z,x_{2n+1})+d(x_{2n+1},Tz)] \\ &= sd(Sx_{2n},Tz)+sd(z,x_{2n+1}) \\ &\leq s\alpha\Big[\frac{d(x_{2n},Sx_{2n})\,d(x_{2n},Tz)+[d(x_{2n},z)]^2+d(x_{2n},Sx_{2n})d(x_{2n},z)}{d(x_{2n},Sx_{2n})+d(x_{2n},z)+d(x_{2n},Tz)}\Big] \\ &\quad +sd(z,x_{2n+1}) \\ &\leq s\alpha\Big[\frac{d(x_{2n},x_{2n+1})\,d(x_{2n},Tz)+[d(x_{2n},z)]^2+d(x_{2n},x_{2n+1})d(x_{2n},z)}{d(x_{2n},x_{2n+1})+d(x_{2n},z)+d(x_{2n},Tz)}\Big] \\ &\quad +sd(z,x_{2n+1}). \end{aligned}$$

Now using the condition of normal cone, we have

$$\begin{aligned} \|d(z,Tz)\| &\leq K\Big\{s\alpha\Big\|\Big[\frac{d(x_{2n},x_{2n+1})\,d(x_{2n},Tz)+[d(x_{2n},z)]^2+d(x_{2n},x_{2n+1})d(x_{2n},z)}{d(x_{2n},x_{2n+1})+d(x_{2n},z)+d(x_{2n},Tz)}\Big]\Big\| \\ &\quad +s\|d(z,x_{2n+1})\|\Big\}. \end{aligned}$$

As $n\to\infty$, we have

$$\|d(z,Tz)\| \leq 0.$$

Hence $\|d(z,Tz)\|=0$. Thus we get $Tz=z$, that is, z is a fixed point of T.

In an exactly the same fashion we can prove that $Sz=z$. Hence $Sz=Tz=z$. This shows that z is a common fixed point of S and T.

For the uniqueness of z, let us suppose that $d(x,Sx)+d(x,y)+d(x,Ty)=0$ implies $d(Sx,Ty)=0$ and let w be another fixed point of S and T in X such that $z\neq w$. Then

$$d(z,Sz)+d(z,w)+d(z,Tw)=0 \;\Rightarrow\; d(Sz,Tw)=0.$$

Therefore, we get

$$d(z,w)=d(Sz,Tw)=0,$$

which implies that $z=w$. This shows that z is the unique common fixed point of S and T. This completes the proof. □

If S is a map which has a fixed point p, then p is a fixed point of S^n for every $n\in\mathbb{N}$ too. However, the converse need not to be true. Jeong and Rhoades [12] discussed the situation and gave examples for metric spaces, while Abbas and Rhoades [1] examined this for cone metric spaces. If a map satisfies $F(S)=F(S^n)$ for each $n\in\mathbb{N}$ then it is said to have property P. If $F(S^n)\cap F(T^n)=F(S)\cap F(T)$ then we say that S and T have property P^*.

We examine the property P^* for those mappings which satisfy inequality (2.1).

Theorem 2.2 *Let (X,d) be a complete cone b-metric space (CCbMS) with the coefficient $s\geq 1$ and P be a normal cone with normal constant K. Suppose that the mappings $S,T\colon X\to X$ satisfy (2.1). Then S and T have the property P^*.*

Proof By the above theorem, we know that S and T have a common fixed in X. Let $z \in F(S^n) \cap F(T^n)$. Then

$$\begin{aligned}
d(z,Tz) &= d(S^z, T^{n+1}z) = d(S(S^{n-1}z, T(T^n z)) \\
&\leq \alpha\Big[\Big(d(S^{n-1}z, S^n z)\, d(S^{n-1}z, T(T^n z)) + [d(S^{n-1}z, T^n z)]^2 \\
&\quad + d(S^{n-1}z, S^n z)\, d(S^{n-1}z, T^n z)\Big) \\
&\quad \times \Big(d(S^{n-1}z, S^n z) + d(S^{n-1}z, T^n z) + d(S^{n-1}z, T(T^n z))\Big)^{-1}\Big] \\
&= \alpha\Big[\Big(d(S^{n-1}z, z)\, d(S^{n-1}z, Tz) + [d(S^{n-1}z, z)]^2 \\
&\quad + d(S^{n-1}z, z)\, d(S^{n-1}z, z)\Big) \\
&\quad \times \Big(d(S^{n-1}z, z) + d(S^{n-1}z, z) + d(S^{n-1}z, Tz)\Big)^{-1}\Big] \\
&= \alpha\, d(S^{n-1}z, z) \times \Big[\frac{d(S^{n-1}z, Tz) + 2d(S^{n-1}z, z)}{2d(S^{n-1}z, z) + d(S^{n-1}z, Tz)}\Big] \\
&= \alpha\, d(S^{n-1}z, z).
\end{aligned}$$

Similarly

$$\begin{aligned}
d(S^n z, T^{n+1}z) &\leq \alpha\, d(S^{n-1}z, T^n z) = \alpha\, d(S(S^{n-2}z), T(T^{n-1}z)) \\
&\leq \alpha\Big[\Big(d(S^{n-2}z, S^{n-1}z)\, d(S^{n-2}z, T^n z) + [d(S^{n-2}z, T^{n-1}z)]^2 \\
&\quad + d(S^{n-2}z, S^{n-1}z)\, d(S^{n-2}z, T^{n-1}z)\Big) \\
&\quad \times \Big(d(S^{n-2}z, S^{n-1}z) + d(S^{n-2}z, T^{n-1}z) + d(S^{n-2}z, T^n z)\Big)^{-1}\Big] \\
&= \alpha\Big[\Big(d(S^{n-2}z, S^{n-1}z)\, d(S^{n-2}z, T^n z) + [d(S^{n-2}z, S^{n-1}z)]^2 \\
&\quad + d(S^{n-2}z, S^{n-1}z)\, d(S^{n-2}z, S^{n-1}z)\Big) \\
&\quad \times \Big(d(S^{n-2}z, S^{n-1}z) + d(S^{n-2}z, S^{n-1}z) + d(S^{n-2}z, T^n z)\Big)^{-1}\Big] \\
&= \alpha\, d(S^{n-2}z, S^{n-1}z) \times \Big[\frac{d(S^{n-2}z, T^n z) + 2d(S^{n-2}z, S^{n-1}z)}{2d(S^{n-2}z, S^{n-1}z) + d(S^{n-2}z, T^n z)}\Big] \\
&= \alpha\, d(S^{n-2}z, S^{n-1}z).
\end{aligned}$$

Continuing this process, we get that

$$d(S^n z, T^{n+1}z) \leq \alpha\, d(S^{n-1}z, T^n z) \leq \alpha^2\, d(S^{n-2}z, T^{n-1}z) \leq \cdots \leq \alpha^n\, d(z, Tz).$$

That is,

$$d(z,Tz) \leq \alpha^n\, d(z,Tz).$$

Using (1.1), the above inequality implies that

$$\|d(z,Tz)\| \leq K\alpha^n\, \|d(z,Tz)\| \to 0 \text{ as } n \to \infty.$$

Hence $\|d(z,Tz)\| = 0$. Thus we get $Tz = z$, that is, z is a fixed point of T. By using Theorem 2.1, we get $Sz = z$, and consequently, S and T have property P^*. This completes the proof. □

Putting $S = T$, we have the following result.

Corollary 2.3 *Let (X,d) be a complete cone b-metric space (CCbMS) with the coefficient $s \geq 1$ and P be a normal cone with normal constant K. Suppose that the mappings $T\colon X \to X$ satisfies the rational contraction:*

$$d(Tx,Ty) \leq \alpha\Big[\frac{d(x,Tx)\,d(x,Ty) + [d(x,y)]^2 + d(x,Tx)d(x,y)}{d(x,Tx) + d(x,y) + d(x,Ty)}\Big] \tag{2.5}$$

for all $x,y \in X$, $\alpha \in [0,1)$ with $s\alpha < 1$ and $d(x,Tx) + d(x,y) + d(x,Ty) \neq 0$. Then T has a fixed point in X. Further if $d(x,Tx) + d(x,y) + d(x,Ty) = 0$ implies that $d(Tx,Ty) = 0$, then T has a unique fixed point in X.

Proof The proof of Corollary 2.3 immediately follows from Theorem 2.1 by taking $S = T$. This completes the proof. □

Theorem 2.4 *Let (X,d) be a complete cone b-metric space (CCbMS) with the coefficient $s \geq 1$ and P be a normal cone with normal constant K. Suppose that the mapping $T\colon X \to X$ satisfies (2.5) with $s\alpha < 1$, where $\alpha \in [0,1)$. Then T has the property P.*

Proof Let $v \in F(T^n)$. Then

$$\begin{aligned}
d(v,Tv) &= d(T^nv, T^{n+1}v) = d(T(T^{n-1}v), T(T^nv)) \\
&\leq \alpha\Big[d(T^{n-1}v,T^nv)\,d(T^{n-1}v,T(T^nv)) + [d(T^{n-1}v,T^nv)]^2 \\
&\quad + d(T^{n-1}v,T^nv)d(T^{n-1}v,T^nv) \\
&\quad \times\big\{d(T^{n-1}v,T^nv) + d(T^{n-1}v,T^nv) + d(T^{n-1}v,T(T^nv))\big\}^{-1}\Big] \\
&= \alpha\Big[d(T^{n-1}v,v)\,d(T^{n-1}v,Tv) + [d(T^{n-1}v,v)]^2 \\
&\quad + d(T^{n-1}v,v)d(T^{n-1}v,v) \\
&\quad \times\big\{d(T^{n-1}v,v) + d(T^{n-1}v,v) + d(T^{n-1}v,Tv)\big\}^{-1}\Big] \\
&= \alpha\, d(T^{n-1}v,v) \times \Big[\frac{d(T^{n-1}v,Tv) + 2d(T^{n-1}v,v)}{2d(T^{n-1}v,v) + d(T^{n-1}v,Tv)}\Big] \\
&= \alpha\, d(T^{n-1}v,v).
\end{aligned}$$

That is

$$d(T^nv, T^{n+1}v) \leq \alpha\, d(T^{n-1}v, T^nv).$$

Similarly

$$
\begin{aligned}
d(T^{n-1}v, T^n v) &= d(T(T^{n-2}v), T(T^{n-1}v)) \\
&\leq \alpha\Big[d(T^{n-2}v, T^{n-1}v)\, d(T^{n-2}v, T^n v) + [d(T^{n-2}v, T^{n-1}v)]^2 \\
&\quad + d(T^{n-2}v, T^{n-1}v) d(T^{n-2}v, T^{n-1}v) \\
&\quad \times \big\{d(T^{n-2}v, T^{n-1}v) + d(T^{n-2}v, T^{n-1}v) + d(T^{n-2}v, T^n v)\big\}^{-1}\Big] \\
&= \alpha\Big[d(T^{n-2}v, T^{n-1}v)\, d(T^{n-1}v, v) + [d(T^{n-2}v, T^{n-1}v)]^2 \\
&\quad + d(T^{n-2}v, T^{n-1}v) d(T^{n-2}v, T^{n-1}v) \\
&\quad \times \big\{d(T^{n-2}v, T^{n-1}v) + d(T^{n-2}v, T^{n-1}v) + d(T^{n-2}v, v)\big\}^{-1}\Big] \\
&= \alpha\, d(T^{n-2}v, T^{n-1}v) \times \Big[\frac{d(T^{n-1}v, v) + 2d(T^{n-2}v, T^{n-1}v)}{2d(T^{n-2}v, T^{n-1}v) + d(T^{n-1}v, v)}\Big] \\
&= \alpha\, d(T^{n-2}v, T^{n-1}v).
\end{aligned}
$$

Continuing this process, we get

$$
d(T^n v, T^{n+1}v) \leq \alpha\, d(T^{n-1}v, T^n v) \leq \alpha^2\, d(T^{n-2}v, T^{n-1}v) \leq \cdots \leq \alpha^n\, d(v, Tv).
$$

That is,

$$
d(v, Tv) \leq \alpha^n\, d(v, Tv).
$$

Using (1.1), the above inequality implies that

$$
\|d(v, Tv)\| \leq K\alpha^n\, \|d(v, Tv)\| \to 0 \text{ as } n \to \infty.
$$

Hence $\|d(v, Tv)\| = 0$. Thus we get $Tv = v$. Thus we conclude that a mapping which satisfies (2.5) has the property P. This completes the proof. □

§3. Applications

The aim of this section is to apply our result to mappings involving contraction of integral type. For this purpose, denote Λ the set of functions $\varphi\colon [0,\infty) \to [0,\infty)$ satisfying the following hypothesis:

(h1) φ is a Lebesgue-integrable mapping on each compact subset of $[0,\infty)$;

(h2) for any $\varepsilon > 0$ we have $\int_0^\varepsilon \varphi(t)\, dt > 0$.

Theorem 3.1 *Let (X, d) be a complete cone b-metric space (CCbMS) with the coefficient $s \geq 1$ and P be a normal cone with normal constant K. Suppose that the mappings $S, T\colon X \to X$*

satisfy the contraction of integral type:

$$\int_0^{d(Sx,Ty)} \psi(t)\,dt \quad \leq \quad \alpha \int_0^{\left[\frac{d(x,Sx)\,d(x,Ty)+[d(x,y)]^2+d(x,Sx)d(x,y)}{d(x,Sx)+d(x,y)+d(x,Ty)}\right]} \psi(t)\,dt$$

for all $x, y \in X$, $\alpha \in [0,1)$ with $s\alpha < 1$ and $\psi \in \Lambda$. Then S and T have a unique common fixed point in X.

If we put $S = T$ in Theorem 3.1, we have the following result.

Theorem 3.2 *Let (X, d) be a complete cone b-metric space (CCbMS) with the coefficient $s \geq 1$ and P be a normal cone with normal constant K. Suppose that the mapping $T\colon X \to X$ satisfies the contraction of integral type:*

$$\int_0^{d(Tx,Ty)} \psi(t)\,dt \quad \leq \quad \alpha \int_0^{\left[\frac{d(x,Tx)\,d(x,Ty)+[d(x,y)]^2+d(x,Tx)d(x,y)}{d(x,Tx)+d(x,y)+d(x,Ty)}\right]} \psi(t)\,dt$$

for all $x, y \in X$, $\alpha \in [0,1)$ with $s\alpha < 1$ and $\psi \in \Lambda$. Then T has a unique fixed point in X.

§4. Conclusion

In this paper, we establish some unique common fixed point theorems for rational contraction in the setting of cone b-metric spaces with normal solid cone. Also, as an application of our result, we obtained some results of integral type for such mappings. Our results extend and generalize several results from the existing literature.

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Binding Number of Some Special Classes of Trees

B.Chaluvaraju[1], H.S.Boregowda[2] and S.Kumbinarsaiah[3]

[1]Department of Mathematics, Bangalore University, Janana Bharathi Campus, Bangalore-560 056, India

[2]Department of Studies and Research in Mathematics, Tumkur University, Tumkur-572 103, India

[3]Department of Mathematics, Karnatak University, Dharwad-580 003, India

E-mail: bchaluvaraju@gmail.com, bgsamarasa@gmail.com, kumbinarasaiah@gmail.com

Abstract: The binding number of a graph $G = (V, E)$ is defined to be the minimum of $|N(X)|/|X|$ taken over all nonempty set $X \subseteq V(G)$ such that $N(X) \neq V(G)$. In this article, we explore the properties and bounds on binding number of some special classes of trees.

Key Words: Graph, tree, realizing set, binding number, Smarandachely binding number.

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§1. Introduction

In this article, we consider finite, undirected, simple and connected graphs $G = (V, E)$ with vertex set V and edge set E. As such $n =| V |$ and $m =| E |$ denote the number of vertices and edges of a graph G, respectively. An edge - induced subgraph is a subset of the edges of a graph G together with any vertices that are their endpoints. In general, we use $\langle X \rangle$ to denote the subgraph induced by the set of edges $X \subseteq E$. A graph G is connected if it has a $u - v$ path whenever $u, v \in V(G)$ (otherwise, G is disconnected). The open neighborhood of a vertex $v \in V(G)$ is $N(v) = \{u \in V : uv \in E(G)\}$ and the closed neighborhood $N[v] = N(v) \cup \{v\}$. The degree of v, denoted by $deg(v)$, is the cardinality of its open neighborhood. A vertex with degree one in a graph G is called pendant or a leaf or an end-vertex, and its neighbor is called its support or cut vertex. An edge incident to a leaf in a graph G is called a pendant edge. A graph with no cycle is acyclic. A tree T is a connected acyclic graph. Unless mentioned otherwise, for terminology and notation the reader may refer Harary [3].

Woodall [7] defined the binding number of G as follows: If $X \subseteq V(G)$, then the open neighborhood of the set X is defined as $N(X) = \bigcup_{x \in X} N(v)$. The binding number of G, denoted $b(G)$, is given by

$$b(G) = min_{x \in F} \frac{|N(X)|}{|X|},$$

where $F = \{X \subseteq V(G) : X \neq \varnothing, N(X) \neq V(G)\}$. We say that $b(G)$ is realized on a set X if $X \in F$ and $b(G) = \frac{|N(X)|}{|X|}$, and the set X is called a realizing set for $b(G)$. Generally, for a given graph H, a *Smarandachely binding number* $b_H(G)$ is the minimum number $b(G)$ on such F with

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$\langle X\rangle_G \ncong H$ for $\forall X \in F$. Clearly, if H is not a spanning subgraph of G, then $b_H(G) = b(G)$.

For complete review and the following existing results on the binding number and its related concepts, we follow [1], [2], [5] and [6].

Theorem 1.1 *For any path P_n with $n \geq 2$ vertices,*

$$b(P_n) = \begin{cases} 1 & \text{if n is even;} \\ \frac{n-1}{n+1} & \text{if n is odd.} \end{cases}$$

Theorem 1.2 *For any spanning subgraph H of a graph G, $b(G) \leq b(H)$.*

In [8], Wayne Goddard established several bounds including ones linking the binding number of a tree to the distribution of its end-vertices $end(G) = \{v \in V(G) : deg(v) = 1\}$. Also, let $\varrho(v) = |N(v) \cap end(G)|$ and $\varrho(G)=$ max $\{\varrho(v) : v \in V(G)\}$. The following result is obviously true if $\varrho(G) = 0$ and if $\varrho(G) = 1$, follows from taking $X = \{N(v) \cap end(G)\}$, where v is a vertex for which $\varrho(v) = \varrho(G)$.

Theorem 1.3 *For any graph G, $\varrho(G).b(G) \leq 1$.*

Theorem 1.4 *For any nontrivial tree T,*

(1) $b(T) \geq 1/\Delta(T)$;
(2) $b(T) \geq 1/\varrho(T) + 1$.

§2. Main Results

Observation 2.1 Let T be a tree with $n \geq 3$ vertices, having $(n-1)$-pendant vertices, which are connected to unique vertex. Then $b(T)$ is the reciprocal of number of vertices connected to unique vertex.

Observation 2.2 Let T be a nontrivial tree. Then $b(T) > 0$.

Observation 2.3 Let T be a tree with $b(T) < 1$. Then every realizing set of T is independent.

Theorem 2.4 *For any Star $K_{1,n-1}$ with $n \geq 2$ vertices,*

$$b(K_{1,n-1}) = \frac{1}{n-1}.$$

Proof Let $K_{1,n-1}$ be a star with $n \geq 2$ vertices. If $K_{1,n-1}$ has $\{v_1, v_2, \cdots, v_n\}$ vertices with $deg(v_1) = n-1$ and $deg(v_2) = deg(v_3) = \cdots = deg(v_n) = 1$. We prove the result by induction on n. For $n = 2$, then $|N(X)| = |X| = 1$ and $b(K_{1,1}) = 1$. For $n = 3$, $|N(X)| < |X| = 2$ and $b(K_{1,2}) = \frac{1}{2}$. Let us assume the result is true for $n = k$ for some k, where k is a positive integer. Hence $b(K_{1,k-1}) = \frac{1}{k-1}$.

Now we shall show that the result is true for $n > k$. Since $(k+1)$- pendant vertices in $K_{1,k+1}$ are connected to the unique vertex v_1. Here newly added vertex v_{k+1} must be adjacent to v_1 only. Otherwise $K_{1,k+1}$ loses its star criteria and v_{k+1} is not adjacent to $\{v_2, v_3, \cdots, v_k\}$, then $K_{1,k+1}$ has k number of pendant vertices connected to vertex v_1. Therefore by Observation 2.1, the desired result follows. □

Theorem 2.5 *Let T_1 and T_2 be two stars with order n_1 and n_2, respectively. Then $n_1 < n_2$ if and only if $b(T_1) > b(T_2)$.*

Proof By Observation 2.1 and Theorem 2.4, we have $b(T_1) = \frac{1}{n_1}$ and $b(T_2) = \frac{1}{n_2}$. Due to the fact of $n_1 < n_2$ if and only if $\frac{1}{n_1} > \frac{1}{n_2}$. Thus the result follows. □

Definition 2.6 *The double star $K^*_{r,s}$ is a tree with diameter 3 and central vertices of degree r and s respectively, where the diameter of graph is the length of the shortest path between the most distanced vertices.*

Theorem 2.7 *For any double star $K^*_{r,s}$ with $1 \le r \le s$ vertices,*

$$b(K^*_{r,s}) = \frac{1}{max\{r, s\} - 1}.$$

Proof Suppose $K^*_{r,s}$ is a double star with $1 \le r \le s$ vertices. Then there exist exactly two central vertices x and y for all $x, y \in V(K^*_{r,s})$ such that the degree of x and y are r and s respectively. By definition, the double star $K^*_{r,s}$ is a tree with diameter 3 having only one edge between x and y. Therefore the vertex x is adjacent to $(r-1)$-pendant vertices and the vertex y is adjacent to $(s-1)$-pendant vertices.

Clearly $max\{r-1, s-1\}$ pendant vertices are adjacent to a unique vertex x or y as the case may be. Therefore $b(K^*_{r,s}) = \frac{1}{max\{r-1,s-1\}}$. Hence the result follows. □

Definition 2.8 *A subdivided star, denoted $K^*_{1,n-1}$ is a star $K_{1,n-1}$ whose edges are subdivided once, that is each edge is replaced by a path of length 2 by adding a vertex of degree 2.*

Observation 2.9 Let $K_{1,n-1}$ be a star with $n \ge 2$ vertices. Then cardinality of the vertex set of $K^*_{1,n-1}$ is $p = 2n - 1$.

Theorem 2.10 *For any subdivided star $K^*_{1,n-1}$ with $n \ge 2$ vertices,*

$$b(K^*_{1,n-1}) = \begin{cases} \frac{1}{2} & \text{if n = 2;} \\ \frac{2}{3} & \text{if n = 3;} \\ 1 & \text{otherwise.} \end{cases}$$

Proof By Observation 2.9, the subdivided star $K^*_{1,n-1}$ has $p = 2n - 1$ vertices. Then the following cases arise:

Case 1. If $n=2$, then by Theorem 1.1, $b(K^*_{1,2-1}) = b(P_3) = \frac{1}{2}$.

Case 2. If $n=3$, then by Theorem 1.1, $b(K^*_{1,3-1}) = b(P_5) = \frac{2}{3}$.

Case 3. If a vertex $v_1 \in V(K_{1,n-1})$ with $deg(v_1) = n-1$ and $deg(N(v_1)) = 1$, where $N(v_1) = \{v_2, v_3, \cdots, v_n\}$. Clearly, each edge $\{v_1v_2, v_1v_3, \cdots, v_1v_n\}$ takes one vertex on each edge having degree 2, so that the resulting graph will be subdivided star $K^*_{1,n-1}$, in which $\{v_1\}$ and $\{v_2, v_3, \cdots, v_n\}$ vertices do not lose their properties. But the maximum degree vertex v_1 is a cut vertex of $K^*_{1,n-1}$. Therefore $b(K_{1,n-1}) < b(K^*_{1,n-1})$ for $n \geq 4$ vertices. Since each newly added vertex $\{u_i\}$ is adjacent to exactly one pendent vertex $\{v_j\}$,where $i=j$ and $2 \leq i, j \leq n$, in $K^*_{1,n-1}$. By the definition of binding number $|N(X)| = |X|$. Hence the result follows. □

Definition 2.11 *A $B_{t,k}$ graph is said to be a Banana tree if the graph is obtained by connecting one pendant vertex of each t-copies of an k-star graph with a single root vertex that is distinct from all the stars.*

Theorem 2.12 *For any Banana tree $B_{t,k}$ with $t \geq 2$ copies and $k \geq 3$ number of stars,*

$$b(B_{t,k}) = \frac{1}{k-2}.$$

Proof Let t be the number of distinct k-stars. Then it has $k-1$-pendant vertices and the binding number of each k-stars is $\frac{1}{k-1}$. But in $B_{t,k}$, each t copies of distinct k-stars are joined by single root vertex. Then the resulting graph is connected and each k-star has $k-2$ number of vertices having degree 1, which are connected to unique vertex. By Observation 2.1, the result follows. □

Definition 2.13 *A caterpillar tree $C^*(T)$ is a tree in which removing all the pendant vertices and incident edges produces a path graph.*

For example, $b(C^*(K_1)) = 0$; $b(C^*(P_2)) = b(C^*(P_4)) = 1$; $b(C^*(P_3)) = \frac{1}{2}$; $b(C^*(P_5)) = \frac{2}{3}$ and $b(C^*(K_{1,n-1})) = \frac{1}{n-1}$.

Theorem 2.14 *For any caterpillar tree $C^*(T)$ with $n \geq 3$ vertices,*

$$b(K_{1,n-1}) \leq b(C^*(T)) \leq b(P_n).$$

Proof By mathematical induction, if $n=3$, then by Theorem 1.1 and Observation 2.1, we have $b(K_{1,2}) = b(C^*(T)) = b(P_3) = \frac{1}{2}$. Thus the result follows. Assume that the result is true for $n=k$. Now we shall prove the result for $n > k$. Let $C^*(T)$ be a Caterpillar tree with $k+1$-vertices. Then the following cases arise:

Case 1. If $k+1$ is odd, then $b(C^*(T)) \leq \frac{k}{k+1}$.

Case 2. If $k+1$ is even, then $b(C^*(T)) \leq 1$.

By above cases, we have $b(C^*(T)) \leq b(P_n)$. Since, k vertices in $C^*(T)$ exist k-stars, which contributed at least $\frac{1}{k-1}$. Hence the lower bound follows. □

Definition 2.15 *The binary tree B^* is a tree like structure that is rooted and in which each vertex has at least two children and child of a vertex is designated as its left or right child.*

To prove our next result we make use of the following conditions of Binary tree B^*.

C_1**:** If B^* has at least one vertex having two children and that two children has no any child.

C_2**:** If B^* has no vertex having two children which are not having any child.

Theorem 2.16 *Let B^* be a Binary tree with $n \geq 3$ vertices. Then*

$$b(B^*) = \begin{cases} \frac{1}{2} & \text{if } B^* \text{ satisfy } C_1; \\ b(P_n) & \text{if } B^* \text{ satisfy } C_2. \end{cases}$$

Proof Let B^* be a Binary tree with $n \geq 3$ vertices. Then the following cases are arises:

Case 1. Suppose binary tree B^* has only one vertex, say v_1 has two children and that two children has no any child. Then only vertex v_1 has two pendant vertices and no other vertex has more than two pendant vertices. That is maximum at most two pendant vertices are connected to unique vertex. There fore $b(B^*) = \frac{1}{2}$ follows.

Case 2. Suppose binary tree B^* has no vertex having two free child. That is each non-pendant vertex having only one child, then this binary tree gives path. This implies that $b(B^*) = b(P_n)$ with $n \geq 3$ vertices. Thus the result follows. □

Definition 2.17 *The t-centipede C_t^* is the tree on $2t$-vertices obtained by joining the bottoms of t - copies of the path graph P_2 laid in a row with edges.*

Theorem 2.18 *For any t-centipede C_t^* with $2t$-vertices,*

$$b(C_t^*) = 1.$$

Proof If $n = 1$, then tree C_1^* is a 1-centipede with 2-vertices. Thus $b(C_1^*) = 1$. Suppose the result is true for $n > 1$ vertices, say $n = t$ for some t, that is $b(C_t^*) = 1$. Further, we prove $n = t+1$, $b(C_{t+1}^*) = 1$. In a $(t+1)$ - centipede exactly one vertex from each of the $(k+1)$- copies of P_2 are laid on a row with edges. Hence the resulting graph must be connected and each such vertex is connected to exactly one pendant vertex. By the definition of binding number $|N(X)| = |X|$. Hence the result follows. □

Definition 2.19 *The Fire-cracker graph $F_{t,s}$ is a tree obtained by the concatenation of t - copies of s - stars by linking one pendant vertex from each.*

Theorem 2.20 *For any Fire-cracker graph $F_{t,s}$ with $t \geq 2$ and $s \geq 3$.*

$$b(F_{t,s}) = \frac{1}{s-1}.$$

Proof If $s = 2$, then Fire-cracker graph $F_{t,2}$ is a t-centipede and $b(F_{t,2}) = 1$. If $t \geq 2$ and $s \geq 3$, then t - copies of s - stars are connected by adjoining one pendant vertex from each s-stars. This implies that the resulting graph is connected and a Fire-cracker graph $F_{t,s}$. Then this connected graph has $(s-2)$-vertices having degree 1, which are connected to unique vertex. Hence the result follows. □

Theorem 2.21 *For any nontrivial tree T,*

$$\frac{1}{n-1} \leq b(T) \leq 1.$$

Further, the lower bound attains if and only if $T = K_{1,n-1}$ and the upper bound attains if the tree T has 1-factor or there exists a realizing set X such that $X \cap N(X) = \phi$.

Proof The upper bound is proved by Woodall in [7]with the fact of $\delta(T) = 1$. Let $X \in F$ and $\frac{|N(X)|}{|X|} = b(G)$. Then $|N(X)| \geq 1$, since the set X is not empty. Suppose, $|N(X)| \geq n - \delta(T) + 1$. If $\delta(T) = 1$, then any vertex of T is adjacent to atleast one vertex in X. This implies that $N(X) = V(T)$, which is a contradiction. There fore $|X| \leq n-1$ and $b(T) = |N(X)|/|X| \geq 1/(n-1)$. Thus the lower bound follows. □

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On the Wiener Index of Quasi-Total Graph and Its Complement

B.Basavanagoud and Veena R.Desai

(Department of Mathematics Karnatak University, Dharwad - 580 003, India)

E-mail: b.basavanagoud@gmail.com, veenardesai6f@gmail.com

Abstract: The *Wiener index* of a graph G denoted by $W(G)$ is the sum of distances between all (unordered) pairs of vertices of G. In practice G corresponds to what is known as the *molecular graph* of an organic compound. In this paper, we obtain the Wiener index of quasi-total graph and its complement for some standard class of graphs, we give bounds for Wiener index of quasi-total graph and its complement also establish Nordhaus-Gaddum type of inequality for it.

Key Words: Wiener index, quasi-total graph, complement of quasi-total graph.

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§1. Introduction

Let G be a simple, connected, undirected graph with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \cdots, e_m\}$. The distance between two vertices v_i and v_j, denoted by $d(v_i, v_j)$ is the length of the shortest path between the vertices v_i and v_j in G. The shortest $v_i - v_j$ path is often called *geodesic*. The *diameter* $diam(G)$ of a connected graph G is the length of any longest geodesic. The *degree* of a vertex v_i in G is the number of edges incident to v_i and is denoted by $d_i = deg(v_i)$ [2].

The *Wiener index* (or *Wiener number*) [8] of a graph G denoted by $W(G)$ is the sum of distances between all (unordered) pairs of vertices of G.

$$W(G) = \sum_{i<j} d(v_i, v_j).$$

The *Wiener index* $W(G)$ of the graph G is also defined by

$$W(G) = \frac{1}{2} \sum_{v_i, v_j \epsilon V(G)} d(v_i, v_j),$$

where the summation is over all possible pairs $v_i, v_j \in V(G)$.

The *Wiener polarity index* [8] of a graph G denoted by $W_P(G)$ is equal to the number of

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unordered vertex pairs of distance 3 of G. In [8], Wiener used a linear formula of $W(G)$ and $W_P(G)$ to calculate the boiling points t_B of the paraffins, i.e.,

$$t_B = aW(G) + bW_P(G) + c,$$

where a, b and c are constants for a given isomeric group.

Line graphs, total graphs and middle graphs are widely studied transformation graphs. Let $G = (V(G), E(G))$ be a graph. The *line graph* $L(G)$ [11] of G is the graph whose vertex set is $E(G)$ in which two vertices are adjacent if and only if they are adjacent in G.

The *middle graph* $M(G)$ [11] of G is the graph whose vertex set is $V(G) \cup E(G)$ in which two vertices x and y are adjacent if and only if at least one of x and y is an edge of G, and they are adjacent or incident in G. The *quasi-total graph* $P(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent if and only if they correspond to two nonadjacent vertices of G or to two adjacent edges of G or one is a vertex and other is an edge incident with it in G. This concept was introduced in [6]. The *complement* of G, denoted by $\overline{G}$, is the graph with the same vertex set as G, but where two vertices are adjacent if and only if they are nonadjacent in G. We denote the *complement of quasi-total graph* $P(G)$ of G by $\overline{P(G)}$. Its vertex set is $V(G) \cup E(G)$ and two vertices are adjacent if and only if they correspond to two adjacent vertices of G or to two nonadjacent edges of G or one is a vertex and other is an edge nonincident with it in G. In [9], it is interesting to see that the transformation graph G^{-++} is exactly the quasi-total graph $P(G)$ of G, and G^{+--} is the complement of $P(G)$. Many papers are devoted to quasi-total graphs [1, 3, 6, 9, 10].

In the following we denote by C_n, P_n, S_n, W_n and K_n the cycle , the path, the star, the wheel and the complete graph of order n respectively. A complete bipartite graph $K_{a,b}$ has $n = a + b$ vertices and $m = ab$ edges. Other undefined notation and terminology can be found in [2].

The following theorem is useful for proving our main results.

Theorem 1.1([7]) *Let G be connected graph with n vertices and m edges. If $diam(G) \leq 2$, then $W(G) = n(n-1) - m$.*

§2. Results

Theorem 2.1 *If S_n is a star graph of order n, then*

$$W(P(S_n)) = 3n^2 - 5n + 2.$$

Proof If S_n is a star graph with n vertices, m edges and $\sum\limits_{i=1}^{n} d_i^2 = (n-1)^2 + (n-1)$, then $P(S_n)$ has $n_1 = n + m = 2n - 1$ vertices and

$$m_1 = \frac{n(n-1)}{2} + \frac{1}{2}\sum_{i=1}^{n} d_i^2 = n^2 - n$$

edges.

In $P(S_n)$ distance between adjacent vertices is one and distance between nonadjacent vertices is two, therefore $diam(P(S_n)) = 2$.

By Theorem 1.1, $W(P(S_n)) = n_1(n_1 - 1) - m_1$. Hence

$$W(P(S_n)) = (2n-1)(2n-2) - n^2 + n = 3n^2 - 5n + 2. \qquad \square$$

Theorem 2.2 *If K_n is a complete graph of order n, then*

$$W(P(K_n)) = \frac{n(n^3 + n - 2)}{4}.$$

Proof If K_n is a complete graph with n vertices, m edges and $\sum\limits_{i=1}^{n} d_i^2 = n(n-1)^2$, then $P(K_n)$ has $n_1 = n + m = \frac{n^2+n}{2}$ vertices and

$$m_1 = \frac{n(n-1)}{2} + \frac{1}{2}\sum_{i=1}^{n} d_i^2 = \frac{n(n^2-n)}{2}$$

edges.

In $P(K_n)$ distance between adjacent vertices is one and distance between nonadjacent vertices is two, therefore $diam(P(K_n)) = 2$. From Theorem 1.1,

$$\begin{aligned} W(P(K_n)) &= n_1(n_1 - 1) - m_1 \\ &= \frac{n^2+n}{2}[\frac{n^2+n}{2} - 1] - \frac{n(n^2-n)}{2} = \frac{n(n^3+n-2)}{4}. \qquad \square \end{aligned}$$

Theorem 2.3 *If W_n is a wheel graph of order n, then*

$$W(P(W_n)) = 2(4n^2 - 9n + 5).$$

Proof If W_n is a wheel graph with n vertices, m edges and $\sum\limits_{i=1}^{n} d_i^2 = n^2 + 7n - 8$, then $P(W_n)$ has $n_1 = n + m = 3n - 2$ vertices and

$$m_1 = \frac{n(n-1)}{2} + \frac{1}{2}\sum_{i=1}^{n} d_i^2 = n^2 + 3n - 4$$

edges.

In $P(W_n)$ distance between adjacent vertices is one and distance between nonadjacent vertices is two, therefore $diam(P(W_n)) = 2$.

From Theorem 1.1, $W(P(W_n)) = n_1(n_1 - 1) - m_1$. Hence,

$$W(P(W_n)) = (3n-2)(3n-2-1) - (n^2 + 3n - 4) = 2(4n^2 - 9n + 5). \qquad \square$$

Theorem 2.4 *If $K_{a,b}$ is a complete bipartite graph of order $n = a + b$, then*

$$W(P(K_{a,b})) = \frac{(a+b+ab-1)(a+b+2ab)}{2}.$$

Proof If $K_{a,b}$ is a complete bipartite graph with $n = a + b$ vertices, $m = ab$ edges and

$$\sum_{i=1}^{n} d_i^2 = ab(a+b),$$

then $P(K_{a,b})$ has $n_1 = n + m = a + b + ab$ vertices and

$$m_1 = \frac{(n+m)(n+m-1)}{2} + \frac{1}{2}\sum_{i=1}^{n} d_i^2 = \frac{(a+b)(a+b+ab-1)}{2}$$

edges.

In $P(K_{a,b})$ distance between adjacent vertices is one and distance between nonadjacent vertices is two, therefore $diam(P(K_{a,b})) = 2$.

From Theorem 1.1, $W(P(K_{a,b})) = n_1(n_1 - 1) - m_1$. Therefore,

$$\begin{aligned} W(P(K_{a,b})) &= (a+b+ab)(a+b+ab-1) - \frac{(a+b)(a+b+ab-1)}{2} \\ &= \frac{(a+b+ab-1)(a+b+2ab)}{2}. \end{aligned}$$ □

Theorem 2.5 *If P_n is a path of order $n \geqslant 4$, then*

$$W(\overline{P(P_n)}) = \frac{5n^2 - 3n - 4}{2}.$$

Proof If P_n is a path with n vertices, m edges and $\sum_{i=1}^{n} d_i^2 = 4n - 6$, then $\overline{P(P_n)}$ has $n_1 = n + m = 2n - 1$ vertices and

$$m_1 = \binom{n+m}{2} - \frac{n(n-1)}{2} - \frac{1}{2}\sum_{i=1}^{n} d_i^2 = \frac{(n-1)(3n-2) - 2(2n-3)}{2}$$

edges.

In $\overline{P(P_n)}$ distance between adjacent vertices is one and distance between nonadjacent vertices is two, therefore $diam(\overline{P(P_n)}) = 2$.

From Theorem 1.1, $W(\overline{P(P_n)}) = n_1(n_1 - 1) - m_1$. So

$$W(\overline{P(P_n)}) = (2n-1)(2n-2) - \frac{(n-1)(3n-2) - 2(2n-3)}{2} = \frac{5n^2 - 3n - 4}{2}.$$ □

Theorem 2.6 *If S_n is a star of order $n \geqslant 4$, then*

$$W(\overline{P(S_n)}) = 3n(n-1).$$

Proof If S_n is a star with n vertices, m edges and $\sum_{i=1}^{n} d_i^2 = (n-1)^2 + n - 1$, then $\overline{P(S_n)}$ has $n_1 = n + m = 2n - 1$ vertices and $m_1 = \binom{n+m}{2} - \frac{n(n-1)}{2} - \frac{1}{2}\sum_{i=1}^{n} d_i^2 = (n-1)^2$ edges.

As $diam(\overline{P(S_n)}) = 3$. Therefore $W(\overline{P(S_n)}) = n_1(n_1 - 1) - m_1 + W_p(\overline{P(S_n)})$, where $W_p(\overline{P(S_n)})$ is Wiener polarity index of $\overline{P(S_n)}$. Hence,

$$\begin{aligned} W(\overline{P(S_n)}) &= (2n-1)(2n-2) - (n-1)^2 + m \\ &= (2n-1)(2n-2) - (n-1)^2 + n - 1 = 3n(n-1). \end{aligned}$$ □

Theorem 2.7 *If K_n is a complete graph of order $n \geqslant 4$, then*

$$W(\overline{P(K_n)}) = \frac{n(n^3 + 6n^2 - 5n - 2)}{8}.$$

Proof If K_n is a complete graph with n vertices, m edges and $\sum_{i=1}^{n} d_i^2 = n(n-1)^2$, then $\overline{P(K_n)}$ has $n_1 = n + m = \frac{n^2+n}{2}$ vertices and

$$m_1 = \binom{n+m}{2} - \frac{n(n-1)}{2} - \frac{1}{2}\sum_{i=1}^{n} d_i^2 = \frac{n(n^3 - 2n^2 + 3n - 2)}{8}$$

edges.

In $\overline{P(K_n)}$ distance between adjacent vertices is one and distance between nonadjacent vertices is two, therefore $diam(\overline{P(K_n)}) = 2$. From Theorem 1.1,

$$\begin{aligned} W(\overline{P(K_n)}) &= n_1(n_1 - 1) - m_1 \\ &= \frac{n^2+n}{2}\left[\frac{n^2+n}{2} - 1\right] - \frac{n(n^3 - 2n^2 + 3n - 2)}{8} \\ &= \frac{n(n^3 + 6n^2 - 5n - 2)}{8}. \end{aligned}$$ □

Theorem 2.8 *If C_n is a cycle of order $n \geqslant 4$, then*

$$W(\overline{P(C_n)}) = \frac{n(5n+1)}{2}.$$

Proof If C_n is a cycle with n vertices, m edges and $\sum_{i=1}^{n} d_i^2 = 4n$, then $\overline{P(C_n)}$ has

$n_1 = n + m = 2n$ vertices and

$$m_1 = \binom{n+m}{2} - \frac{n(n-1)}{2} - \frac{1}{2}\sum_{i=1}^{n} d_i^2 = \frac{n(3n-5)}{2}$$

edges.

In $\overline{P(C_n)}$ distance between adjacent vertices is one and distance between nonadjacent vertices is two, therefore $diam(\overline{P(C_n)}) = 2$.

From Theorem 1.1, $W(\overline{P(C_n)}) = n_1(n_1 - 1) - m_1$. So,

$$W(\overline{P(C_n)}) = 2n(2n-1) - \frac{n(3n-5)}{2} = \frac{n(5n+1)}{2}. \qquad \square$$

Theorem 2.9 *If $K_{a,b}$ is a complete bipartite graph of order $n = a + b$, then*

$$W(\overline{P(K_{a,b})}) = \frac{(a+b+ab-1)[2(a+b+ab)-ab]}{2}.$$

Proof If $K_{a,b}$ is a complete bipartite graph with $n = a + b$ vertices, $m = ab$ edges and

$$\sum_{i=1}^{n} d_i^2 = ab(a+b),$$

then $\overline{P(K_{a,b})}$ has $n_1 = n + m = a + b + ab$ vertices and

$$m_1 = \binom{n+m}{2} - \frac{(n+m)(n+m-1)}{2} - \frac{1}{2}\sum_{i=1}^{n} d_i^2 = \frac{ab(a+b+ab-1)}{2}$$

edges.

In $\overline{P(K_{a,b})}$ distance between adjacent vertices is one and distance between nonadjacent vertices is two, therefore $diam(\overline{P(K_{a,b})}) = 2$.

By Theorem 1.1,

$$\begin{aligned} W(\overline{P(K_{a,b})}) &= n_1(n_1-1) - m_1 \\ &= (a+b+ab)(a+b+ab-1) - \frac{ab(a+b+ab-1)}{2} \\ &= \frac{(a+b+ab-1)[2(a+b+ab)-ab]}{2}. \end{aligned} \qquad \square$$

Theorem 2.10 *If G is a connected graph of order n, then $W(G) < W(P(G))$.*

Proof If G is graph with n vertices and m edges then $P(G)$ is a quasi-total graph of G with $n + m$ vertices and

$$\frac{n(n-1)}{2} + \frac{1}{2}\sum_{i=1}^{n} d_i^2$$

edges.

Wiener index of graph increases when new vertices are added to the graph G. Therefore $W(G) < W(P(G))$. □

Lemma 2.11 *If G is connected graph of order n, then*

$$3n^2 - 5n + 2 \leq W(P(G)) \leq \frac{n(n^3+n-2)}{4},$$

and the upper bound attain if G is a complete graph and lower bound attain if G is a star graph.

Proof Let $P(G)$ is a quasi-total graph of G with $n+m$ vertices and

$$\frac{n(n-1)}{2} + \frac{1}{2}\sum_{i=1}^{n} d_i^2$$

edges.

G has maximum edges if and only if $G \cong K_n$, $P(G)$ has maximum number of vertices if and only if $G \cong K_n$.

Wiener index of a graph increases when new vertices are added to the graph and $P(K_n)$ has maximum number of vertices compared with any other $P(G)$. Therefore $W(P(G)) \leq W(P(K_n))$.

From Theorem 2.2, $W(P(K_n)) = \frac{n(n^3+n-2)}{4}$. Therefore

$$W(P(G)) \leq \frac{n(n^3+n-2)}{4} \tag{1}$$

with equality holds if and only if $G \cong K_n$.

For any graph G has minimum edges if and only if $G \cong T$ and $P(G)$ has minimum number of vertices if and only if $G \cong T$. Wiener index of a graph increases when new vertices are added to the graph and $P(T)$ has minimum number of vertices compared with any other $P(G)$. Therefore $W(P(T)) \leq W(P(G))$. In the case of tree $W(P(S_n)) \leq W(P(T))$. Therefore $W(P(S_n)) \leq W(P(G))$.

From Theorem 2.1, $W(P(S_n)) = 3n^2 - 5n + 2$. Hence,

$$3n^2 - 5n + 2 \leq W(P(G)) \tag{2}$$

with equality if and only if $G \cong S_n$.

From equations (1) and (2), we get that

$$3n^2 - 5n + 2 \leq W(P(G)) \leq \frac{n(n^3+n-2)}{4}.$$ □

Lemma 2.12 *For any connected graph G of order $n \geq 4$,*

$$\frac{5n^2 - 3n - 4}{2} \leq W(\overline{P(G)}) \leq \frac{n(n^3+6n^2-5n-2)}{8},$$

and the upper bound attain if G is a complete graph and lower bound attain if G is a path.

Proof Let G be connected graph with $n \geq 4$ vertices and m edges. Then $P(G)$ has $n+m$ vertices and

$$\frac{n(n-1)}{2} + \frac{1}{2}\sum_{i=1}^{n} d_i^2$$

edges. $\overline{P(K_n)}$ has $n+m$ vertices and

$$\binom{n+m}{2} - (\frac{n(n-1)}{2} + \frac{1}{2}\sum_{i=1}^{n} d_i^2)$$

edges.

G has maximum edges if and only if $G \cong K_n$, $\overline{P(G)}$ has maximum number of vertices if and only if $G \cong K_n$. Wiener index of a graph increases when new vertices are added to the graph and $\overline{P(K_n)}$ has maximum number of vertices compared to any other $\overline{P(G)}$. Therefore $W(\overline{P(G)}) \leq W(\overline{P(K_n)})$. From Theorem 2.7,

$$W(\overline{P(K_n)}) = \frac{n(n^3+6n^2-5n-2)}{8}.$$

Therefore

$$W(\overline{P(G)}) \leq \frac{n(n^3+6n^2-5n-2)}{8}. \tag{3}$$

For any connected graph G with $n \geq 4$ vertices, G has minimum number of vertices if and only if $G \cong T$. Wiener index of a graph increases when new vertices are added to a graph and $\overline{P(T)}$ has minimum number of vertices compared to any other $\overline{P(G)}$. Thus, $W(\overline{P(T)}) \leq W(\overline{P(G)})$.

In case of tree $W(\overline{P(P_n)}) \leq W(\overline{P(T)})$. Therefore $W(\overline{P(P_n)}) \leq W(\overline{P(G)})$. By Theorem 2.5, $W(\overline{P(P_n)}) = \frac{5n^2-3n-4}{2}$. Therefore

$$\frac{5n^2-3n-4}{2} \leq W(\overline{P(G)}). \tag{4}$$

From equations (3) and (4), we get that

$$\frac{5n^2-3n-4}{2} \leq W(\overline{P(G)}) \leq \frac{n(n^3+6n^2-5n-2)}{8}.$$

□

The following theorem gives the Nordhaus-Gaddum type inequality for Wiener index of quasi-total graph.

Theorem 2.13 *For any graph G with $n \geq 4$,*

$$\frac{n(11n-13)}{2} \leq W(P(G)) + W(\overline{P(G)}) \leq \frac{3n(n^3+2n^2-n-2)}{8}.$$

Proof From Lemmas 2.11 and 2.12, we have

$$3n^2-5n+2+\frac{5n^2-3n-4}{2} \leq W(P(G))+W(\overline{P(G)}) \leq \frac{n^4+n^2-2n}{4}+\frac{n^4+6n^3-5n^2-2n}{8}.$$

Thus,

$$\frac{n(11n-13)}{2} \leq W(P(G))+W(\overline{P(G)}) \leq \frac{3n(n^3+2n^2-n-2)}{8}.$$ □

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Clique Partition of Transformation Graphs

Chandrakala S.B

(Nitte Meenakshi Institute of Technology, Bangalore, India)

K.Manjula

(Bangalore Institute of Technology, Bangalore, India)

E-mail: chandrakalasb@yahoo.co.in, manju_chandru2005@rediffmail.com

Abstract: A *clique* in a graph G is a complete subgraph of G. A *clique partition* of G is a collection C of cliques such that each edge of G occurs in exactly one clique in C. The clique partition number $cp(G)$ is the minimum size of a clique partition of G. In this paper upper bounds for the clique partition number of the transformation graphs G^{++-} and G^{+++} for some standard class of graphs is obtained.

Key Words: Transformation graph, clique, clique partition.

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§1. Introduction

All graphs G considered here are finite, undirected and simple. We refer to [1] for unexplained terminology and notations. In 2001 Wu and Meng introduced some new graphical transformations which generalizes the concept of the total graph. As is the case with the total graph, these generalizations referred to as *transformation graphs* G^{xyz} have $V(G)\cup E(G)$ as the vertex set. The adjacency of two of its vertices is determined by adjacency and incidence nature of the corresponding elements in G.

Let α, β be two elements of $V(G)\cup E(G)$. Then associativity of α and β is taken as $+$ if they are adjacent or incident in G, otherwise $-$. Let xyz be a 3-permutation of the set $\{+,-\}$. The pair α and β is said to correspond to x or y or z of xyz if α and β are both in $V(G)$ or both are in $E(G)$, or one is in $V(G)$ and the other is in $E(G)$ respectively. Thus the *transformation graph* G^{xyz} of G is the graph whose vertex set is $V(G)\cup E(G)$ and two of its vertices α and β are adjacent if and only if their associativity in G is consistent with the corresponding element of xyz.

In particular G^{++-} and G^{+++} are defined as:

Definition 1.1 *The transformation graph G^{++-} of G is the graph with vertex set $V(G)\cup E(G)$ in which the vertices u and v are joined by an edge if one of the following holds*

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(1) *both* $u, v \in V(G)$ *and* u *and* v *are adjacent in* G*;*

(1) *both* $u, v \in E(G)$ *and* u *and* v *are adjacent in* G*;*

(3) *one is in* $V(G)$ *and the other is in* $E(G)$ *and they are not incident with each other in* G.

Definition 1.2 *The transformation graph* G^{+++} *(total graph) of* G *is the graph with vertex set* $V(G) \cup E(G)$ *in which the vertices* u *and* v *are joined by an edge if one of the following holds*

(1) *both* $u, v \in V(G)$ *and* u *and* v *are adjacent in* G*;*

(2) *both* $u, v \in E(G)$ *and* u *and* v *are adjacent in* G*;*

(3) *one is in* $V(G)$ *and the other is in* $E(G)$ *and they are incident with each other in* G.

The transformation graphs are investigated in [2], [3] and [4].

For convenience, the transformation graph G^{xyz} is partitioned into $G^{xyz} = S_x(G) \cup S_y(G) \cup S_z(G)$ where $S_x(G)$, $S_y(G)$ and $S_z(G)$ are the edge-induced subgraphs of G^{xyz}. The edge set of each of which is respectively determined by x, y and z of the permutation xyz. $S_x(G) \cong G$ when x is $+$ and $S_x(G) \cong \overline{G}$ when x is $-$. $S_y(G) \cong L(G)$ when y is $+$ and $S_y(G) \cong \overline{L(G)}$ when y is $-$. When z is $+$, $\alpha, \beta \in V(G^{xyz})$ are adjacent in $S_z(G)$ if they are incident with each other in G. When z is $-$, α, β are adjacent in $S_z(G)$ if they are not incident in G.

A *clique partition* of G is a collection C of cliques such that each edge of G occurs in exactly one clique in C. The clique partition number $cp(G)$ is the minimum size of a clique partition of G.

In this paper the upper bounds for clique partition number of transformation graphs G^{++-} and G^{+++} of some class of graphs such as path, cycle, star, wheel, etc, are obtained.

§2. Clique Partition of P_n^{++-} and C_n^{++-}

We note that the size of P_n^{++-} and C_n^{++-} are $n^2 - n - 1$ and n^2 respectively; the clique numbers of P_n^{++-} and C_n^{++-} is 4. Therefore no clique partition of P_n^{++-} and C_n^{++-} can contain $K_t (t \geqslant 5)$.

Theorem 2.1 *For a path* P_n $(n \geq 8)$*,* $cp(P_n^{++-}) \leq n^2 - 6n + 7$.

Proof Consider the path $P_n : v_1 - v_2 - v_3 - \cdots - v_n$. Let $e_i = v_i v_{i+1} (1 \leq i \leq n-1)$ be the edges of P_n. The edge set of P_n^{++-} is partitioned into K_4, K_3 and K_2's. Vertex sets of K_4's and K_3's are listed as elements of the sets B_j.

When $n \equiv 0 (mod\ 4)$,

$B_1 = \{\{v_i,\ v_{i+1},\ e_{i+2},\ e_{i+3}\} : i = 1, 3, 5, \cdots, \frac{n}{2} - 1\}$,

$B_2 = \{\{v_i,\ v_{i+1},\ e_{i-3},\ e_{i-2}\} : i = \frac{n}{2} + 1,\ \frac{n}{2} + 3, \cdots, n-3,\ n-1\}$,

$B_3 = \{\{v_i,\ v_{i+1},\ e_{(\frac{n}{2}+1+i)},\ e_{(\frac{n}{2}+2+i)}\} : i = 2, 4, 6, \cdots, \frac{n}{2} - 4\}$,

$B_4 = \{\{v_i,\ v_{i+1},\ e_{(i-\frac{n}{2}-2)},\ e_{(i-\frac{n}{2}-1)}\} : i = \frac{n}{2} + 4,\ \frac{n}{2} + 6, \cdots, n-2\}$,

$B_5 = \{\{v_{(\frac{n}{2}+2)},\ v_{(\frac{n}{2}+3)},\ e_1,\ e_2\},\ \{v_{(\frac{n}{2}-2)},\ v_{(\frac{n}{2}-1)},\ e_{n-2}, e_{n-1}\},\ \{v_{(\frac{n}{2})},\ v_{(\frac{n}{2}+1)}, e_1\}\}$.

When $n \equiv 2 (mod\ 4)$,

$B_1 = \{\{v_i,\ v_{i+1},\ e_{i+2},\ e_{i+3}\} : i = 1, 3, 5, \cdots, \frac{n}{2} - 2\}$,
$B_2 = \{\{v_i,\ v_{i+1},\ e_{i-3},\ e_{i-2}\} : i = n-1,\ n-3,\ n-5,\ \cdots, \frac{n}{2} - 2\}$,
$B_3 = \{\{v_i,\ v_{i+1},\ e_{(\frac{n}{2}+i)},\ e_{(\frac{n}{2}+i+1)}\} : i = 2, 4, 6, \cdots, \frac{n}{2} - 3\}$,
$B_4 = \{\{v_{(\frac{n}{2}+i)},\ v_{(\frac{n}{2}+i+1)},\ e_{i-1},\ e_i\} : i = 3, 5, 7, \cdots, \frac{n}{2} - 2\}$,
$B_5 = \{\{v_{(\frac{n}{2}-1)},\ v_{(\frac{n}{2})},\ e_1,\ e_2\},\ \{v_{(\frac{n}{2}+1)},\ v_{(\frac{n}{2}+2)},\ e_{n-2},\ e_{n-1}\},\ \{v_{(\frac{n}{2})},\ v_{(\frac{n}{2}+1)},\ e_3\}\}$

When $n \equiv 1, 3(mod\ 4)$,

$B_1 = \{\{v_i,\ v_{i+1},\ e_{i+2},\ e_{i+3}\} : i = 1, 3, 5 \cdots, n-2\}$,
$B_2 = \{\{v_i,\ v_{i+1},\ e_{i-6},\ e_{i-5}\} : i = n-3,\ n-5,\ n-7, \cdots, 12, 10, 8\}$,
$B_3 = \{\{v_{n-1},\ v_n,\ e_{n-5},\ e_{n-4}\},\ \{v_{n-2},\ v_{n-1},\ e_{n-7},\ e_{n-8}\},\ \{v_6,\ v_7,\ e_1,\ e_2\},$
$\{v_2,\ v_3,\ e_{n-3},\ e_{n-2}\},\ \{v_4,\ v_5,\ e_1\}\}$

In each case there are $n-2$ K_4's and one K_3. These cover all the edges of S_x, S_y and $4n-6$ edges of S_z. The remaining $(n^2 - 7n + 8)$ edges of S_z are covered by K_2's.

Therefore $P_n^{++-} = (n-2)K_4 \cup K_3 \cup (n^2 - 7n + 8)K_2$ and $cp(P_n^{++-}) \leq n^2 - 6n + 7$. □

Theorem 2.2 *For a cycle C_n $(n \geq 8)$, $cp(C_n^{++-}) \leq n^2 - 5n$.*

Proof Consider the cycle $C_n : v_1 - v_2 - v_3 - \cdots - v_n - v_1$. Let $e_i = v_i v_{i+1}$ $(1 \leq i \leq n-1)$ and $e_n = v_n v_1$ be the edges of C_n. Edge set of C_n^{++-} is partitioned into K_4's and K_2's. Vertex sets of K_4's are listed as elements of the sets B_j as follows:

When n is even,

$B_1 = \{\{v_i,\ v_{i+1},\ e_j,\ e_k\}:$ for each $i = 1, 3, 5, \cdots, n-3,\ n-1,\ j \equiv i + 2(mod\ n)$ and $k \equiv i + 3(mod\ n)\}$,

$B_2 = \{\{v_i,\ v_j,\ e_k,\ e_l\}\}$ for each $i = 2, 4, 6, \cdots, n-4,\ n-2,\ j \equiv i + 1(mod\ n)\}$ with

$$k \equiv \begin{cases} \frac{n}{2} + 1 + i(mod\ n) & \text{when } \frac{n}{2} \text{ is odd} \\ \frac{n}{2} + i(mod\ n) & \text{when } \frac{n}{2} \text{ is even} \end{cases} \quad \text{and } l \equiv k + 1(mod\ n)$$

When n is odd,

$B_1 = \{\{v_i,\ v_{i+1},\ e_{i+2},\ e_j\}:$ for each $i = 1, 3, 5, \cdots, n-4,\ n-2$ $j \equiv i + 3(mod\ n)\}$,

$B_2 = \{v_i,\ v_{i+1},\ e_j,\ e_k\}:$ for each $i = 2, 4, 6, \cdots, n-7,\ n-5,$ $j \equiv i + 6(mod\ n)$ $k \equiv i + 7(mod\ n)\}$,

$B_3 = \{\{v_n,\ v_1,\ e_6,\ e_7\},\ \{v_{n-3},\ v_{n-2},\ e_2,\ e_3\}, \{v_{n-1},\ v_n,\ e_4,\ e_5\}\}$.

In these sets v_0 and e_0 are taken as v_n and e_n respectively.

In both the cases there are nK_4's. These cover all the edges of S_x, S_y and some edges of S_z. Remaining edges of S_z are listed as K_2's. Therefore $C_n^{++-} = nK_4 \cup (n^2 - 6n)K_2$ and $cp(C_n^{++-}) \leq n^2 - 5n$. □

§3. Clique Partition of G^{++-} with G isomorphic to Comb or Sunlet graphs

The Comb graph $G \cong P_n \odot K_1$ is the graph with path on n vertices and each vertex of path is adjacent to a pendant vertex. The Sunlet graph $S_n \cong C_n \odot K_1$ is a graph with the cycle on n

vertices and each vertex of the cycle is adjacent to a pendant vertex.

For a comb graph $G \cong P_n \odot K_1$, let $v_i (1 \leq i \leq n)$ denote the vertices of P_n with v_1 and v_n as its end vertices and $e_i = v_i v_{i+1}$ be the edges of P_n and v'_i be the pendant vertices adjacent to each of v_i and $e'_i = v_i v'_i$ be the pendant edges of G. We note that order and size of $V(G^{++-})$ is $4n-1$ and $4n^2 - n - 3$ respectively and the clique number is 5.

For the sunlet graph $S_n \cong C_n \odot K_1$, let $v_i (1 \leq i \leq n)$ denote the vertices of C_n and v'_i be the pendant vertex adjacent to v_i, $e_i = v_i v_{i+1}$ be the n edges of C_n and $e'_i = v_i v'_i$ be the pendant edges of S_n. We note that order and size of S_n^{++-} is $4n$ and $4n^2 + n$ respectively and the clique number is 5.

Theorem 3.1 *Let $G \cong P_n \odot K_1 (n \geq 6)$ be the comb graph. Then $cp(G^{++-}) \leq 4n^2 - 12n + 7$.*

Proof Consider the comb $G \cong P_n \odot K_1$. Edge set of G^{++-} is partitioned into K_5, K_4, K_3 and K_2's. The vertex sets of these cliques are listed as elements of sets B_j are given below:

$B_1 = \{\{v_i,\ v'_i,\ e_{i+1},\ e'_{i+2}\} : i = 1, 2, 3, \cdots, n-3\}$,

$B_2 = \{\{v_{n-2},\ v'_{n-2},\ e_{n-1},\ e'_n\},\ \{v_{n-1},\ v'_{n-2},\ e_1,\ e'_1\},\ \{v_n,\ v'_n,\ e_1,\ e_2,\ e'_2\}\}$,

$B_3 = \{\{\{v_i,\ v_{i+1},\ e_j\} : i = 1, 2, 3, \cdots, n-5\ ;\ j = i+4\}, \{\{v_i,\ v_{i+1},\ e_j\} : i = n-4, n-3, n-2, n-1;\ j = i-(n-5)\}\}$.

The sets B_1, B_2 and B_3 cover all the edges of S_x, S_y and some edges of S_z while remaining edges of S_z are covered by K_2's.

$B_4 = \{\{\{v_i,\ e'_j\}$: for each $1 \leq i \leq n; 1 \leq j \leq n$ and $j \neq i+2\}$, $\{\{v_1,\ e_i\} : i = 4\ \ 6 \leq i \leq n-1\}$, $\{\{v_n,\ e_i\} : i = 3\ \ 5 \leq i \leq n-2\}$, $\{v_{n-2},\ e_i\} :\ i = 1\ \ 4 \leq i \leq n-4\}$, $\{\{v_{n-1},\ e_i\} : 4 \leq i \leq n-3\}$, $\{\{v_i,\ e_j\}$: for each $2 \leq i \leq n-3; 1 \leq j \leq n-1$ *and* $j \neq i-1, i, i+1, i+2, i+3, i+4\}\}$,

$B_5 = \{\{v'_i,\ e_j\}$: for each $1 \leq i \leq n-3; 1 \leq j \leq n-1$ and $j \neq i+1, i+2\}$, $\{\{v'_{n-2},\ e_i\} :$ $i = n$ and $1 \leq i \leq n-2\}$, $\{\{v'_{n-1},\ e_i\} : 2 \leq i \leq n-1\}$, $\{\{v'_n,\ e_i\} : 3 \leq i \leq n-1\}$, $\{\{v'_i,\ e'_j\} :$ for each $1 \leq i \leq n-2; 1 \leq j \leq n$ and $j \neq i, i+2\}$, $\{\{v'_{n-1},\ e'_j\} : i = n\ \ 2 \leq i \leq n-2\}$, $\{\{v'_n,\ e_i\} :$ $i = 1\ \ 3 \leq i \leq n-1\}\}$.

Thus, $G^{++-} = (n-2)K_5 \cup 2K_4 \cup (n-1)K_3 \cup (4n^2 - 14n + 8)K_2$ and hence $cp(G^{++-}) \leq 4n^2 - 12n + 7$. □

Theorem 3.2 *For $S_n \cong C_n \odot K_1 (n \geq 6)$ a sunlet graph, $cp(S_n^{++-}) \leq 4n^2 - 10n$.*

Proof Consider the sunlet graph $S_n \cong C_n \odot K_1$. Edge set of S_n^{++-} is partitioned into K_5, K_3 and K_2's where,

$B_1 = \{\{v_i,\ v'_i,\ e_j,\ e_k,\ e'_k\}$: for each $1 \leq i \leq n,\ j \equiv i+1 (mod\ n),\ k \equiv i+2 (mod\ n)\}$,

$B_2 = \{\{v_i,\ v_j,\ e_k\}$: for each $1 \leq i \leq n,\ j \equiv i+1 (mod\ n),\ k \equiv i+4 (mod\ n)\}$,

$B_3 = \{\{v_i,\ e'_j\},\ \{v'_i,\ e'_j\}$: for each $1 \leq i \leq n,\ 1 \leq j \leq n$ and $j \neq i, i+2 (mod\ n)\} \cup$ $\{\{v'_i,\ e_j\}$: *for each* $1 \leq i \leq n,\ 1 \leq j \leq n-1$ and $j \neq i+1, i+2 (mod\ n)\} \cup$ $\{\{v_i,\ e_j\}$: for each $1 \leq i \leq n,\ 1 \leq j \leq n-1$ and $j \neq i-1, i, i+1, i+2, i+3, i+4 (mod\ n)\}$.

Thus, $S_n^{++-} = nK_5 \cup nK_3 \cup (4n^2 - 12n)K_2$ and $cp(S_n^{++-}) \leq 4n^2 - 10n$. □

§4. Clique Partition of Transformation Graphs $K_{1,n}^{++-}$ and W_{n+1}^{++-}

For the star graph $K_{1,n}$, let v_0 be the central vertex, $v_i(1 \leq i \leq n)$ be the pendant vertices and $e_i = v_0v_i$ be the pendant edges. We note that $|V(K_{1,n}^{++-})| = 2n+1$, $|E(K_{1,n}^{++-})| = n(3n-1)/2$ and the clique number is n.

For the wheel graph $W_{n+1} = C_n + K_1$, let v_0 be the central vertex, v_i be the vertices, $e_i = v_0v_i(1 \leq i \leq n)$ be the spokes and $e'_i = v_iv_j(1 \leq i \leq n,\ j = i+1(mod\ n))$ be the hubs of W_{n+1}. Then, $V(W_{n+1}^{++-}) = V(W_{n+1}) \cup E(W_{n+1})$, $|V(W_{n+1}^{++-})| = 3n+1$, $|E(W_{n+1}^{++-})| = 5n(n+1)/2$ and the clique number is n.

Theorem 4.1 *For $n \geq 3$, $cp(K_{1,n}^{++-}) \leq n^2+1$.*

Proof Here $S_y = L(K_{1,n}) \cong K_n$. The clique K_n covers all the edges of S_y; $S_x = K_{1,n}$ and $S_z = nK_{1,n-1}$, which are covered by $n + n(n-1)$ K'_2s.

$\{\{v_0,\ v_i\} : 1 \leq i \leq n\}$ and

$\{\{v_i,\ e_j\} :\ for\ each\ 1 \leq i \leq n,\ 1 \leq j \leq n\ and\ j \neq i\}$

Therefore, $K_{1,n}^{++-} = K_n \cup n^2K_2$ and hence $cp(K_{1,n}^{++-}) \leq n^2+1$. □

Theorem 4.2 *For $n \geq 6$, $cp(W_{n+1}^{++-}) \leq 2n^2-6n+1$.*

Proof The edge set of W_{n+1}^{++-} is partitioned into a K_n, n K_4's, $2n$ K_3's and $(2n^2-9n)$ K_2's. Here,

$B_1 = \{\{e_1,\ e_2,\ e_3,\ \cdots,e_{n-1},\ e_n\}\}$,

$B_2 = \{\{v_i,\ e'_i,\ e'_k,\ e_k\} :$ for each $1 \leq i \leq n,\ j \equiv i+1(mod\ n),\ k \equiv i+2(mod\ n)\}$,

$B_3 = \{\{v_0,\ v_i,\ e'_j\} :$ for each $1 \leq i \leq n,\ j =\equiv i+3(mod\ n)\}$,

$B_4 = \{\{v_i,\ v_j,\ e'_k\} :$ for each $1 \leq i \leq n,\ j \equiv i+1(mod\ n),\ k \equiv i+5(mod\ n)\}$,

$B_5 = \{\{v_i,\ e_j\} :$ for each $1 \leq i \leq n,\ 1 \leq j \leq n$ and $j \neq i, i+2(mod\ n)\}$ $\cup$ $\{\{v_i,\ e'_j\} :$ for each $1 \leq i \leq n,\ 1 \leq j \leq n$ and $j \neq i-1, i, i+1, i+2, i+3, i+4, i+5(mod\ n)\}$.

(In the above sets v_0, e_0 and e'_0 are taken as v_n, e_n and e'_n.)

Thus $(W_{n+1}^{++-}) = K_n \cup nK_4 \cup 2nK_3 \cup (2n^2-9n)K_2$ and hence $cp(W_{n+1}^{++-}) \leq 2n^2-6n+1$.□

§5. Clique Partition of Transformation Graphs P_n^{+++}, C_n^{+++}, $K_{1,n}^{+++}$, W_{n+1}^{+++} and K_n^{+++}

Theorem 5.1 *For $n \geq 3$, $cp(P_n^{+++}) \leq 2n-3$.*

Proof Consider the path $P_n : v_1 - v_2 - v_3 - \cdots - v_n$. Let $e_i = v_iv_{i+1}$ be the edges of P_n. We note that order, size and clique number of P_n^{+++} are $2n-1$, $4n-5$ and 3 respectively. The edges of subgraphs S_x and S_z are partitioned into K'_3s and that of S_y by K'_2s:

$\{\{v_i,\ v_{i+1},\ e_i\} : 1 \leq i \leq n-1\}$ and $\{\{e_j,\ e_{j+1}\} : 1 \leq j \leq n-2\}$.

Therefore, $P_n^{+++} = (n-1)K_3 \cup (n-2)K_2$ and $cp(P_n^{+++}) \leq 2n-3$. □

Theorem 5.2 *For* $n \geq 3$, $cp(C_n^{+++}) \leq 2n$.

Theorem 5.3 *For* $n \geq 3$, $cp(K_{1,n}^{+++}) \leq n+1$.

Theorem 5.4 *For* $n \geq 6$, $cp(W_{n+1}^{+++}) \leq 3n+1$.

Proof The order, size and clique number of W_{n+1}^{+++} are $3n+1$, $(n^2+17n)/2$ and $n+1$. The edge set of W_{n+1}^{+++} is partitioned into a K_n, $3nK_3's$. Here,

$B_1 = \{\{e_1,\ e_2,\ e_3,\ \cdots,\ e_{n-1},\ e_n\}\}$,
$B_2 = \{\{e_i,\ e_i',\ e_j'\}:$ for each $1 \leq i \leq n,\ j \equiv i-1(mod\ n)\}$,
$B_3 = \{\{v_i,\ v_j,\ e_i'\},\ \{v_0,\ v_i,\ e_i\}:$ for each $1 \leq i \leq n,\ j \equiv i+1(mod\ n)\}$.

Here each edge of subgraphs S_x and S_z are present in exactly one clique of B_3 and each edge of S_y is in exactly one clique of B_1 or B_2. Thus, $W_{n+1}^{+++} = K_n \cup 3nK_3$ and hence $cp(W_{n+1}^{+++}) \leq 3n+1$. □

Theorem 5.5 *For* $n \geqslant 4$, $cp(K_n^{+++}) \leq n+1$.

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Probabilistic Bounds
On Weak and Strong Total Domination in Graphs

M.H.Akhbari

(Department of Mathematics, Estahban Branch, Islamic Azad University, Estahban, Iran)

E-mail: mhakhbari20@gmail.com, akhbari@iauest.ac.ir

Abstract: A set D of vertices in a graph $G = (V, E)$ is a total dominating set if every vertex of G is adjacent to some vertex in D. A total dominating set D of G is said to be weak if every vertex $v \in V - D$ is adjacent to a vertex $u \in D$ such that $d_G(v) \geq d_G(u)$. The weak total domination number $\gamma_{wt}(G)$ of G is the minimum cardinality of a weak total dominating set of G. A total dominating set D of G is said to be strong if every vertex $v \in V - D$ is adjacent to a vertex $u \in D$ such that $d_G(v) \leq d_G(u)$. The strong total domination number $\gamma_{st}(G)$ of G is the minimum cardinality of a strong total dominating set of G. We present probabilistic upper bounds on weak and strong total domination number of a graph.

Key Words: Weak total domination, strong total domination, pigeonhole property, probability.

AMS(2010): 05C69.

§1. Introduction

We consider finite, undirected, simple graphs. Let G be a graph, with vertex set V and edge set E. The *open neighborhood* of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. For a subset $S \subseteq V$, the *open neighborhood* is $N(S) = \cup_{v \in S} N(v)$ and the *closed neighborhood* is $N[S] = N(S) \cup S$. If v is a vertex of V, then the *degree* of v denoted by $d_G(v)$, is the cardinality of its open neighborhood. By $\Delta(G) = \Delta$ and $\delta(G) = \delta$ we denote the *maximum* and *minimum degree* of a graph G, respectively. A subset $S \subseteq V$ is a *dominating set* of G if every vertex in $V - S$ has a neighbor in S and is a *total dominating set* (td-set) if every vertex in V has a neighbor in S. The *domination number* $\gamma(G)$ (respectively, *total domination number* $\gamma_t(G)$) is the minimum cardinality of a dominating set (respectively, total dominating set) of G. Total domination was introduced by Cockayne, Dawes and Hedetniemi [2].

In [10], Sampathkumar and Pushpa Latha have introduced the concept of weak and strong domination in graphs. A subset $D \subseteq V$ is a *weak dominating set* (wd-set) if every vertex $v \in V - S$ is adjacent to a vertex $u \in D$, where $d_G(v) \geq d_G(u)$. The subset D is a *strong dominating set* (sd-set) if every vertex $v \in V - S$ is adjacent to a vertex $u \in D$, where $d_G(u) \geq$

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$d_G(v)$. The *weak* (*strong*, respectively) *domination number* $\gamma_w(G)$ ($\gamma_s(G)$, respectively) is the minimum cardinality of a wd-set (an sd-set, respectively) of G. Strong and weak domination have been studied for example in [4, 5, 7, 8, 9]. For more details on domination in graphs and its variations, see [6].

Chellali et al. [3] have introduced the concept of weak total domination in graphs. A total dominating set D of G is said to be *weak* if every vertex $v \in V - D$ is adjacent to a vertex $u \in D$ such that $d_G(v) \geq d_G(u)$. The *weak total domination number* $\gamma_{wt}(G)$ of G is the minimum cardinality of a weak total dominating set of G. The concept of *strong total domination* can be defined analogously. A total dominating set D of G is said to be *strong* if every vertex $v \in V - D$ is adjacent to a vertex $u \in D$ such that $d_G(v) \leq d_G(u)$. The *strong total domination number* $\gamma_{st}(G)$ of G is the minimum cardinality of a strong total dominating set of G.

We obtain probabilistic upper bounds on weak and strong total domination number of a graph.

§2. Results

We adopt the notations of [1]. Let $W = W(G)$ be the set of all vertices $v \in V(G)$ such that $deg(v) < deg(u)$ for every $u \in N(v)$. Note that $W(G)$ may be empty, and if $W(G) \neq \emptyset$, then $W(G)$ is independent and is contained in every weak total dominating set of G. For any vertex $v \in V(G)$ let $\deg_w(v) = \{u \in N(v) | \deg(u) \leq \deg(v)\}$.

Theorem 2.1 *Let G be a graph with $V(G) - N[W] \neq \emptyset$. If $\delta_w = \min\{\deg_w(v) | v \in V(G) - N[W]\}$, then*

$$\gamma_{wt}(G) \leq 2|W| + 2(n - |W|)\left(1 - \frac{\delta_w}{(1+\delta_w)^{1+\frac{1}{\delta_w}}}\right).$$

Proof For each vertex $w \in W$ consider a vertex $w' \in N(w)$, and let $W' = \{w' | w \in W\}$. Clearly $|W'| \leq |W|$. Let A be a set formed by an independent choice of vertices of $G - W$, where each vertex is selected with probability

$$p = 1 - \left(\frac{1}{1+\delta_w}\right)^{\frac{1}{\delta_w}}.$$

Let $B \subseteq V(G) - (A \cup N[W])$ be the set of vertices that have not a weak neighbor in A. Clearly $E(|A|) \leq (n - |W|)p$. Each vertex of B has at least δ_w weak neighbors in $V(G) - W$. It is easy to show that

$$Pr(v \in B) = (1-p)^{1+\deg_w(v)} \leq (1-p)^{1+\delta_w}.$$

Thus $E(|B|) \leq (n - |W|)(1-p)^{\delta_w+1}$. For each $a \in A$ let $a' \in N(a)$, and let $A' = \{a' | a \in A\}$. Similarly for each $b \in B$ let $b' \in N(b)$, and let $B' = \{b' | b \in B\}$. Then clearly $|A'| \leq |A|$ and $|B'| \leq |B|$. It is obvious that $D = W \cup W' \cup A \cup A' \cup B \cup B'$ is a weak total dominating set for

G. The expectation of $|D|$ is

$$\begin{aligned} E(|D|) &\leq 2E(|W|) + 2E(|A|) + 2E(|B|) \\ &\leq 2|W| + 2(n - |W|)p + 2(n - |W|)(1-p)^{\delta_w + 1} \\ &\leq 2|W| + 2(n - |W|)\left(1 - \frac{\delta_w}{(1+\delta_w)^{1+\frac{1}{\delta_w}}}\right). \end{aligned}$$

By the pigeonhole property of expectation there exists a desired weak total dominating set. □

The proof of Theorem 2.1 implies the following upper bound, which is asymptotically same as the bound of Theorem 2.1.

Corollary 2.2 *Let G be a graph with $V(G) - N[W] \neq \emptyset$. If $\delta_w = \min\{\deg_w(v) | v \in V(G) - N[W]\}$, then*

$$\gamma_{wt}(G) \leq 2|W| + 2(n - |W|)\left(\frac{1 + \ln(\delta_w + 1)}{\delta_w + 1}\right).$$

Proof We use the proof of Theorem 2.1. Using the inequality $1 - p \leq e^{-p}$ we obtain that

$$\begin{aligned} E(|D|) &\leq 2|W| + 2(n - |W|)p + 2(n - |W|)(1-p)^{\delta_w + 1} \\ &\leq 2|W| + 2(n - |W|)p + 2(n - |W|)e^{-p(\delta_w + 1)}. \end{aligned}$$

If we put $p = \dfrac{\ln(1+\delta_w)}{1+\delta_w}$ then

$$E(|D|) \leq 2|W| + 2(n - |W|)\left(\frac{1 + \ln(\delta_w + 1)}{\delta_w + 1}\right).$$

By the pigeonhole property of expectation there exists a desired weak total dominating set. □

Next we obtain probabilistic upper bounds for strong total domination number. Let $S = S(G)$ be the set of all vertices $v \in V(G)$ such that $deg(v) > deg(u)$ for every $u \in N(v)$. Note that $S(G)$ may be empty, and if $S(G) \neq \emptyset$, then $S(G)$ is independent and is contained in every strong total dominating set of G. For any vertex $v \in V(G)$ let $\deg_s(v) = \{u \in N(v) | \deg(u) \geq \deg(v)\}$. The following can be proved similar to Theorem 2.1 and Corollary 2.2, and thus we omit the proofs.

Theorem 2.3 *Let G be a graph with $V(G) - N[S] \neq \emptyset$. If $\delta_s = \min\{\deg_s(v) | v \in V(G) - N[S]\}$, then*

$$\gamma_{st}(G) \leq 2|S| + 2(n - |S|)\left(1 - \frac{\delta_s}{(1+\delta_s)^{1+\frac{1}{\delta_s}}}\right).$$

Corollary 2.4 *Let G be a graph with $V(G) - N[S] \neq \emptyset$. If $\delta_s = \min\{\deg_s(v) | v \in V(G) - N[S]\}$,* *then*

$$\gamma_{st}(G) \leq 2|S| + 2(n - |S|)\left(\frac{1 + \ln(\delta_s + 1)}{\delta_s + 1}\right).$$

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Quotient Cordial Labeling of Graphs

R.Ponraj

(Department of Mathematics, Sri Paramakalyani College, Alwarkurichi-627412, India)

M.Maria Adaickalam

(Department of Economics and Statistics, Government of Tamilnadu, Chennai- 600 006, India)

R.Kala

(Department of Mathematics, Manonmaniam Sundaranar University, Tirunelveli-627012, India)

E-mail: ponrajmaths@gmail.com, mariaadaickalam@gmail.com, karthipyi91@yahoo.co.in

Abstract: In this paper we introduce quotient cordial labeling of graphs. Let G be a (p,q) graph. Let $f : V(G) \to \{1,2,\cdots,p\}$ be a $1-1$ map. For each edge uv assign the label $\left[\frac{f(u)}{f(v)}\right]$ (or) $\left[\frac{f(v)}{f(u)}\right]$ according as $f(u) \geq f(v)$ or $f(v) > f(u)$. f is called a quotient cordial labeling of G if $|e_f(0) - e_f(1)| \leq 1$ where $e_f(0)$ and $e_f(1)$ respectively denote the number of edges labelled with even integers and number of edges labelled with odd integers. A graph with a quotient cordial labeling is called a quotient cordial graph. We investigate the quotient cordial labeling behavior of path, cycle, complete graph, star, bistar etc.

Key Words: Path, cycle, complete graph, star, bistar, quotient cordial labeling, Smarandachely quotient cordial labeling.

AMS(2010): 05C78.

§1. Introduction

Graphs considered here are finite and simple. Graph labeling is used in several areas of science and technology like coding theory, astronomy, circuit design etc. For more details refer Gallian [2]. The union of two graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. Cahit [1], introduced the concept of cordial labeling of graphs. Recently Ponraj et al. [4], introduced difference cordial labeling of graphs. Motivated by these labelings we introduce quotient cordial labeling of graphs. Also in this paper we investigate the quotient cordial labeling behavior of path, cycle, complete graph, star, bistar etc. In [4], Ponraj et al. investigate the quotient cordial labeling behavior of subdivided star $S(K1,n)$, subdivided bistar $S(B_{n,n})$ and union of some star related graphs. $[x]$ denote the smallest integer less than or equal to x. Terms are not defined here follows from Harary [3].

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§2. Quotient Cordial Labeling

Definition 2.1 *Let G be a (p,q) graph. Let $f : V(G) \to \{1, 2, \cdots, p\}$ be an injective map. For each edge uv assign the label $\left[\frac{f(u)}{f(v)}\right]$ (or) $\left[\frac{f(v)}{f(u)}\right]$ according as $f(u) \geq f(v)$ or $f(v) > f(u)$. Then f is called a quotient cordial labeling of G if $|e_f(0) - e_f(1)| \leq 1$ where $e_f(0)$ and $e_f(1)$ respectively denote the number of edges labelled with even integers and number of edges labelled with odd integers. A graph with a quotient cordial labeling is called a quotient cordial graph.*

Generally, a Smarandachely quotient cordial labeling of G respect to $S \subset V(G)$ is such a labelling of G that it is a quotient cordial labeling on $G \setminus S$. Clearly, a quotient cordial labeling is a Smarandachely quotient cordial labeling of G respect to $S = \emptyset$.

A simple example of quotient cordial graph is given in Figure 1.

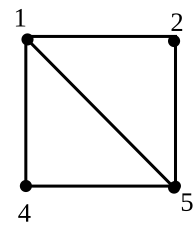

Figure 1.

§3. Main Results

First we investigate the quotient cordial labeling behavior of path.

Theorem 3.1 *Any path is quotient cordial.*

Proof Let P_n be the path $u_1u_2\cdots u_n$. Assign the label 1 to u_1. Then assign $2, 4, 8, \cdots$ $(\leq n)$ to the consecutive vertices until we get $\left[\frac{n-1}{2}\right]$ edges with label 0, then choose the least number $\leq n$ that is not used as a label. That is consider the label 3. Assign the label to the next non labelled vertices consecutively by $3, 6, 12, \ldots (\leq n)$ until we get $\left[\frac{n-1}{2}\right]$ edges with label 0. If not, consider the next least number $\leq n$ that is not used as a label. That is choose 5. Then label the vertices $5, 10, 20, \cdots$ $(\leq n)$ consecutively. If the total number of edges with label 0 is $\left[\frac{n-1}{2}\right]$, then stop this process, otherwise repeat the same until we get the $\left[\frac{n-1}{2}\right]$ edges with label 0. Let S be the set of integer less than or equal to n that are not used as a label. Let t be the least integer such that u_t is not labelled. Then assign the label to the vertices $u_t, u_{t+1}, \cdots, u_n$ from the set S in descending order. Clearly the above vertex labeling is a quotient cordial labeling. □

Illustration 3.2 A quotient cordial labeling of P_{15} is given in Figure 2.

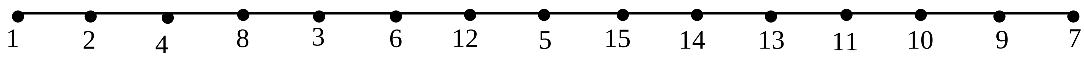

Figure 2

Here, $S=\{5,7,9,10,11,13,14,15\}$

Corollary 3.3 *If n is odd then the cycle C_n is quotient cordial.*

Proof The quotient cordial labeling of path P_n, n odd, given in Theorem 3.1 is obviously a quotient cordial labeling of the cycle C_n. □

Next is the complete graph.

Theorem 3.4 *The complete graph K_n is quotient cordial iff $n \leq 4$.*

Proof Obviously $K_n, n \leq 4$ is quotient cordial. Assume $n > 4$. Suppose f is a quotient cordial labeling of K_n.

Case 1. n is odd.

Consider the sets,

$$
\begin{aligned}
S_1 &= \left\{\left[\frac{n}{n-1}\right], \left[\frac{n}{n-2}\right], \ldots, \left[\frac{n}{\frac{n+1}{2}}\right]\right\} \cup \left\{\left[\frac{n}{1}\right]\right\} \\
S_2 &= \left\{\left[\frac{n-2}{n-3}\right], \left[\frac{n-2}{n-4}\right], \ldots, \left[\frac{n-2}{\frac{n-1}{2}}\right]\right\} \cup \left\{\left[\frac{n-2}{1}\right]\right\} \\
&\vdots \\
S_{\frac{n-1}{2}} &= \left\{\left[\frac{3}{2}\right]\right\} \cup \left\{\left[\frac{3}{1}\right]\right\}
\end{aligned}
$$

Clearly, S_1 contains $\frac{n+1}{2}$ integers. S_2 contains $\frac{n-1}{2}$ integers. S_3 contains $\frac{n-3}{2}$ integers. $\ldots$, $S_{\frac{n-2}{2}}$ contains 2 integers. Each S_i obviously contributes edges with label 1.T herefore

$$
\begin{aligned}
e_f(1) &\geq |S_1| + |S_2| + \ldots + \left|S_{\frac{n-1}{2}}\right| \\
&= \frac{n+1}{2} + \frac{n-1}{2} + \frac{n-3}{2} + \ldots + 2 \\
&= 2 + 3 + \ldots + \frac{n+1}{2} \\
&= \left[1 + 2 + 3 + \ldots + \frac{n+1}{2}\right] - 1 \\
&= \frac{\left(\frac{n+1}{2}\right)\left(\frac{n+1}{2}+1\right)}{2} - 1 = \frac{(n+1)(n+3)}{8} - 1 \qquad (1)
\end{aligned}
$$

Next consider the sets,

$$\begin{aligned} S'_1 &= \left\{ \left[\frac{n-1}{n-2}\right], \left[\frac{n-1}{n-3}\right], \ldots, \left[\frac{n-1}{\frac{n+1}{2}}\right] \right\} \\ S'_2 &= \left\{ \left[\frac{n-3}{n-4}\right], \left[\frac{n-3}{n-5}\right], \ldots, \left[\frac{n-3}{\frac{n-1}{2}}\right] \right\} \\ &\vdots \\ S'_{\frac{n-3}{2}} &= \left\{ \left[\frac{4}{3}\right] \right\} \end{aligned}$$

Clearly each of the sets S'_i also contributes edges with label 1. Therefore

$$\begin{aligned} e_f(1) &\geq |S'_1| + |S'_2| + \ldots + \left|S'_{\frac{n-3}{2}}\right| \\ &= \frac{n-3}{2} + \frac{n-5}{2} + \frac{n-7}{2} + \ldots + 1 \\ &= 1 + 2 + 4 + \ldots + \frac{n-3}{2} \\ &= \frac{\left(\frac{n-3}{2}\right)\left(\frac{n-3}{2}+1\right)}{2} = \frac{(n-3)(n-1)}{8} \end{aligned} \qquad (2)$$

From (1) and (2), we get

$$\begin{aligned} e_f(1) &\geq \frac{(n+1)(n+3)}{8} - 1 + \frac{(n-3)(n-1)}{8} \\ &\geq \frac{n^2+4n+3+n^2-4n+3-8}{8} \\ &\geq \frac{2n^2-2}{8} \geq \frac{n^2-1}{4} > \left[\frac{n(n-1)}{4}\right] + 1, \end{aligned}$$

a contradiction to that f is a quotient cordial labeling.

Case 2. n is even.

Similar to Case 1, we get a contradiction. □

Theorem 3.5 *Every graph is a subgraph of a connected quotient cordial graph.*

Proof Let G be a (p, q) graph with $V(G) = \{u_i : 1 \leq i \leq p\}$. Consider the complete graph K_p with vertex set $V(G)$. Let $f(u_i) = i$, $1 \leq i \leq p$. By Theorem 3.4, we get $e_f(1) > e_f(0)$. Let $e_f(1) = m + e_f(0)$, $m \in \mathbb{N}$. Consider the two copies of the star $K_{1,m}$. The super graph G^* of G is obtained from K_p as follows: Take one star $K_{1,m}$ and identify the central vertex of the star with u_1. Take another star $K_{1,m}$ and identify the central vertex of the same with u_2. Let $S_1 = \{x : x \text{ is an even number and } p < x < p + 2m\}$ and $S_2 = \{x : x$ is an odd number and $p <$

$x < p + 2m\}$. Assign the label to the pendent vertices adjacent to u_1 from the set S_1in any order and then assign the label to the pendent vertices adjacent to u_2 from the set S_2. Clearly this vertex labeling is a quotient cordial labeling of G^*. □

Illustration 3.6 K_5 is not quotient cordial but it is a subgraph of quotient cordial graph G^* given in Figure 3.

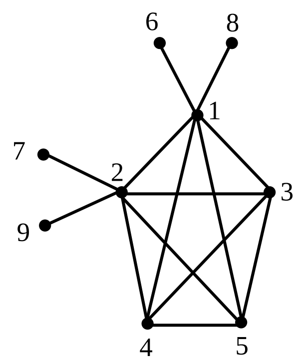

Figure 3

Theorem 3.7 *Any star $K_{1,n}$ is quotient cordial.*

Proof Let $V(K_{1,n}) = \{u, u_i : 1 \leq i \leq n\}$ and $E(K_{1,n}) = \{uu_i : 1 \leq i \leq n\}$. Assign the label 1 to the central vertex u and then assign the labels $2, 3, \cdots, n+1$ to the pendent vertices $u_1, u_2, \cdots, u_n$. f is a quotient cordial labeling follows from the following Table 1. □

Nature of n	$e_f(0)$	$e_f(1)$
even	$\frac{n}{2}$	$\frac{n}{2}$
odd	$\frac{n+1}{2}$	$\frac{n-1}{2}$

Table 1

Now we investigate the complete bipartite graph $K_{2,n}$.

Theorem 3.8 *$K_{2,n}$ is quotient cordial.*

Proof Let $V(K_{2,n}) = \{u, v, u_i : 1 \leq i \leq n\}$ and $E(K_{2,n}) = \{uu_i, vv_i : 1 \leq i \leq n\}$. Assign the label $1, 2$ respectively to the vertices u, v. Then assign the label $3, 4, 5, \cdots, m+2$ to the remaining vertices. Clearly f is a quotient cordial labeling since $e_f(0) = m+1$, $e_f(1) = m$. □

Theorem 3.9 *$K_{1,n} \cup K_{1,n} \cup K_{1,n}$ is quotient cordial.*

Proof Let $V(K_{1,n} \cup K_{1,n} \cup K_{1,n}) = \{u, u_i, v, v_i, w, w_i : 1 \leq i \leq n\}$ and $E(K_{1,n} \cup K_{1,n} \cup K_{1,n}) = \{uu_i, vv_i, ww_i : 1 \leq i \leq n\}$. Define a map $f : V(K_{1,n} \cup K_{1,n} \cup K_{1,n}) \rightarrow \{1, 2, 3, \ldots, 3n\}$ by $f(u) = 1$, $f(v) = 2$, $f(w) = 3$,

$$\begin{aligned} f(u_i) &= 3i+1, \quad 1 \le i \le n \\ f(v_i) &= 3i+3, \quad 1 \le i \le n \\ f(w_i) &= 3i+2, \quad 1 \le i \le n. \end{aligned}$$

Clearly Table 3 shows that f is a quotient cordial labeling. □

Nature of n	$e_f(0)$	$e_f(1)$
n is even	$\frac{3n}{2}$	$\frac{3n}{2}$
$n \equiv 1 \pmod 4$	$\frac{3n-1}{2}$	$\frac{3n+1}{2}$
$n \equiv 3 \pmod 4$	$\frac{3n+1}{2}$	$\frac{3n-1}{2}$

Table 3

Next is the bistar $B_{n,n}$.

Theorem 3.10 *The bistar $B_{n,n}$ is quotient cordial.*

Proof Let $V(B_{n,n}) = \{u, u_i, v, v_i : 1 \le i \le n\}$ and $E(B_{n,n}) = \{uv, uu_i, vv_i : 1 \le i \le n\}$. Assign the label 1 to u and assign the label 2 to v. Then assign the labels $3, 4, 5, \ldots, n+2$ to the vertices $u_1, u_2, \cdots, u_n$. Next assign the label $n+3, n+4, \ldots, 2n+2$ to the pendent vertices $v_1, v_2, \cdots, v_n$. The edge condition is given in Table 2.

Nature of n	$e_f(0)$	$e_f(1)$
$n \equiv 0, 1, 2 \pmod n$	$n+1$	n
$n \equiv 3 \pmod n$	n	$n+1$

Table 3

Hence f is a quotient cordial labeling. □

The final investigation is about the graph obtained from a triangle and three stars.

Theorem 3.11 *Let C_3 be the cycle $u_1u_2u_3u_1$. Let G be a graph obtained from C_3 with $V(G) = V(C_3) \cup \{v_i, w_i, z_i : 1 \le i \le n\}$ and $E(G) = E(C_3) \cup \{u_1v_i, u_2w_i, u_3z_i : 1 \le i \le n\}$. Then G is quotient cordial.*

Proof Define $f : V(G) \to \{1, 2, 3, \cdots, 3n+3\}$ by $f(u_1) = 1$, $f(u_2) = 2$, $f(u_3) = 3$.

Case 1. $n \equiv 0, 2, 3 \pmod 4$.

Define

$$\begin{aligned} f(v_i) &= 3i+1, \quad 1 \le i \le n \\ f(w_i) &= 3i+3, \quad 1 \le i \le n \\ f(z_i) &= 3i+2, \quad 1 \le i \le n. \end{aligned}$$

Case 2. $n \equiv 1 \pmod 4$.

Define

$$\begin{aligned} f(v_i) &= 3i+2, \quad 1 \le i \le n \\ f(w_i) &= 3i+1, \quad 1 \le i \le n \\ f(z_i) &= 3i+3, \quad 1 \le i \le n. \end{aligned}$$

The Table 4 shows that f is a quotient cordial labeling.

values of n	$e_f(0)$	$e_f(1)$
$n \equiv 1, 3 \pmod 4$	$\frac{3n+3}{2}$	$\frac{3n+3}{2}$
$n \equiv 0, 2 \pmod 4$	$\frac{3n+2}{2}$	$\frac{3n+4}{2}$

Table 3

□

Illustration 3.12 A quotient cordial labeling of G obtained from C_3 and $K_{1,7}$ is given in Figure 4.

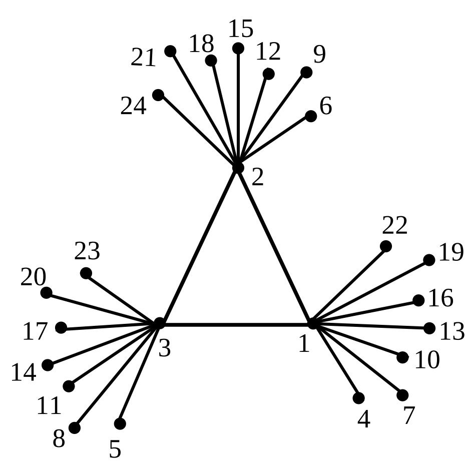

Figure 4

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Nonholonomic Frames for Finsler Space with (α,β)–Metrics

Brijesh Kumar Tripathi[1], K.B.Pandey[2] and R.B. Tiwari[3]

1. Department of Mathematics, L.E. College, Morbi(Gujrat) 363642, India

2. Department of Mathematics, K.N.I.T., Sultanpur,U.P., 228118, India

3. Department of Mathematics, M.K.R.E.C.I.T., Ambedkar Nagar(U.P.), India

E-mail: brijeshkumartripathi4@gmail.com, kunjbiharipandey05@gmail.com, tiwarirambharat@yahoo.in

Abstract: The purpose of present paper to determine the two special Finsler spaces due to deformations of some special Finsler space with help of (α,β)-metrics. Consequently, we obtain the non-holonomic frame for the (α,β)-metrics, such as (I) $L = \left(\frac{\alpha^2}{\alpha-\beta}\right)\frac{\beta^2}{\alpha} = \frac{\alpha\beta^2}{(\alpha-\beta)}$ i.e. product of Matsumoto metric and Kropina metric and (II) $L = (\alpha+\beta)\frac{\beta^2}{\alpha} = \beta^2 + \frac{\beta^3}{\alpha}$ i.e. product of Randers metric and Kropina metric.

Key Words: Finsler Space, (α,β)–metrics, Randers metric, Kropina metric, Matsumoto metric, GL-metric, Non-holonomic Finsler frame.

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§1. Introduction

In 1982, P.R. Holland [1] and [2] studies a unified formalism that uses a nonholonomic frame on space time arising from consideration of a charged particle moving in an external electromagnetic field. In fact, R.S. Ingarden [3] was the first to point out that the Lorentz force law can be written in this case as geodesic equation on a Finsler space called Randers space. The author R.G. Beil [5], [6] have studied a gauge transformation viewed as a nonholonomic frame on the tangent bundle of a four dimensional base manifold. The geometry that follows from these considerations gives a unified approach to gravitation and gauge symmetries. In the present paper we have used the common Finsler idea to study the existence of a nonholonomic frame on the vertical sub bundle VTM of the tangent bundle of a base manifold M.

In this paper, the fundamental tensor field of a Finsler space might be considered as the deformations of two different special Finsler spaces from the (α,β)-metrics. Further we obtain corresponding frame for each of these two Finsler deformations. Consequently, a nonholonomic frame for a Finsler space with special (α,β)-metrics such as first is the product of Matsumoto metric[11] and kropina metric[11] and second is the product of Randers metric[11] and Kropina metric. This is an extension work of Ioan Bucataru and Radu Miron [10] and also second extension work of S.K. Narasimhamurthy [14].

Consider, $a_{ij}(x)$ the components of a Riemannian metric on the base manifold M, $a(x,y) >$

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0 and $b(x,y) \geq 0$ Two functions on TM and $B(x,y) = B_i(x,y)(dx^i)$ a vertical 1-form on TM. Then

$$g_{ij}(x,y) = a(x,y)a_{ij}(x) + b(x,y)B_i(x,y)B_j(x,y) \tag{1.1}$$

is a generalized Lagrange metric, called the Beil metric. The metric tensor g_{ij} is also known as a Beil deformation of the Riemannian metric a_{ij}. It has been studied and applied by R. Miron and R.K. Tavakol in General Relativity for $a(x,y) = exp(2\sigma(x,y))$ and b = 0. The case a(x, y) = 1 with various choices of b and B_i was introduced and studied by R.G. Beil for constructing a new unified field theory [6].

§2. Preliminaries

An important class of Finsler spaces is the class of Finsler spaces with(α,β)–metrics [11]. The first Finsler spaces with (α,β)–metrics were introduced by the physicist G.Randers in 1940, are called Randers spaces [4]. Recently, R.G. Beil suggested a more general case and considered the class of Lagrange spaces with (α,β)–metric, which was discussed in [12]. A unified formalism which uses a nonholonomic frame on space time, a sort of plastic deformation, arising from consideration of a charged particle moving in an external electromagnetic field in the background space time viewed as a strained mechanism studied by P. R. Holland [1], [2]. If we do not ask for the function L to be homogeneous of order two with respect to the (α,β) variables, then we have a Lagrange space with (α,β)–metric. Next we defined some different Finsler space with (α,β)–metrics.

Definition 2.1 *A Finsler space $F^n = (M, F(x,y))$ is called with (α,β)–metric if there exists a 2-homogeneous function L of two variables such that the Finsler metric $F : TM \to R$ is given by*

$$F^2(x,y) = L(\alpha(x,y), \beta(x,y)), \tag{2.1}$$

where $\alpha^2(x,y) = a_{ij}(x)y^iy^j, \alpha$ is a Riemannian metric on the manifold M, and $\beta(x,y) = b_i(x)y^i$ is a 1-form on M.

Consider $g_{ij} = \dfrac{1}{2}\dfrac{(\partial^2 F^2)}{(\partial y^i \partial y^j)}$ the fundamental tensor of the Randers space(M,F). Taking into account the homogeneity of a and F we have the following formulae:

$$p^i = \frac{1}{a}y^i = a^{ij}\frac{\partial\alpha}{\partial y^j}; \quad p_i = a_{ij}p^j = \frac{\partial\alpha}{\partial y^i};$$

$$l^i = \frac{1}{L}y^i = g^{ij}\frac{\partial l}{\partial y^i}; l_i = g_{ij}l^j = \frac{\partial L}{\partial y^i} = P_i + b_i, \tag{2.2}$$

$$l^i = \frac{1}{L}p^i; l^il_i = p^ip_i = 1; l^ip_i = \frac{\alpha}{L}; p^il_i = \frac{L}{\alpha};$$

$$b_iP^i = \frac{\beta}{\alpha}; b_il^i = \frac{\beta}{L}$$

with respect to these notations, the metric tensors (α_{ij}) and (g_{ij}) are related by [13],

$$g_{ij}(x,y) = \frac{L}{\alpha}a_{ij} + b_iP_j + P_ib_j - \frac{\beta}{\alpha}p_ip_j = \frac{L}{\alpha}(a_{ij} - p_ip_j) + l_il_j. \tag{2.3}$$

Theorem 2.1([10]) *For a Finsler space (M,F) consider the matrix with the entries:*

$$Y_i^j = \sqrt{\frac{\alpha}{L}}(\delta_j^i - l^il_j + \sqrt{\frac{\alpha}{L}}p^ip_j) \tag{2.4}$$

defined on TM. Then $Y_j = Y_j^i(\frac{\partial}{\partial y^i}), j \in 1,2,3,\ldots,n$ *is a non holonomic frame.*

Theorem 2.2([7]) *With respect to frame the holonomic components of the Finsler metric tensor* $\alpha_{\alpha\beta}$ *is the Randers metric* g_{ij}*, i.e,*

$$g_{ij} = Y_i^\alpha Y_j^\beta \alpha_{\alpha\beta}. \tag{2.5}$$

Throughout this section we shall rise and lower indices only with the Riemannian metric $\alpha_{ij}(x)$ that is $y_i = \alpha_{ij}y^j, \beta^i = \alpha^{ij}\beta_j$, and so on. For a Finsler space with (α,β)–metric $F^2(x,y) = L(\alpha(x,y),\beta(x,y))$ we have the Finsler invariants [13]

$$\rho_1 = \frac{1}{2\alpha}\frac{\partial L}{\partial \alpha}; \rho_0 = \frac{1}{2}\frac{\partial^2 L}{\partial \beta^2}; \rho_{-1} = \frac{1}{2\alpha}\frac{\partial^2 L}{\partial\alpha\partial\beta}; \rho_{-2} = \frac{1}{2\alpha^2}(\frac{\partial^2 L}{\partial\alpha^2} - \frac{1}{\alpha}\frac{\partial L}{\partial \alpha}) \tag{2.6}$$

where subscripts 1, 0, -1, -2 gives us the degree of homogeneity of these invariants.

For a Finsler space with metric we have,

$$\rho_{-1}\beta + \rho_{-2}\alpha^2 = 0 \tag{2.7}$$

With respect to the notations we have that the metric tensor g_{ij} of a Finsler space with (α,β)–metric is given by [13]

$$g_{ij}(x,y) = \rho\alpha_{ij}(x) + \rho_0 b_i(x) + \rho_{-1}(b_i(x)y_j + b_j(x)y_i) + \rho_{-2}y_iy_j. \tag{2.8}$$

From (2.8) we can see that $g_{ij}$is the result of two Finsler deformations

$$I. \qquad a_{ij} \to h_{ij} = \rho a_{ij} + \frac{1}{\rho_{-2}}(\rho_{-1}b_i + \rho_{-2}y_i)(\rho_{-1}b_j + \rho_{-2}y_j)$$

$$II. \quad h_{ij} \to g_{ij} = h_{ij} + \frac{1}{\rho_{-2}}(\rho_0\rho_{-1} - \rho_{-1}^2)b_ib_j. \tag{2.9}$$

The nonholonomic Finsler frame that corresponding to the I^{st} deformation (2.9) is according to the Theorem 7.9.1 in [10], given by

$$X_j^i = \sqrt{\rho}\delta_j^i - \frac{1}{\beta^2}(\sqrt{\rho} + \sqrt{\rho + \frac{\beta^2}{\rho_{-2}}})(\rho_{-1}b^i + \rho_{-2}y^i)(\rho_{-1}b_j + \rho_{-2}y_j), \tag{2.10}$$

where $B^2 = a_{ij}(\rho_{-1}b^i + \rho_{-2}y^i)(\rho_{-1}b_j + \rho_{-2}y_j) = \rho_{-1}^2 b^2 + \beta\rho_{-1}\rho_{-2}$.

This metric tensor a_{ij} and h_{ij} are related by,

$$h_{ij} = X_i^k X_j^l a_{kl}. \tag{2.11}$$

Again the frame that corresponds to the II_{nd} deformation (2.9) given by,

$$Y_j^i = \delta_j^i - \frac{1}{C^2}\left(1 \pm \sqrt{1 + \left(\frac{\rho_{-2}C^2}{\rho_0\rho_{-2-\rho_{-1}^2}}\right)}\right) b^i b_j, \tag{2.12}$$

where $C^2 = h_{ij}b^ib^j = \rho b^2 + \frac{1}{\rho_{-2}}(\rho_{-2}b^2 + \rho_{-2}\beta)^2$.

The metric tensor h_{ij} and g_{ij} are related by the formula

$$g_{mn} = Y_m^i Y_n^j h_{ij}. \tag{2.13}$$

Theorem 2.3([10]) *Let $F^2(x,y) = L(\alpha(x,y), \beta(x,y))$ be the metric function of a Finsler space with (α,β) metric for which the condition (2.7) is true. Then*

$$V_j^i = X_k^i Y_j^k$$

is a nonholonomic Finsler frame with X_k^i and Y_j^k are given by (2.10) and (2.12) respectively.

§3. Nonholonomic Frames for Finsler Space with (α,β)–Metrics

In this section we consider two cases of non-holonomic Finlser frames with (α,β)-metrics, such a I^{st} Finsler frame product of Matusmoto metric and Kropina metric and II^{nd} Finsler frame product of Randers metric and Kropina metric.

(I) Nonholonomic Frames for $L = \left(\frac{\alpha^2}{\alpha-\beta}\right)\frac{\beta^2}{\alpha} = \frac{\alpha\beta^2}{\alpha-\beta}$

In the first case, for a Finsler space with the fundamental function

$$L = \left(\frac{\alpha^2}{\alpha-\beta}\right)\frac{\beta^2}{\alpha} = \frac{\alpha\beta^2}{\alpha-\beta},$$

the Finsler invariants (2.6) are given by

$$\rho_1 = -\frac{\beta^3}{2\alpha(\alpha-\beta)^2}; \rho_0 = \frac{1}{2}\frac{(2\alpha^3 - \alpha\beta^2)}{(\alpha-\beta)^3};$$

$$\rho_{-1} = \frac{1}{2\alpha}\frac{\beta^2(\beta-3\alpha)}{(\alpha-\beta)^3}; \rho_{-2} = \frac{\beta^3(3\alpha-\beta)}{2\alpha^3(\alpha-\beta)^3};$$

$$B^2 = \frac{\beta^2(1-3\alpha)^2b^2 + \beta^5(\alpha-\beta)(1-3\alpha)(3\alpha-\beta)}{4\alpha^4(\alpha-\beta)^6}. \tag{3.1}$$

Using (3.1) in (2.10) we have,

$$X^i_j = \sqrt{-\frac{\beta^3}{2\alpha(\alpha-\beta)^2}}\quad \delta^i_j - \frac{\beta^4}{4\alpha^4(\alpha-\beta)^5}\left[\sqrt{\frac{-\beta^3}{2\alpha}} + \sqrt{\frac{4\alpha^4(\alpha-\beta)^5 - \beta^4(3\alpha-\beta)}{2\alpha\beta(3\alpha-\beta)}}\right] \times(b^i - \frac{(3\alpha-\beta)}{\alpha^2(\alpha-\beta)}y^i)(b_j - \frac{(3\alpha-\beta)}{\alpha^2(\alpha-\beta)}y_j). \tag{3.2}$$

Again using (3.1) in (2.12) we have,

$$Y^i_j = \delta^i_j - \frac{1}{C^2}\left(1 \pm \sqrt{1 + \frac{2(\alpha-\beta)^3C^2}{\alpha^2(2\alpha-3\beta)}}\right) b^i b_j, \tag{3.3}$$

where $C^2 = -\dfrac{\beta^3}{2\alpha(\alpha-\beta)^2}b^2 + \dfrac{\beta(3\alpha-\beta)}{2\alpha^3(\alpha-\beta)^3}(\alpha^2b^2-\beta^2)^2.$

Theorem 3.1 *Consider Finsler space $L = \left(\frac{\alpha^2}{\alpha-\beta}\right)\frac{\beta^2}{\alpha} = \frac{\alpha\beta^2}{\alpha-\beta}$, for which the condition (2.7) is true. Then*

$$V^i_j = X^i_k Y^k_j$$

is non-holomic Finsler Frame with X^i_k and Y^k_j are given by (3.2) and (3.3) respectively.

(II) Nonholonomic Frames for $L = (\alpha+\beta)(\dfrac{\beta^2}{\alpha}) = \beta^2 + \dfrac{\beta^3}{\alpha}$

In the second case, for a Finsler space with the fundamental function $L = (\alpha+\beta)(\frac{\beta^2}{\alpha})$ the Finsler invariants (2.6) are given by:

$$\rho_1 = -\frac{\beta^3}{2\alpha^3}; \rho_0 = \frac{3\beta+\alpha}{\alpha}; \rho_{-1} = -\frac{3}{2}\frac{\beta^2}{\alpha^3}; \rho_{-2} = \frac{3}{2}\frac{\beta^3}{\alpha^5};$$

$$B^2 = \frac{9}{4}\frac{\beta^4}{\alpha^8}(\alpha^2b^2-\beta^2), \tag{3.4}$$

$$X^i_j = \sqrt{-\frac{\beta^3}{2\alpha^3}}\quad \delta^i_j - \frac{9}{4}\frac{\beta^2}{\alpha^6}\left[\sqrt{-\frac{\beta^3}{2\alpha^3}} + \sqrt{-\frac{\beta^3}{2\alpha^3} + \frac{2}{3}\frac{\alpha^5}{\beta}}\right](b^i - \frac{\beta}{\alpha^2}y^i)(b_j - \frac{\beta}{\alpha^2}y_j). \tag{3.5}$$

Again using (3.4) in (2.12) we have

$$y^i_j = \delta^i_j - \frac{1}{c^2}\left[1 \pm \sqrt{1 + (\frac{2\alpha c^2}{2\alpha+3\beta}}\right] b^i b_j, \tag{3.6}$$

where $C^2 = -\dfrac{\beta^3}{2\alpha^3}b^2 + \dfrac{3}{2}\dfrac{\beta}{\alpha^5}[\alpha^2b^2-\beta^2]^2.$

Theorem 3.2 *Consider a Finsler space $L = (\alpha+\beta)(\frac{\beta^2}{\alpha}) = \beta^2 + \frac{\beta^3}{\alpha}$, for which the condition 2.7 is true. Then*

$$V^i_j = X^i_k Y^k_j$$

is non-holomic Finsler Frame with X^i_k and Y^k_j are given by (3.5) and (3.6) respectively.

§4. **Conclusions**

Non-holonomic frame relates a semi-Riemannian metric (the Minkowski or the Lorentz metric) with an induced Finsler metric. Antonelli P.L., Bucataru I. ([7][8]), has been determined such a non-holonomic frame for two important classes of Finsler spaces that are dual in the sense of Randers and Kropina spaces [9]. As Randers and Kropina spaces are members of a bigger class of Finsler spaces, namely the Finsler spaces with(α,β)–metric, it appears a natural question: Does how many Finsler space with(α,β)–metrics have such a nonholonomic frame? The answer is yes, there are many Finsler space with(α,β)–metrics.

In this work, we consider the two special Finsler metrics and we determine the non-holonomic Finsler frames. Each of the frames we found here induces a Finsler connection on TM with torsion and no curvature. But, in Finsler geometry, there are many(α,β)–metrics, in future work we can determine the frames for them also.

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On b-Chromatic Number of Some Line, Middle and Total Graph Families

Vernold Vivin J.

Department of Mathematics, University College of Engineering Nagercoil

(Anna University, Constituent College) Konam, Nagercoil - 629 004, Tamilnadu, India

Venkatachalam.M.

Department of Mathematics, RVS Educational Trust's Group of Institutions

RVS Faculty of Engineering, Coimbatore - 641 402, Tamilnadu, India

Mohanapriya N.

Research & Development Centre, Bharathiar University, Coimbatore-641 046 and Department of Mathematics, RVS Technical Campus, RVS Faculty of Engineering, Coimbatore - 641 402, Tamilnadu, India

E-mail: vernoldvivin@yahoo.in, venkatmaths@gmail.com, n.mohanamaths@gmail.com

Abstract: A proper coloring of the graph assigns colors to the vertices, edges, or both so that proximal elements are assigned distinct colors. Concepts and questions of graph coloring arise naturally from practical problems and have found applications in many areas, including information theory and most notably theoretical computer science. A b-coloring of a graph G is a proper coloring of the vertices of G such that there exists a vertex in each color class joined to at least one vertex in each other color class. The b-chromatic number of a graph G, denoted by $\varphi(G)$, is the largest integer k such that G may have a b-coloring with k colors. In this paper, the authors obtain the b-chromatic number for line, middle and total graph of some families such as cycle, helm and gear graphs.

Key Words: b-coloring, Helm graph, gear Graph, middle graph, total graph, line graph.

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§1. Introduction

All graphs considered in this paper are non-trivial, simple and undirected. Let G be a graph with vertex set V and edge set E. A k-coloring of a graph G is a partition $P = \{V_1, V_2, \cdots, V_k\}$ of V into independent sets of G. The minimum cardinality k for which G has a k-coloring is the chromatic number $\chi(G)$ of G. The b-chromatic number $\varphi(G)$ ([16,19,20]) of a graph G is the largest positive integer k such that G admits a proper k-coloring in which every color class has a representative adjacent to at least one vertex in each of the other color classes. Such a coloring is called a b-coloring. The b-chromatic number was introduced by Irving and Manlove in [11] by considering proper colorings that are minimal with respect to a partial order defined

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on the set of all partitions of $V(G)$. They have shown that determination of $\varphi(G)$ is NP-hard for general graphs, but polynomial for trees. There has been an increasing interest in the study of b-coloring since the publication of [11]. They also proved the following upper bound of $\varphi(G)$

$$\varphi(G) \leq \Delta(G) + 1. \tag{1.1}$$

Kouider and Mahéo [12] gave some lower and upper bounds for the b-chromatic number of the cartesian product of two graphs. Kratochvíl et al. [13] characterized bipartite graphs for which the lower bound on the b-chromatic number is attained and proved the NP-completeness of the problem to decide whether there is a dominating proper b-coloring even for connected bipartite graphs with $k = \Delta(G) + 1$. Effantin and Kheddouci studied in [3, 5, 6] the b-chromatic number for the complete caterpillars, the powers of paths, cycles, and complete k-ary trees. Faik [7] was interested in the continuity of the b-coloring and proved that chordal graphs are b-continuous. Corteel et al. [2] proved that the b-chromatic number problem is not approximable within $120/133 - \epsilon$ for any $\epsilon > 0$, unless $P = NP$. Hoáng and Kouider characterized in [10], the bipartite graphs and the P_4-sparse graphs for which each induced subgraph H of G has $\varphi(H) = \chi(H)$. Kouider and Zaker [14] proposed some upper bounds for the b-chromatic number of several classes of graphs in function of other graph parameters (clique number, chromatic number, biclique number). Kouider and El Sahili proved in [15] by showing that if G is a d-regular graph with girth 5 and without cycles of length 6, then $\varphi(G) = d + 1$. Effantin and Kheddouci [4] proposed a discussion on relationships between this parameter and two other coloring parameters (the Grundy and the partial Grundy numbers). The property of the dominating nodes in a b-coloring is very interesting since they can communicate directly with each partition of the graph.

There have been lots of works on various properties of line graphs, middle graphs and total graphs of graphs [1, 9, 17, 18].

For any integer $n \geq 4$, the wheel graph W_n is the $n-$vertex graph obtained by joining a vertex v_1 to each of the $n-1$ vertices $\{w_1, w_2, \cdots, w_{n-1}\}$ of the cycle graph C_{n-1}. Where $V(W_n) = \{v, v_1, v_2, \cdots v_{n-1}\}$ and $E(W_n) = \{e_1, e_2, \cdots, e_n\} \cup \{s_1, s_2, \cdots, s_n\}$.

The Helm graph H_n is the graph obtained from an n-wheel graph by adjoining a pendent edge at each node of the cycle.

The Gear graph G_n, also known as a bipartite wheel graph, is a wheel graph with a graph vertex added between each pair of adjacent graph vertices of the outer cycle.

The line graph [8] of G, denoted by $L(G)$ is the graph with vertices are the edges of G with two vertices of $L(G)$ adjacent whenever the corresponding edges of G are adjacent.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The middle graph [8] of G, denoted by $M(G)$ is defined as follows. The vertex set of $M(G)$ is $V(G) \cup E(G)$. Two vertices x, y in the vertex set of $M(G)$ are adjacent in $M(G)$ in case one of the following holds: (i) x, y are in $E(G)$ and x, y are adjacent in G; (ii) x is in $V(G)$, y is in $E(G)$, and x, y are incident in G.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The total graph [8] of G, denoted by $T(G)$ is defined in the following way. The vertex set of $T(G)$ is $V(G) \cup E(G)$. Two vertices x, y in the vertex set of $T(G)$ are adjacent in $T(G)$ in case one of the following holds:

(i) x, y are in $V(G)$ and x is adjacent to y in G; (ii) x, y are in $E(G)$ and x, y are adjacent in G; (iii) x is in $V(G)$, y is in $E(G)$, and x, y are incident in G.

§2. b-Coloring of Some Line, Middle and Total Graph Families

Lemma 2.1 *If $n \geq 8$ then b-chromatic number on middle graph of cycle $M(C_n)$ is $\varphi(M(C_n)) = 5$.*

Proof Let $V(C_n) = \{v_1, v_2, \cdots, v_n\}$ and let $V(M(C_n)) = \{v_1, v_2, \cdots, v_n\} \cup \{u_1, u_2, \cdots, u_n\}$ where u_i is the vertex of $T(C_n)$ corresponding to the edge $v_i v_{i+1}$ of C_n $(1 \leq i \leq n-1)$.

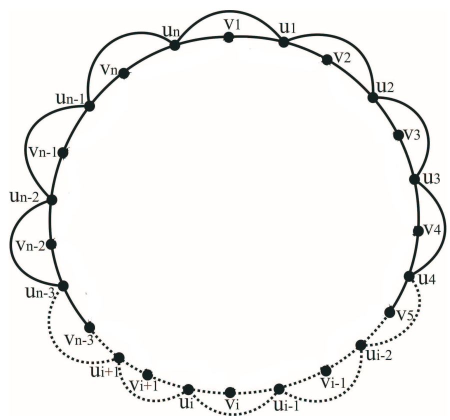

Fig.1 Middle Graph of Cycle $M(C_n)$

Consider the following 5-coloring $(c_1, c_2, c_3, c_4, c_5)$ of $M(C_n)$ as b-chromatic:

Assign the color c_1 to v_1, c_3 to u_1, c_4 to v_2, c_1 to u_2, c_5 to v_3, c_2 to u_3, c_4 to v_4, $c_3$to u_4, c_1 to v_5, c_5 to u_5, c_2 to v_6, c_4 to u_6, c_1 to v_7, c_3 to u_7. For $8 \leq i \leq n$, assign to vertex v_i one of the allowed colors-such color exists, because $\deg(v_i) = 2$. For $8 \leq i \leq n-1$, if any, assign to vertex u_i one of the allowed colors-such color exists, because $\deg(u_i) = 4$. An easy check shows that this coloring is a b-coloring. Therefore, $\varphi(M(C_n)) \geq 5$. Since $\Delta(M(C_n)) = 4$, using (1.1) we get that $\varphi(M(C_n)) \leq 5$. Hence, $\varphi(M(C_n)) = 5$, $\forall\ n \geq 5$. □

Theorem 2.2 *If $n \geq 5$ then b-chromatic number on total graph of cycle $T(C_n)$ is $\varphi(T(C_n)) = 5$.*

Proof Let $V(C_n) = \{v_1, v_2, \cdots, v_n\}$ and let $V(T(C_n)) = \{v_1, v_2, \cdots, v_n\} \cup \{u_1, u_2, \cdots, u_n\}$ where u_i is the vertex of $T(C_n)$ corresponding to the edge $v_i v_{i+1}$ of C_n $(1 \leq i \leq n-1)$.

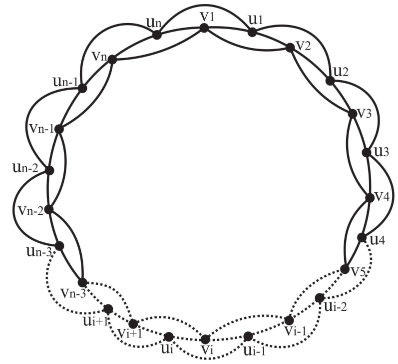

Fig.2 Total Graph of Cycle $T(C_n)$

Consider the following 5-coloring $(c_1, c_2, c_3, c_4, c_5)$ of $T(C_n)$ as b-chromatic: assign the color c_4 to v_1, c_5 to u_1, c_1 to v_2, c_2 to u_2, c_3 to v_3,c_4 to u_3, c_5 to v_4, c_1 to u_4, c_2 to v_5. For $6 \leq i \leq n$, assign to vertex v_i one of the allowed colors-such color exists, because $\deg(v_i) = 4$. For $5 \leq i \leq n-1$, if any, assign to vertex u_i one of the allowed colors-such color exists, because $\deg(u_i) = 4$. An easy check shows that this coloring is a b-coloring. Therefore, $\varphi(T(C_n)) \geq 5$. Since $\Delta(T(C_n)) = 4$, using (1.1), we get that $\varphi(T(C_n)) \leq 5$. Hence, $\varphi(T(C_n)) = 5$, $\forall\ n \geq 5$. □

Lemma 2.3 *If $n \geq 6$ then b-chromatic number on helm graph H_n is $\varphi(H_n) = 5$.*

Proof Let H_n be the Helm graph obtained by attaching a pendant edge at each vertex of the cycle. Let $V(H_n) = \{v\} \cup \{v_1, v_2, \cdots, v_n\} \cup \{u_1, u_2, \cdots, u_n\}$ where v_i's are the vertices of cycles taken in cyclic order and u_i's are pendant vertices such that each v_iu_i is a pendant edge and v is a hub of the cycle.

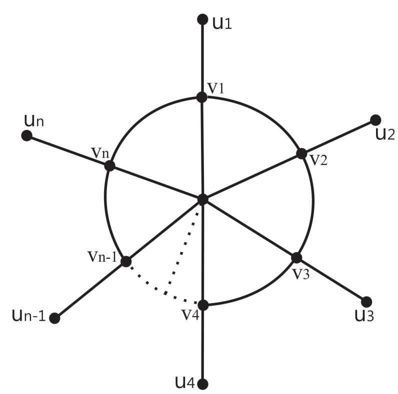

Fig.3 Helm Graph H_n

Consider the following 5-coloring $(c_1, c_2, c_3, c_4, c_5)$ of H_n as b-chromatic:

For $1 \leq i \leq 4$, assign the color c_i to v_i and assign the colors c_5 to v, c_1 to v_5, c_3 to v_n, c_4 to u_1, c_4 to u_2, c_1 to u_3, c_2 to u_4. For $6 \leq i \leq n$, assign to vertex v_i one of the allowed colors-such color exists, because $\deg(v_i) = 4$. For $5 \leq i \leq n$, if any, assign the color c_4 to the vertex u_i. An easy check shows that this coloring is a b-coloring. Therefore, $\varphi(H_n) \geq 5$.

Let us assume that $\varphi(H_n)$ is greater than 5, i.e. $\varphi(H_n) = 6$, $\forall\ n \geq 6$, there must be at least 6 vertices of degree 5 in H_n, all with distinct colors, and each adjacent to vertices of all of the other colors. But then these must be the vertices $v, \{v_i : 1 \leq i \leq n\}$, since these are only ones with degree at least 4. This is the contradiction, b-coloring with 6 colors is impossible. Thus, we have $\varphi(H_n) \leq 5$. Hence, $\varphi(H_n) = 5$, $\forall\ n \geq 6$. □

Lemma 2.4 *If $n \geq 7$ then b-chromatic number on line graph of Helm graph $L(H_n)$ is $\varphi(L(H_n)) = n$.*

Proof Let $V(H_n) = \{v\} \cup \{v_1, v_2, \cdots, v_n\} \cup \{u_1, u_2, \cdots, u_n\}$ and $E(H_n) = \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq n-1\} \cup \{e'_n\} \cup \{s_i : 1 \leq i \leq n\}$ where e_i is the edge vv_i $(1 \leq i \leq n)$, e'_i is the edge v_iv_{i+1} $(1 \leq i \leq n-1)$, e'_n is the edge v_nv_1 and s_i is the edge v_iu_i $(1 \leq i \leq n)$. By the definition of line graph $V(L(H_n)) = E(H_n) = \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\}$.

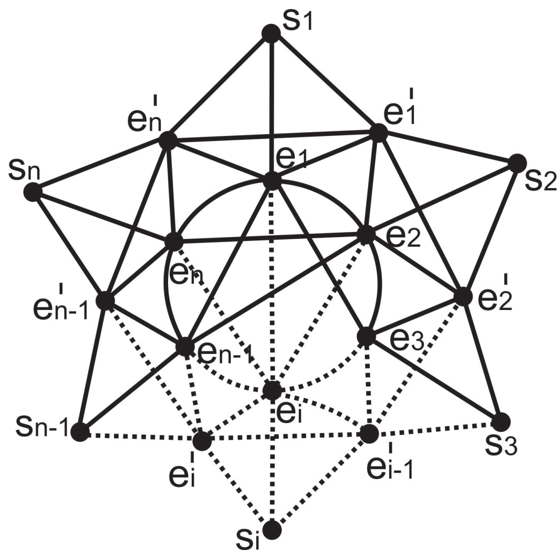

Fig.4 Line Graph of Helm Graph $L(H_n)$

Consider the following n-coloring of $L(H_n)$ as b-chromatic: For $1 \leq i \leq n$, assign the color c_i to e_i. For $1 \leq i \leq n$, assign to vertices s_i and e'_i, one of the allowed colors-such color exists, because $\deg(s_i) = 3$ and $\deg(e'_i) = 6$. An easy check shows that this coloring is a b-coloring. Therefore, $\varphi(L(H_n)) \geq n$.

Let us assume that $\varphi(L(H_n))$ is greater than n, $\varphi(L(H_n)) = n + 1$, $\forall n \geq 7$, there must be at least $n + 1$ vertices of degree n in $L(H_n)$, all with distinct colors, and each adjacent to vertices of all of the other colors. But then these must be the vertices $e_i (1 \leq i \leq n)$, since these are only ones with degree at least n. This is the contradiction, b-coloring with $n + 1$ colors is impossible. Thus, we have $\varphi(L(H_n)) \leq n$. Hence, $\varphi(L(H_n)) = n$, $\forall\ n \geq 7$. □

Theorem 2.5 *If $n \geq 8$ then b-chromatic number on middle graph of Helm graph $M(H_n)$ is $\varphi(M(H_n)) = n + 1$.*

Proof Let $V(H_n) = \{v\} \cup \{v_1, v_2, \cdots, v_n\} \cup \{u_1, u_2, \cdots, u_n\}$ and $E(H_n) = \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq n-1\} \cup \{e'_n\} \cup \{s_i : 1 \leq i \leq n\}$ where e_i is the edge vv_i $(1 \leq i \leq n)$, e'_i is the edge v_iv_{i+1} $(1 \leq i \leq n-1)$, e'_n is the edge v_nv_1 and s_i is the edge v_iu_i $(1 \leq i \leq n)$. By the definition of middle graph $V(M(H_n)) = \{v\} \cup V(H_n) \cup E(H_n) = \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\}$.

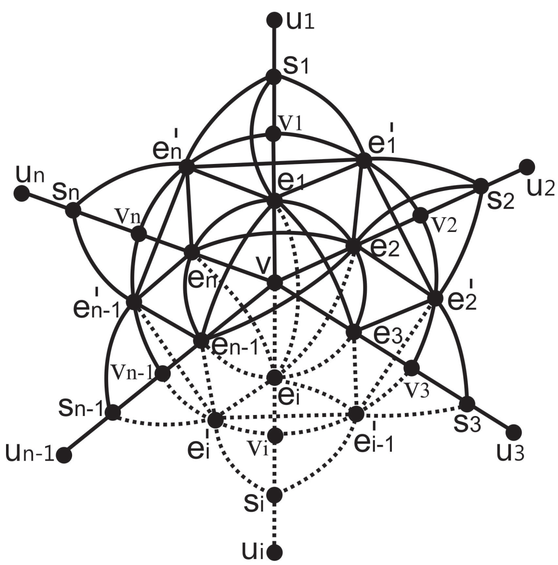

Fig.5 Middle Graph of Helm Graph $M(H_n)$

Consider the following $n+1$-coloring of $M(H_n)$ as b-chromatic: For $1 \leq i \leq n$, assign the color c_i to e_i and assign the color c_{n+1} to v. For $1 \leq i \leq n$, assign to vertices u_i, v_i, s_i, e'_i, one of the allowed colors-such color exists, because $deg(u_i) = 1$, $\deg(v_i) = 4$, $\deg(s_i) = 3$ and $\deg(e'_i) = 8$. An easy check shows that this coloring is a b-coloring. Therefore, $\varphi(M(H_n)) \geq n+1$.

Let us assume that $\varphi(M(H_n))$ is greater than $n+1$, i.e. $\varphi(M(H_n)) = n+2$, $\forall\, n \geq 8$, there must be at least $n+2$ vertices of degree $n+1$ in $M(H_n)$, all with distinct colors, and each adjacent to vertices of all of the other colors. But then these must be the vertices $v, \{e_i : 1 \leq i \leq n\}$, since these are only ones with degree at least $(n-1)+3$. This is the contradiction, b-coloring with $n+2$ colors is impossible. Thus, we have $\varphi(M(H_n)) \leq n+1$. Hence, $\varphi(M(H_n)) = n+1$, $\forall\, n \geq 8$. □

Proposition 2.6 *If $n \geq 8$ then b-chromatic number on total graph of Helm graph $T(H_n)$ is* $\varphi(T(H_n)) = n+1$.

Proof Consider the coloring of $M(H_n)$ introduced on the proof of Theorem 5. An easy check shows that this coloring is a b-coloring of $T(H_n)$. Hence, $\varphi(T(H_n)) = n+1$, $\forall\, n \geq 8$. □

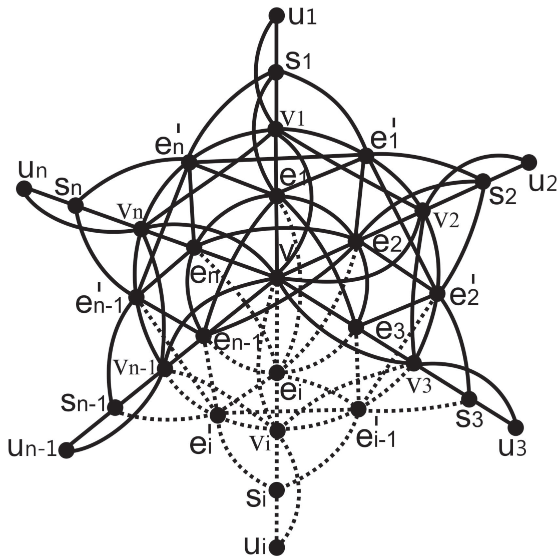

Fig.6 Total Graph of Helm Graph $T(H_n)$.

Lemma 2.7 *If $n \geq 4$ then b-chromatic number of gear graph G_n is $\varphi(G_n) = 4$.*

Proof Let $V(G_n) = \{v\} \cup \{v_1, v_2, \cdots, v_{2n}\}$ where v_i's are the vertices of cycles taken in cyclic order and v is adjacent with $v_{2i-1}(1 \leq i \leq n)$.

Consider the following 4-coloring (c_1, c_2, c_3, c_4) of G_n as b-chromatic:

Assign the colors c_1 to v_1, c_3 to v_2, c_2 to v_3, c_1 to v_4, c_3 to v_5, c_2 to v_6, c_4 to v and c_2 to v_{2n}. For $7 \leq i \leq 2n-1$, if any, assign to vertex v_i one of the allowed colors-such color exists, because $2 \leq \deg(v_i) \leq 3$. An easy check shows that this coloring is a b-coloring. Therefore, $\varphi(G_n) \geq 4$.

Let us assume that $\varphi(G_n)$ is greater than 4, i.e. $\varphi(G_n) = 5$, $\forall\ n \geq 4$, there must be at least 5 vertices of degree 4 in G_n, all with distinct colors, and each adjacent to vertices of all of the other colors. But then these must be the vertices $v, \{v_{2i-1} : 1 \leq i \leq n\}$, since these are only ones with degree at least 3. This is the contradiction, b-coloring with 5 colors is impossible. Thus, we have $\varphi(G_n) \leq 4$. Hence, $\varphi(G_n) = 4$, $\forall\ n \geq 4$. □

Lemma 2.8 *If $n \geq 4$ then b-chromatic number on line graph of Gear graph $L(G_n)$ is $\varphi(L(G_n)) = n$.*

Proof Let $V(G_n) = \{v\} \cup \{v_1, v_2, \ldots, v_{2n}\}$ and $E(G_n) = \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq 2n-1\} \cup \{e'_n\}$ where e_i is the edge vv_{2i-1} $(1 \leq i \leq n)$, e'_i is the edge v_iv_{i+1} $(1 \leq i \leq 2n-1)$, and e'_{2n} is the edge $v_{2n-1}v_1$. By the definition of line graph $V(L(G_n)) = E(G_n) = \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq 2n\}$.

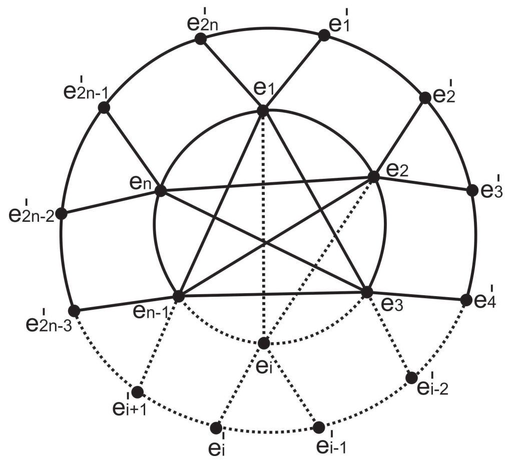

Fig.7 Line graph of Gear Graph $L(G_n)$.

Consider the following n-coloring of $L\left(G_n\right)$ as b-chromatic: For $1 \leq i \leq n$, assign the color c_i to e_i. For $1 \leq i \leq 2n$, assign to vertices e'_i, one of the allowed colors-such color exists, because $\deg(e'_i) = 3$. An easy check shows that this coloring is a b-coloring. Therefore, $\varphi(L(G_n)) \geq n$.

Let us assume that $\varphi(L(G_n))$ is greater than n, $\varphi(L(G_n)) = n+1$, $\forall\ n \geq 4$, there must be at least $n+1$ vertices of degree n in $L(G_n)$, all with distinct colors, and each adjacent to vertices of all of the other colors. But then these must be the vertices $e_i (1 \leq i \leq n)$, since these are only ones with degree at least n. This is the contradiction, b-coloring with $n+1$ colors is impossible. Thus, we have $\varphi(L(G_n)) \leq n$. Hence, $\varphi(L(G_n)) = n$, $\forall\ n \geq 4$. □

Theorem 2.9 *If $n \geq 5$ then b-chromatic number on middle graph of Gear graph $M(G_n)$ is $\varphi(M(G_n)) = n+1$.*

Proof Let $V(G_n) = \{v\} \cup \{v_1, v_2, \cdots, v_{2n}\}$ and $E(G_n) = \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq 2n-1\} \cup \{e'_n\}$ where e_i is the edge vv_{2i-1} $(1 \leq i \leq n)$, e'_i is the edge v_iv_{i+1} $(1 \leq i \leq 2n-1)$, and e'_{2n} is the edge $v_{2n-1}v_1$. By the definition of middle graph $V(M(G_n)) = V(G_n) \cup E(G_n) = \{v\} \cup \{v_i : 1 \leq i \leq 2n\} \cup \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq 2n\}$.

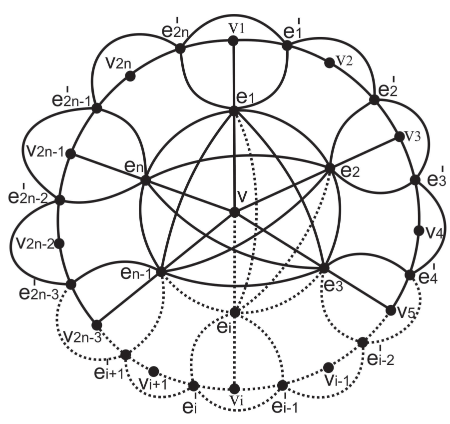

Fig.8 Middle graph of Gear Graph $M(G_n)$.

Consider the following $n+1$-coloring of $M(G_n)$ as b-chromatic:

For $1 \leq i \leq n$, assign the color c_i to e_i and assign the color c_{n+1} to v. For $1 \leq i \leq 2n$, assign to vertices v_i and e_i', one of the allowed colors-such color exists, because $2 \leq \deg(v_i) \leq 3$ and $\deg(e_i') = 5$. An easy check shows that this coloring is a b-coloring. Therefore, $\varphi(M(G_n)) \geq n+1$.

Let us assume that $\varphi(M(G_n))$ is greater than $n+1$, i.e. $\varphi(M(G_n)) = n+2$, $\forall\ n \geq 5$, there must be at least $n+2$ vertices of degree $n+1$ in $M(G_n)$, all with distinct colors, and each adjacent to vertices of all of the other colors. But then these must be the vertices $v, \{e_i : 1 \leq i \leq n\}$, since these are only ones with degree at least $n+1$. This is the contradiction, b-coloring with $n+2$ colors is impossible. Thus, we have $\varphi(M(G_n)) \leq n+1$. Hence, $\varphi(M(G_n)) = n+1$, $\forall\ n \geq 5$. □

Proposition 2.10 *If $n \geq 6$ then b-chromatic number on total graph of Gear graph $T(G_n)$ is $\varphi(T(G_n)) = n+1$.*

Proof Consider the coloring of $M(G_n)$ introduced on the proof of Theorem 9. An easy check shows that this coloring is a b-coloring of $T(G_n)$. Hence, $\varphi(T(G_n)) = n+1$, $\forall\ n \geq 6$. □

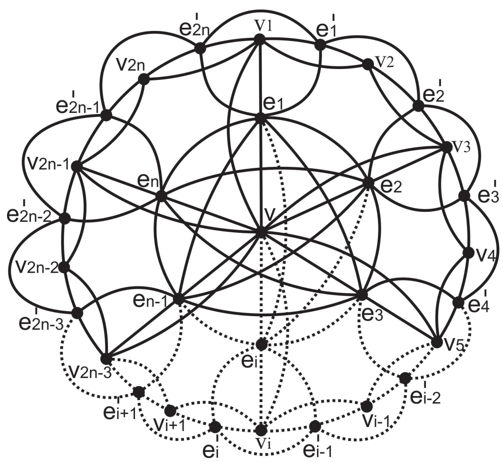

Fig.9 Total graph of Gear Graph $T(G_n)$.

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A Note on the Strong Defining Numbers in Graphs

Z.Tahmasbzadehbaee

(Department of Basic Science, Technical and Vocational University, Babol branch-Alzahra, Babol, I.R. Iran)

H.Abdollahzadeh Ahangar

(Department of Basic Science, Babol University of Technology, Babol, I.R. Iran)

D.A.Mojdeh

(Department of Mathematics, University of Mazandaran, Babolsar, I.R. Iran)

E-mail: ztahmasb@yahoo.com, ha.ahangar@nit.ac.ir, damojdeh@umz.ac.ir

Abstract: A *defining set* (of vertex coloring) of a graph $G = (V, E)$ is a set of vertices S with an assignment of colors to its elements which has a unique extension to a proper coloring of G. A defining set S is called a *strong defining set* if there exists an ordering set $\{v_1, v_2, \cdots, v_{|V|-|S|}\}$ of the vertices of $G-S$ such that in the induced list of colors in each of the subgraphs $G-S, G-(S\cup\{v_1\}), G-(S\cup\{v_1, v_2\}), \cdots, G-(S\cup\{v_1, v_2, \cdots, v_{|V|-|S|-1}\})$ there exists at least one vertex whose list of colors is of cardinality 1. The strong defining number, denoted $sd(G, k)$, of G is the cardinality of its smallest strong defining set, where $k \geqslant \chi(G)$. In the paper, [D.A. Mojdeh and A.P. Kazemi, Defining numbers in some of the Harary graphs, Appl. Math. Lett. **22** (2009), 922-926], the authors have studied the strong defining number in Harary graphs and posed the following problem: $\mathrm{sd}(H_{2m,3m+2}, \chi) = 2m$ if m is even and $\mathrm{sd}(H_{2m,3m+2}, \chi) = 2m + 1$ when m is odd. In this note we prove this problem.

Key Words: Defining set, strong defining set, Harary graphs.

AMS(2010): 05C15, 05C38.

§1. Introduction and Preliminaries

Let $G = (V, E)$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$ (briefly V and E, respectively). The order $n = n(G)$ of G is the number of its vertices. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V \mid uv \in E\}$ and its *closed neighborhood* is the set $N[v] = N(v)\cup\{v\}$. A proper k-coloring of G is an assignment of k different colors to the vertices of G, such that no two adjacent vertices receive the same color. The vertex chromatic number of G, $\chi(G)$, is the minimum number k, for which there exists a k-coloring for G. Let $\chi(G) \leq k \leq |V(G)|$. A set S of the vertices of G with an assignment of colors to them is called a *defining set* of vertex coloring of G, if there exists a unique extension of S to a proper k-coloring of G. A defining set with minimum cardinality is called a minimum defining set and its cardinality

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is the defining number, denoted by $d(G,k)$. If $k=\chi(G)$, then defining number is denoted by $d(G,\chi)$. Let G be a graph with n vertices. A defining set S, with an assignment of colors in G, is called a *strong defining* set of the vertex coloring of G with k colors if there exists an ordering set $\{v_1, v_2, \cdots, v_{n-|S|}\}$ of the vertices of $G-S$ such that in the induced list of colors in each of the subgraphs $G-S, G-(S\cup\{v_1\}), G-(S\cup\{v_1,v_2\}), \cdots, G-(S\cup\{v_1,v_2,\cdots,v_{n-|S|-1}\})$ there exists at least one vertex whose list of colors is of cardinality 1. The *strong defining number* of G, $\mathrm{sd}(G,k)$, is the cardinality of its smallest strong defining set. The strong defining number in graphs was introduced by Mahmoodian and Mendelsohn in [5] and has been studied by several authors. For more details, we refer the readers to [1-4, 6, 7].

For $2\le k<n$, the Harary graph $H_{k,n}$ on n vertices is defined as follows. Place n vertices around a circle, equally spaced. If k is even, $H_{k,n}$ is formed by making each vertex adjacent to the nearest $\frac{k}{2}$ vertices in each direction around the circle. If k is odd and n is even, $H_{k,n}$ is formed by making each vertex adjacent to the nearest $\frac{k-1}{2}$ vertices in each direction around the circle and to the diametrically opposite vertex. In both cases, $H_{k,n}$ is k-regular. If both k and n are odd, $H_{k,n}$ is constructed as follows. It has vertices $0,\ 1,\ \cdots, n-1$ and is constructed from $H_{k-1,n}$ by adding edges joining vertex i to vertex $i+\frac{n-1}{2}$ for $0\le i\le\frac{n-1}{2}$ (see [9]).

Mojdeh and Kazemi [8] have studied the defining and strong defining number in Harary graphs. In their paper, they showed that

$$\chi(H_{2m,3m+2})=\lceil\frac{3m+2}{2}\rceil$$

for $m\geqslant 2$, and posed the following conjecture.

Conjecture A *If $n=3m+2$, then*

$$\mathrm{sd}(H_{2m,3m+2},\chi)=\begin{cases}2m & if \quad n\ is\ even\\ 2m+1 & if \quad n\ is\ odd.\end{cases}$$

In this note, we prove that it is true.

§2. Main Results

Now we prove Conjecture A as the following Theorem.

Theorem 2.1 *If $n=3m+2$, then*

$$\mathrm{sd}(H_{2m,3m+2},\chi)=\begin{cases}2m & if \quad n\ is\ even\\ 2m+1 & if \quad n\ is\ odd.\end{cases}$$

Proof Let $V(H_{2m,3m+2}) = \{x_1, x_2, \cdots, x_{3m+2}\}$. First we show that

$$\mathrm{sd}(H_{2m,3m+2}, \chi) \leqslant \begin{cases} 2m & \text{if } n \text{ is even} \\ 2m+1 & \text{if } n \text{ is odd.} \end{cases}$$

Define the coloring function f by

$$f(x_i) = i \text{ for } 1 \leqslant i \leqslant \lceil \frac{3m}{2} \rceil + 1,$$
$$f(x_i) = i - \lceil \frac{3m}{2} \rceil - 1 \text{ for } \lceil \frac{3m}{2} \rceil + 2 \leqslant i \leqslant 2m+2,$$
$$f(x_i) = i - m - 1 \text{ for } 2m+3 \leqslant i \leqslant \lceil \frac{5m}{2} \rceil + 2 \text{ and}$$
$$f(x_i) = i - 2m - 1 \text{ for } \lceil \frac{5m}{2} \rceil + 3 \leqslant i \leqslant 3m+2.$$

We now consider the following cases.

Case 1. m is even.

Let $D = \{x_2, \cdots, x_{m+1}, x_{\lceil \frac{3m}{2} \rceil + 2}, \cdots, x_{\lceil \frac{5m}{2} \rceil + 2}\} \setminus \{x_{2m+2}\}$. Clearly $|D| = 2m$. Consider the function $g = f|_D$ as an assignment of colors to D in $H_{2m,3m+2}$. It is easy to see that the ordering

$$x_1, x_{3m+2}, x_{3m+1}, \cdots, x_{\lceil \frac{5m}{2} \rceil + 3}, x_{2m+2}, x_{\lceil \frac{3m}{2} \rceil + 1}, \cdots, x_{m+2}$$

of $V(H_{2m,3m+2}) - D$ satisfies the condition on definition of strong defining set and so D is a strong defining set of $H_{2m,3m+2}$. Hence, $\mathrm{sd}(H_{2m,3m+2}, \chi) \leqslant 2m$ in this case.

Case 2. m is odd.

Let $D = \{x_1, x_2, \cdots, x_{m+1}, x_{\lceil \frac{3m}{2} \rceil + 2}, \cdots, x_{\lceil \frac{5m}{2} \rceil + 2}\} \setminus \{x_{2m+2}\}$. Then $|D| = 2m+1$. Let $g = f|_D$ be an assignment of colors to D in $H_{2m,3m+2}$. Clearly f is the unique extension of g to a $\chi(H_{2m,3m+2})$-coloring. It is not hard to see that the ordering set

$$\{x_1, x_{3m+2}, x_{3m+1}, \cdots, x_{\lceil \frac{5m}{2} \rceil + 3}, x_{\lceil \frac{3m}{2} \rceil + 1}, \cdots, x_{m+2}\}$$

of $V(H_{2m,3m+2}) - D$ satisfies the condition on definition of strong defining set and so D is a strong defining set of $H_{2m,3m+2}$. Hence, $\mathrm{sd}(H_{2m,3m+2}, \chi) \leqslant 2m+1$ when m is odd.

Now it will be shown that

$$\mathrm{sd}(H_{2m,3m+2}, \chi) \geqslant \begin{cases} 2m & \text{if } n \text{ is even} \\ 2m+1 & \text{if } n \text{ is odd.} \end{cases}$$

Let S be a minimum strong defining set of $H = H_{2m,3m+2}$. Assume that $x \in V(H_{2m,3m+2}) - S$, then it takes the color uniquely if $N(x)$ has at least $\lceil \frac{3m}{2} \rceil$ coloring vertices. Therefore, any $\lceil \frac{3m}{2} \rceil$ vertices in S may be caused at most $\lfloor \frac{m}{2} \rfloor + 1$ of vertices in $V(H_{2m,3m+2}) - S$ take their colors uniquely, if these vertices in S are $S' = \{x_{i+1}, x_{i+2}. \cdots, x_{i+m}, x_{i+\lfloor \frac{3m}{2} \rfloor + 2}, \cdots x_{i+2m+1}\}$.

Now let m be even and S has at most $2m-1$ vertices, that is $S - S'$ has $\frac{m}{2} - 1$ vertices.

Then any vertex $x \in V(H_{2m,3m+2}) - S \cup \{x_{i+m+1}, x_{i+m+2}, \cdots, x_{i+\frac{3m}{2}+1}\}$ has at most

$$\frac{3m}{2} - 1 = \chi(H) - 2$$

coloring vertices in $N(x)$. This shows that the vertex x cannot take its color uniquely, a contradiction. Thus $|S| \geqslant 2m$.

Let m be odd and S has at most $2m$ vertices, that is $S - S'$ has $\lceil \frac{m}{2} \rceil - 1$ vertices. Then any vertex $x \in V(H_{2m,3m+2}) - S \cup \{x_{i+m+1}, x_{i+m+2}, \cdots, x_{i+\lfloor \frac{3m}{2} \rfloor+1}\}$ has at most

$$\lceil \frac{3m}{2} \rceil - 1 = \chi(H) - 2$$

coloring vertices in $N(x)$. This shows that the vertex x cannot take its color uniquely, a contradiction. Thus $|S| \geqslant 2m + 1$ and the proof is completed. □

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BIOGRAPHY

Mathematics for Everything with Combinatorics on Nature – A Report on the Promoter Dr.Linfan Mao of Mathematical Combinatorics

Florentin Smarandache

Mathematics & Science Department

University of New Mexico

705 Gurley Ave., Gallup, NM 87301, USA)

http://fs.gallup.unm.edu/FlorentinSmarandache.htm

E-mail: fsmarandache@gmail.com

The science's function is realizing the natural world, developing our society in coordination with natural laws and the mathematics provides the quantitative tool and method for solving problems helping with that understanding. Generally, understanding a natural thing by mathematical ways or means to other sciences are respectively establishing mathematical model on typical characters of it with analysis first, and then forecasting its behaviors, and finally, directing human beings for hold on its essence by that model.

As we known, the contradiction between things is generally kept but a mathematical system must be homogenous without contradictions in logic. The great scientist Albert Einstein complained classical mathematics once that "As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality." Why did it happens? It is in fact result in the consistency on mathematical systems because things are full of contradictions in nature in the eyes of human beings, which implies also that the classical mathematics for things in the nature is local, can not apply for hold on the behavior of things in the world completely. Thus, turning a mathematical system with contradictions to a compatible one and then establish an envelope mathematics matching with the nature is a proper way for understanding the natural reality of human beings. The *mathematical combinatorics* on Smarandache multispaces, proposed by Dr.Linfan Mao in mathematical circles nearly 10 years is just around this notion for establishing such an envelope theory. As a matter of fact, such a notion is praised highly by the Eastern culture, i.e., to hold on the global behavior of natural things on the understanding of individuals, which is nothing else but the essence of combinatorics.

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Linfan Mao was born in December 31, 1962, a worker's family of China. After graduated from Wanyuan school, he was beginning to work in the first company of —it China Construction Second Engineering Bureau at the end of December 1981 as a scaffold erector first, then appointed to be technician, technical adviser, director of construction management department, and then finally, the general engineer in construction project, respectively. But he was special preference for mathematics. He obtained an undergraduate diploma in applied mathematics and Bachelor of Science of *Peking University* in 1995, also postgraduate courses, such as those of graph theory, combinatorial mathematics, $\cdots$, etc. through self-study, and then began his career of doctoral study under the supervisor of Prof.Yanpei Liu of *Northern Jiaotong University* in 1999, finished his doctoral dissertation "A census of maps on surface with given underlying graph" and got his doctor's degree in 2002. He began his postdoctoral research on automorphism groups of surfaces with co-advisor Prof.Feng Tian in *Chinese Academy of Mathematics and System Science* from 2003 to 2005. After then, he began to apply combinatorial notion to mathematics and other sciences cooperating with some professors in USA. Now he has formed his own unique notion and method on scientific research. For explaining his combinatorial notion, i.e., *any mathematical science can be reconstructed from or made by combinatorization, and then extension mathematical fields for developing mathematics*, he addressed a report "Combinatorial speculations and the combinatorial conjecture for mathematics" in *The 2nd Conference on Combinatorics and Graph Theory of China* on his postdoctoral report "On automorphism groups of maps, surfaces and Smarandache geometries" in 2006. It is in this report he pointed out that the motivation for developing mathematics in 21th century is combinatorics, i.e., establishing an envelope mathematical theory by combining different branches of classical mathematics into a union one such that the classical branch is its special or local case, or determining the combinatorial structure of classical mathematics and then extending classical mathematics under a given combinatorial structure, characterizing and finding its invariants, which is called the *CC conjecture* today. Although he only reported with 15 minutes limitation in this conference but his report deeply attracted audiences in combinatorics or graph theory because most of them only research on a question or a problem in combinatorics or graph theory, never thought the contribution of combinatorial notion to mathematics and the whole science. After the full text of his report published in journal, Prof.L.Lovasz, the chairman of *International Mathematical Union* (IMU) appraise it "an interesting paper", and said "I agree that combinatorics, or rather the interface of combinatorics with classical mathematics, is a major theme today and in the near future" in one of his letter to Dr.Linfan Mao. This paper was listed also as a reference for the terminology *combinatorics* in Hungarian on Wikipedia, a free encyclopedia on the internet. After CC conjecture appeared 10 years, Dr.Linfan Mao was invited to make a plenary report "Mathematics after CC conjecture – combinatorial notions and achievements" in the *International Conference on Combinatorics, Graph Theory, Topology and Geometry* in January, 2015, surveying its roles in developing mathematics and mathematical sciences, such as those of its contribution to algebra, topology, Euclidean geometry or differential geometry, non-solvable differential equations or classical mathematical systems with contradictions to mathematics, quantum fields and gravitational field. His report was highly valued by mathematicians coming from USA, France, Germany and China. They surprisingly found that most results in his report are finished by himself in the past 10 years.

Generally, the understanding on nature by human beings is originated from observation, particularly, characterizing behaviors of natural things by solution of differential equation established on those of observed data. However, the uncertainty of microscopic particles, or different positions of the observer standing on is resulted in different equations. For example, if the observer is in the interior of a natural thing, we usually obtain non-solvable differential equations but each of them is solvable. How can we understand this strange phenomenon? There is an ancient poetry which answer this thing in China, i.e., "Know not the real face of Lushan mountain, Just because you are inside the mountain". Hence, all contradictions are artificial, not the nature of things, which only come from the boundedness or unilateral knowing on natural things of human beings. Any thing inherits a combinatorial structure in the nature. They are coherence work and development. In fact, there are no contradictions between them in the nature. Thus, extending a contradictory system in classical mathematics to a compatible one and establishing an envelope theory for understanding natural things motivate Dr.Linfan Mao to extend classical mathematical systems such as those of Banach space and Hilbert space on oriented graphs with operators, i.e., action flows with conservation on each vertex, apply them to get solutions of action flows with geometry on systems of algebraic equations, ordinary differential equations or partial differential equations, and construct combinatorial model for microscopic particles with a mathematical interpretation on the uncertainty of things. For letting more peoples know his combinatorial notion on contradictory mathematical systems, he addressed a report "Mathematics with natural reality – action flows" with philosophy on the *National Conference on Emerging Trends in Mathematics and Mathematical Sciences of India* as the chief guest and got highly praised by attendee in December of last year.

After finished his postdoctoral research in 2005, Dr.Linfan Mao always used combinatorial notion to the nature and completed a number of research works. He has found a natural road from combinatorics to topology, topology to geometry, and then from geometry to theoretical physics and other sciences by combinatorics and published 3 graduate textbooks in mathematics and a number of collection of research papers on mathematical combinatorics for the guidance of young teachers and post-graduated students understanding the nature. He is now the president of the *Academy of Mathematical Combinatorics & Applications* (USA), also the editor-in-chief of *International Journal of Mathematical Combinatorics* (ISSN 1937-1055, founded in 2007).

Go your own way. "Now the goal is that the horizon, Leaving the world can be only your back". Dr.Linfan Mao is also the vice secretary-general of *China Tendering & Bidding Association* at the same time. He is also busy at the research on bidding purchasing policy and economic optimization everyday, but obtains his benefits from the research on mathematics and purchase both. As he wrote in the postscript "My story with multispaces" for the *Proceedings of the First International Conference on Smarandache Multispace & Multistructure* (USA) in 2013, he said: "For multispaces, a typical example is myself. My first profession is the industrial and civil buildings, which enables me worked on architecture technology more than 10 years in a large construction enterprise of China. But my ambition is mathematical research, which impelled me learn mathematics as a doctoral candidate in the *Northern Jiaotong University* and then, a postdoctoral research fellow in the *Chinese Academy of Sciences*. It was a very strange for search my name on the internet. If you search my name *Linfan Mao* in Google, all items are related with my works on mathematics, including my monographs and papers published in English journals. But if you search my name *Linfan Mao in Chinese* on Baidu, a Chinese search engine in China, items are nearly all of my works on bids because I am simultaneously the vice secretary-general of China Tendering & Bidding Association. Thus, I appear 2 faces in front of the public: In the eyes of foreign peoples I am a mathematician, but in the eyes of Chinese, I am a scholar on theory of bidding and purchasing. So I am a multispace myself." He also mentioned in this postscript: "There is a section in my monograph *Combinatorial Geometry with Applications to Fields* published in USA with a special discussion on scientific notions appeared in *TAO TEH KING*, a well-known Chinese book, applying topological graphs as the inherited structure of things in the nature, and then hold on behavior of things by combinatorics on space model and gravitational field, gauge field appeared in differential geometry and theoretical physics. This is nothing else but examples of applications of mathematical combinatorics. Hence, it is not good for scientific research if you don't understand Chinese philosophy because it is a system notion on things for Chinese, which is in fact the Smarandache multispace in an early form. There is an old saying, i.e., philosophy gives people wisdom and mathematics presents us precision. The organic combination of them comes into being the scientific notion for multi-facted nature of natural things on Smarandache multispaces, i.e., mathematical combinatorics. This is a kind of sublimation of scientific research and good for understanding the nature."

This is my report on Dr.Linfan Mao with his combinatorial notion. We therefore note that Dr.Linfan Mao is working on a way conforming to the natural law of human understanding. As he said himself: "mathematics can not be existed independent of the nature, and only those of mathematics providing human beings with effective methods for understanding the nature should be the search aim of mathematicians!" As a matter of fact, the mathematical combinatorics initiated by him in recent decade is such a kind of mathematics following with researchers, and there are journals and institutes on such mathematics. We believe that mathematicians would provide us more and more effective methods for understanding the nature following his combinatorial notion and prompt the development of human society in harmony with the nature.

Lead to something new and better. No man can sever the bonds that unite him to his society simply by averting his eyes. He must ever be receptive and sensitive to the new; and have sufficient courage and skill to novel facts and to deal with them.

By Franklin Roosevelt, an American President.

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