# Formulas for the Somos Constant

Edgar Valdebenito

abstract

We show some formulas for the somos constant

$\sigma = 1.66168\ldots$

# Algunas Fórmulas para la Constante $\sigma$ de Somos

**EDGAR VALDEBENITO**
**(2007)**

**Resumen.** Se muestran algunas fórmulas para la constante $\sigma$ de Somos.

## 1 Introducción

La constante (sigma) $\sigma$ de Somos se define por:

$$\sigma = 1^{1/2} 2^{1/4} 3^{1/8} \ldots = \prod_{n=1}^{\infty} n^{2^{-n}} \tag{1.1}$$

$$\sigma = 1.66168794\ldots \tag{1.2}$$

En esta nota se muestran algunas fórmulas para la constante $\sigma$

## 2 Fórmulas

$$\ln(\sigma) = \sum_{n=1}^{\infty} \sum_{k=0}^{2n} (-1)^k \binom{2n-1}{k-1} \ln(k+1) \tag{2.1}$$

$$\sigma = \prod_{n=1}^{\infty} \prod_{k=0}^{2n} (k+1)^{(-1)^k \binom{2n-1}{k-1}} \tag{2.2}$$

$$\ln(\sigma) = \frac{\ln(2)}{2} + \sum_{n=1}^{\infty} (-1)^n \sum_{k=1}^{n} \frac{(-1)^k}{2^{k+1}(n-k+1)(k+1)^{n-k+1}} \tag{2.3}$$

$$\sigma = \prod_{n=1}^{\infty} \left( \frac{2n(2n+1)}{(2n-1)^2} \right)^{2^{-2n}} \tag{2.4}$$

$$\sigma = \sqrt{2} \prod_{n=1}^{\infty} \left( \frac{(n+1)(2n+1)}{2n^2} \right)^{2^{-2n-1}} \tag{2.5}$$

$$\sigma = \left( \prod_{n=1}^{\infty} (2n+1)^{3/2^{2n+1}} \right) \left( \prod_{n=1}^{\infty} \left( 1 - \frac{1}{2n+1} \right)^{2^{-2n}} \right) \tag{2.6}$$

$$\sigma = \left( \prod_{n=1}^{\infty} (2n)^{3/2^{2n+1}} \right) \left( \prod_{n=1}^{\infty} \left( 1 + \frac{1}{2n} \right)^{2^{-2n-1}} \right) \tag{2.7}$$

$$\begin{aligned} \sigma &= \sqrt[4]{2} + \sum_{n=2}^{\infty} \sqrt[4]{2} \ldots \sqrt[2^n]{n} \left( \sqrt[2^{n+1}]{n+1} - 1 \right) = \\ &= \sqrt[4]{2} + \sqrt[4]{2} \left( \sqrt[8]{3} - 1 \right) + \sqrt[4]{2} \sqrt[8]{3} \left( \sqrt[16]{4} - 1 \right) + \ldots \end{aligned} \tag{2.8}$$

$$\begin{aligned} \sigma &= \sqrt{2} + \sum_{n=2}^{\infty} \sqrt[4]{2} \ldots \sqrt[2^n]{n} \left( \sqrt[2^n]{n+1} - \sqrt[2^n]{n} \right) = \\ &= \sqrt{2} + \sqrt[4]{2} \left( \sqrt[4]{3} - \sqrt[4]{2} \right) + \sqrt[4]{2} \sqrt[8]{3} \left( \sqrt[8]{4} - \sqrt[8]{3} \right) + \ldots \end{aligned} \tag{2.9}$$

$$ln(\sigma) = \sum_{n=0}^{\infty} \frac{c_n}{n+1} = 1 - \frac{1}{2}\frac{3}{2} + \frac{1}{3}\frac{17}{12} - \frac{1}{4}\frac{35}{24} + \ldots \tag{2.10}$$

$$c_0 = 1, c_n = -\sum_{k=1}^{n} \frac{2k+1}{k(k+1)} c_{n-k}, n = 1,2,3,\ldots$$

$$ln(\sigma) = \int_0^\infty \frac{(1-e^{-x})e^{-x}}{(2-e^{-x})x}dx = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n} Li_n\left(\frac{1}{2}\right) \qquad (2.11)$$

$Li_n\left(\frac{1}{2}\right)$ , es la función polilogaritmo

$$ln(\sigma) = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n}\left(\sum_{k=0}^\infty \frac{\gamma(n,(k+1)a)}{2^{k+1}(k+1)^n}\right) + \frac{Ie(a)}{2} - \sum_{n=0}^\infty \frac{Ie((n+2)a)}{2^{n+2}} \qquad (2.12)$$

$$a > 0$$

$\gamma(n,(k+1)a)$, es la función gamma incompleta

$Ie((n+2)a)$, es la integral exponencial: $Ie(x) = \int_x^\infty \frac{e^{-u}}{u}du$

$$ln(\sigma) = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n}\left(\sum_{m=1}^\infty \frac{1}{2^{m+1}}\left(\frac{m}{m+1}\right)^n\right) = \sum_{n=1}^\infty \sum_{m=1}^n \frac{(-1)^{n-m}}{2^{m+1}(n-m+1)}\left(\frac{m}{m+1}\right)^{n-m+1} \qquad (2.13)$$

$$\sigma = m^{\frac{1}{2^m-1}} \prod_{n=1}^\infty \left(n \prod_{k=1}^{m-1} (mn-m+k)^{2^{m-k}}\right)^{2^{-mn}} \quad , m = 1,2,3,... \qquad (2.14)$$

$$\sigma = \sqrt[3]{2} \prod_{n=1}^\infty \left(n(2n-1)^2\right)^{2^{-2n}} \qquad (2.15)$$

$$\sigma = \sqrt[7]{3}\prod_{n=1}^{\infty}\left(n\left(3n-2\right)^{4}\left(3n-1\right)^{2}\right)^{2^{-3n}} \tag{2.16}$$

$$ln\left(\sigma\right) = ln\left(2\right) + \sum_{n=1}^{\infty}\left(-1\right)^{n} c_{n}\left(ln\left(2\right) - \sum_{k=1}^{n}\frac{\left(-1\right)^{k-1}}{k}\right) \tag{2.17}$$

$$c_{0} = 1, c_{n} = -\sum_{k=1}^{n}\frac{c_{n-k}}{k+1}, n \in \mathbb{N}$$

## 3 Referencias


1) Abramowitz, M. e I.A. Stegun, Handbook of Mathematical Functions. Nueva York: Dover , 1965.
2) I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products (A. Jeffrey) , Academic Press, New York, London, and Toronto, 1980.
3) M. R. Spiegel, Mathematical Handbook, McGraw-Hill Book Company, New York, 1968.
4) E. Valdebenito, Pi Handbook, manuscript, unpublished, 1989 , ( 20000 fórmulas).