New Mathematics and Natural Computation
Vol. 8, No. 3 (2012) 343–359

DOI: 10.1142/S179300571250007X

AN INTRODUCTION TO DSmT FOR INFORMATION FUSION

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The management and combination of uncertain, imprecise, fuzzy and even paradoxical or highly conflicting sources of information has always been, and still remains today, of primal importance for the development of reliable modern information systems involving artificial and approximate reasoning. In this short paper, we present an introduction of our recent theory of plausible and paradoxical reasoning, known as Dezert-Smarandache Theory (DSmT), developed to deal with imprecise, uncertain and conflicting sources of information. We focus our presentation on the foundations of DSmT and on its most important rules of combination, rather than on browsing specific applications of DSmT available in literature. Several simple examples are given throughout this presentation to show the efficiency and the generality of this new theory.

Keywords: Dezert-Smarandache Theory; DSmT; quantitative and qualitative reasoning; information fusion.

1. Introduction

The management and combination of uncertain, imprecise, fuzzy and even paradoxical or highly conflicting sources of information has always been, and still remains today, of primal importance for the development of reliable modern information systems involving approximate reasoning. The combination (fusion) of information arises in many fields of applications nowadays (especially in defense, medicine, finance, geo-science, economy, etc). When several sensors, observers or experts have to be combined together to solve a problem, or if one wants to update our current estimation of solutions for a given problem with some new information available, we need powerful and solid mathematical tools for the fusion, specially when the information one has to deal with is imprecise and uncertain. In this paper, we present a short introduction of our recent theory of plausible and paradoxical reasoning, known as Dezert-Smarandache Theory (DSmT) in the literature, developed for approximate reasoning with imprecise, uncertain and conflicting sources of information. Recent publications have shown the interest and the ability of DSmT to solve problems where other approaches fail, especially when conflict between sources becomes high. We focus here on the foundations of DSmT, and on the main important rules of combination, than on browsing specific applications of DSmT available in literature. Successful applications of DSmT in target tracking, satellite surveillance, situation analysis, robotics, medicine, biometrics, etc, can be found in Parts II of Refs. 20, 23, 25 and on the world wide web.[26] Due to space limitation, we cannot give detailed examples of our formulas, but they can be easily found in Refs. 20, 23, 25 and freely available on the world wide web. An extended version of this paper is available in Refs. 7, 25.

2. Foundations of DSmT

The development of DSmT (Dezert-Smarandache Theory of plausible and paradoxical reasoning[4,20]) arises from the necessity to overcome the inherent limitations of DST (Dempster-Shafer Theory[18]) which are closely related with the acceptance of Shafer's model for the fusion problem under consideration (i.e. the frame of *discernment* Θ is implicitly defined as a finite set of *exhaustive* and *exclusive* hypotheses θ_i, $i = 1, \ldots, n$ since the masses of belief are defined only on the power set of Θ, see Sec. 2.1), the third middle excluded principle (i.e. the existence of the complement for any elements/propositions belonging to the power set of Θ), and the acceptance of Dempster's rule of combination (involving normalization) as the framework for the combination of independent sources of evidence. Discussions on limitations of DST and presentation of some alternative rules to Dempster's rule of combination can be found in Refs. 8, 10−13, 15, 17, 20, 27, 30, 33−36 and therefore they will be not reported in details in this introduction. We argue that these three fundamental conditions of DST can be removed and another new mathematical approach for combination of evidence is possible. This is the purpose of DSmT.

The basis of DSmT is the refutation of the principle of the third excluded middle and Shafer's model, since for a wide class of fusion problems the intrinsic nature of hypotheses can be only vague and imprecise in such a way that precise refinement is just impossible to obtain in reality so that the exclusive elements θ_i cannot be properly identified and precisely separated. Many problems involving fuzzy continuous and relative concepts described in natural language and having no absolute interpretation like tallness/smallness, pleasure/pain, cold/hot, Sorites paradoxes, etc, enter in this category. DSmT starts with the notion of *free DSm model*, denoted $\mathcal{M}^f(\Theta)$, and considers Θ only as a frame of exhaustive elements θ_i, $i = 1, \ldots, n$ which can potentially overlap. This model is *free* because no other assumption is done on the hypotheses, but the weak exhaustivity constraint which can always be satisfied according to the closure principle.[20] No other constraint is involved in the free DSm model. When the free DSm model holds, the commutative and associative classical

DSm rule of combination, denoted by DSmC, corresponding to the conjunctive consensus defined on the free Dedekind's lattice is performed.

Depending on the intrinsic nature of the elements of the fusion problem under consideration, it can however happen that the free model does not fit the reality because some subsets of Θ can contain elements known to be truly exclusive but also truly non existing at all at a given time (specially when working on dynamic fusion problem where the frame Θ varies with time with the revision of the knowledge available). These integrity constraints are then explicitly and formally introduced into the free DSm model $\mathcal{M}^f(\Theta)$ in order to adapt it properly to fit as close as possible with the reality and permit to construct a *hybrid DSm model* $\mathcal{M}(\Theta)$ on which the combination will be efficiently performed. Shafer's model, denoted $\mathcal{M}^0(\Theta)$, corresponds to a very specific hybrid DSm model including all possible exclusivity constraints. DST has been developed for working only with $\mathcal{M}^0(\Theta)$ while DSmT has been developed for working with any kind of hybrid model (including Shafer's model and the free DSm model), to manage as efficiently and precisely as possible imprecise, uncertain and potentially highly conflicting sources of evidence while keeping in mind the possible dynamicity of the information fusion problematic. The foundations of DSmT are therefore totally different from those of all existing approaches managing uncertainties, imprecisions and conflicts. DSmT provides a new interesting way to attack the information fusion problematic with a general framework in order to cover a wide variety of problems.

DSmT refutes also the idea that sources of evidence provide their beliefs with the same absolute interpretation of elements of the same frame Θ and the conflict between sources arises not only because of the possible unreliability of sources, but also because of possible different and relative interpretation of Θ, e.g. what is considered as good for somebody can be considered as bad for somebody else. There is some unavoidable subjectivity in the belief of assignments provided by the sources of evidence. Otherwise, it would mean that all bodies of evidence have a same objective and universal interpretation (or measure) of the phenomena under consideration, which unfortunately rarely occurs in reality, and only when basic belief assignments (bba's) are based on some *objective probabilities* transformations. But in this last case, probability theory can handle properly and efficiently the information, and DST, as well as DSmT, becomes useless. If we now get out of the probabilistic background argumentation for the construction of bba, we claim that in most of cases, the sources of evidence provide their beliefs about elements of the frame of the fusion problem based only on their own limited knowledge and experience without reference to the (inaccessible) absolute truth of the space of possibilities.

2.1. *The power set, hyper-power set and super-power set*

In DSmT, we take care about the model associated with the set Θ of hypotheses where the solution of the problem is assumed to belong to. In particular, the three main sets (power set, hyper-power set and super-power set) can be used depending on their ability to fit adequately with the nature of hypotheses. In the following, we assume that $\Theta = \{\theta_1, \ldots, \theta_n\}$ is a finite set (called frame) of n exhaustive elements.[a] If $\Theta = \{\theta_1, \ldots, \theta_n\}$ is *a priori* not closed (Θ is said to be an open world/frame), one can always include in it a closure element, say θ_{n+1} in such away that we can work with a new closed world/frame $\{\theta_1, \ldots, \theta_n, \theta_{n+1}\}$. So without loss of generality, we will always assume that we work in a closed world by considering the frame Θ as a finite set of exhaustive elements. Before introducing the power set, the hyper-power set and the super-power set, it is necessary to recall that subsets are regarded as propositions in Dempster-Shafer Theory (see Chapter 2[18]) and we adopt the same approach in DSmT.

- **Subsets as propositions:** Glenn Shafer in pages 35–37[18] considers the subsets as propositions in the case we are concerned with the true value of some quantity θ taking its possible values in Θ. Then the propositions $\mathcal{P}_\theta(A)$ of interest are those of the form[b]:

  $$\mathcal{P}_\theta(A) \triangleq \textit{The true value of } \theta \textit{ is in a subset } A \textit{ of } \Theta.$$

  Any proposition $\mathcal{P}_\theta(A)$ is thus in one-to-one correspondence with the subset A of Θ. Such correspondence is very useful since it translates the logical notions of conjunction $\wedge$, disjunction $\vee$, implication $\Rightarrow$ and negation $\neg$ into the set-theoretic notions of intersection $\cap$, union $\cup$, inclusion $\subset$ and complementation $c(.)$. Indeed, if $\mathcal{P}_\theta(A)$ and $\mathcal{P}_\theta(B)$ are two propositions corresponding to subsets A and B of Θ, then the conjunction $\mathcal{P}_\theta(A) \wedge \mathcal{P}_\theta(B)$ corresponds to the intersection $A \cap B$ and the disjunction $\mathcal{P}_\theta(A) \vee \mathcal{P}_\theta(B)$ corresponds to the union $A \cup B$. A is a subset of B if and only if $\mathcal{P}_\theta(A) \Rightarrow \mathcal{P}_\theta(B)$ and A is the set-theoretic complement of B with respect to Θ (written $A = c_\Theta(B)$) if and only if $\mathcal{P}_\theta(A) = \neg\mathcal{P}_\theta(B)$.
- **Canonical form of a proposition:** In DSmT, we consider all propositions/sets in a canonical form. We take the disjunctive normal form, which is a disjunction of conjunctions, and it is unique in Boolean algebra and simplest. For example, $X = A \cap B \cap (A \cup B \cup C)$ it is not in a canonical form, but we simplify the formula and $X = A \cap B$ is in a canonical form.
- **The power set:** $2^\Theta \triangleq (\Theta, \cup)$
  Besides Dempster's rule of combination, the power set is one of the corner stones of Dempster-Shafer Theory (DST) since the basic belief assignments to combine are defined on the power set of the frame Θ. In mathematics, given a set Θ, the power set of Θ, written 2^Θ, is the set of all subsets of Θ. In Zermelo–Fraenkel set theory with the axiom of choice (ZFC), the existence of the power set of any set is postulated by the axiom of power set. In other words, Θ generates the power set 2^Θ with the $\cup$ (union) operator only. More precisely, the power set 2^Θ is defined as the

[a] We do not assume here that elements θ_i are necessary exclusive, unless specified. There is no restriction on θ_i but the exhaustivity.
[b] We use the symbol $\triangleq$ to mean *equals by definition*; the right-hand side of the equation is the definition of the left-hand side.

set of all composite propositions/subsets built from elements of Θ with $\cup$ operator such that:

(1) $\emptyset, \theta_1, \ldots, \theta_n \in 2^\Theta$.
(2) If $A, B \in 2^\Theta$, then $A \cup B \in 2^\Theta$.
(3) No other elements belong to 2^Θ, except those obtained by using rules 1 and 2.

- **The hyper-power set:** $D^\Theta \triangleq (\Theta, \cup, \cap)$
  One of the cornerstones of DSmT is the free Dedekind's lattice[2] denoted as *hyper-power set* in DSmT framework. Let $\Theta = \{\theta_1, \ldots, \theta_n\}$ be a finite set (called frame) of n exhaustive elements. The hyper-power set D^Θ is defined as the set of all composite propositions/subsets built from elements of Θ with $\cup$ and $\cap$ operators such that:

  (1) $\emptyset, \theta_1, \ldots, \theta_n \in D^\Theta$.
  (2) If $A, B \in D^\Theta$, then $A \cap B \in D^\Theta$ and $A \cup B \in D^\Theta$.
  (3) No other elements belong to D^Θ, except those obtained by using rules 1 or 2.

  Therefore by convention, we write $D^\Theta = (\Theta, \cup, \cap)$ which means that Θ generates D^Θ under operators $\cup$ and $\cap$. The dual (obtained by switching $\cup$ and $\cap$ in expressions) of D^Θ is itself. There are elements in D^Θ which are self-dual (dual to themselves), for example α_8 for the case when $n = 3$ in the following example. The cardinality of D^Θ is majored by 2^{2^n} when the cardinality of Θ equals n, i.e. $|\Theta| = n$. The generation of hyper-power set D^Θ is closely related with the famous Dedekind's problem[1,2] on enumerating the set of isotone Boolean functions. The generation of the hyper-power set is presented.[20] Since for any given finite set Θ, $|D^\Theta| \geq |2^\Theta|$ we call D^Θ the *hyper-power set* of Θ. The cardinality of the hyper-power set D^Θ for $n \geq 1$ follows the sequence of Dedekind's numbers,[19] i.e. 1, 2, 5, 19, 167, 7580, 7828353,... and analytical expression of Dedekind's numbers has been obtained recently by Tombak[29] (see Ref. 20 for details on generation and ordering of D^Θ). Interesting investigations on the programming of the generation of hyper-power sets for engineering applications have been done in Chapter 15.[23,25]

  Shafer's model of a frame: More generally, when all the elements of a given frame Θ are known (or are assumed to be) truly exclusive, then the hyper-power set D^Θ reduces to the classical power set 2^Θ. Therefore, working on power set 2^Θ as Glenn Shafer has proposed in his Mathematical Theory of Evidence[18] is equivalent to work on hyper-power set D^Θ with the assumption that all elements of the frame are exclusive. This is what we call *Shafer's model of the frame* Θ, written $\mathcal{M}^0(\Theta)$, even if such model/assumption has not been clearly stated explicitly by Shafer himself in his milestone book.

- **The super-power set:** $S^\Theta \triangleq (\Theta, \cup, \cap, c(.))$
  The notion of super-power set has been introduced by Smarandache in Chapter 8.[23] It corresponds actually to the theoretical construction of the power set of the minimal[c] refined frame Θ^{ref} of Θ. Θ generates S^Θ under operators $\cup$, $\cap$ and

[c] The minimality refers here to the cardinality of the refined frames.

complementation $c(.)$. $S^{\Theta} = (\Theta, \cup, \cap, c(.))$ is a Boolean algebra with respect to the union, intersection and complementation. Therefore, working with the super-power set is equivalent to work with a minimal theoretical refined frame Θ^{ref} satisfying Shafer's model. More precisely, S^{Θ} is defined as the set of all composite propositions/subsets built from elements of Θ with $\cup$, $\cap$ and $c(.)$ operators such that:

(1) $\emptyset, \theta_1, \ldots, \theta_n \in S^{\Theta}$.
(2) If $A, B \in S^{\Theta}$, then $A \cap B \in S^{\Theta}$, $A \cup B \in S^{\Theta}$.
(3) If $A \in S^{\Theta}$, then $c(A) \in S^{\Theta}$.
(4) No other elements belong to S^{Θ}, except those obtained by using rules 1, 2 and 3.

As reported in Ref. 21, a similar generalization has been previously used in 1993 by Guan and Bell[9] for the Dempster-Shafer rule using propositions in sequential logic and reintroduced in 1994 by Paris in his book.[14, p. 4] In summary, DSmT offers truly the possibility to build and to work on refined frames and to deal with the complement whenever necessary, but in most applications either the frame Θ is already built/chosen to satisfy Shafer's model or the refined granules have no clear physical meaning. This does not allow to be considered/assessed individually so that working on the hyper-power set is usually sufficient to deal with uncertain imprecise (quantitative or qualitative) and highly conflicting sources of evidences. Working with S^{Θ} is actually very similar to working with 2^{Θ} in the sense that in both cases we work with classical power sets; the only difference is that when working with S^{Θ} we have implicitly switched from the original frame Θ representation to a minimal refinement Θ^{ref} representation. Therefore, in the sequel, we focus our discussions based mainly on hyper-power set rather than (super-) power set which has already been the basis for the development of DST. But as already mentioned, DSmT can easily deal with belief functions defined on 2^{Θ} or S^{Θ} similarly as those defined on D^{Θ}. In the sequel, we use the generic notation G^{Θ} to denote the sets (power set, hyper-power set and super-power set) on which the belief functions are defined.

The main distinctions between DSmT and DST are summarized by the following points:

(1) The refinement is not always (physically) possible, especially for elements from the frame of discernment whose frontiers are not clear, such as: colors, vague sets, unclear hypotheses, etc. in the frame of discernment; DST does not fit well for working in such cases, while DSmT does;
(2) Even in the case when the frame of discernment can be refined (i.e. the *atomic* elements of the frame have all a distinct physical meaning), it is still easier to use DSmT than DST since in DSmT framework the refinement is done automatically by the mathematical construction of the super-power set;
(3) DSmT offers better fusion rules, for example Proportional Conflict redistribution Rule # 5 (PCR5) — presented in the sequel — is better than Dempster's

rule; hybrid DSm rule (DSmH) works for the dynamic fusion, while Dubois-Prade fusion rule does not (DSmH is an extension of Dubois-Prade rule);

(4) DSmT offers the best qualitative operators (when working with labels) giving the most accurate and coherent results;

(5) DSmT offers new interesting quantitative conditioning rules (BCRs) and qualitative conditioning rules (QBCRs), different from Shafer's conditioning rule (SCR). SCR can be seen simply as a combination of a prior mass of belief with the mass $m(A) = 1$ whenever A is the conditioning event;

(6) DSmT proposes a new approach for working with imprecise quantitative or qualitative information and not limited to interval-valued belief structures as proposed generally in the literature.[3,31]

2.2. *Notion of free and hybrid DSm models*

Free DSm model: The elements θ_i, $i = 1, \ldots, n$ of Θ constitute the finite set of hypotheses/concepts characterizing the fusion problem under consideration. When there is no constraint on the elements of the frame, we call this model the *free DSm model*, written $\mathcal{M}^f(\Theta)$. This free DSm model allows to deal directly with fuzzy concepts which depict a continuous and relative intrinsic nature and which cannot be precisely refined into finer disjoint information granules having an absolute interpretation because of the unreachable universal truth. In such case, the use of the hyper-power set D^Θ (without integrity constraints) is particularly well adapted for defining the belief functions one wants to combine.

Shafer's model: In some fusion problems involving discrete concepts, all the elements θ_i, $i = 1, \ldots, n$ of Θ can be truly exclusive. In such case, all the exclusivity constraints on θ_i, $i = 1, \ldots, n$ have to be included in the previous model to characterize properly the true nature of the fusion problem and to fit it with the reality. By doing this, the hyper-power set D^Θ as well as the super-power set S^Θ reduce naturally to the classical power set 2^Θ and this constitutes what we have called *Shafer's model*, denoted $\mathcal{M}^0(\Theta)$. Shafer's model corresponds actually to the most restricted hybrid DSm model.

Hybrid DSm models: Between the class of fusion problems corresponding to the free DSm model $\mathcal{M}^f(\Theta)$ and the class of fusion problems corresponding to Shafer's model $\mathcal{M}^0(\Theta)$, there exists another wide class of hybrid fusion problems involving Θ in both fuzzy continuous concepts and discrete hypotheses. In such (hybrid) class, some exclusivity constraints and possibly some non-existential constraints (especially when working on dynamic[d] fusion) have to be taken into account. Each hybrid fusion problem of this class will then be characterized by a proper hybrid DSm model denoted $\mathcal{M}(\Theta)$ with $\mathcal{M}(\Theta) \neq \mathcal{M}^f(\Theta)$ and $\mathcal{M}(\Theta) \neq \mathcal{M}^0(\Theta)$.

[d] i.e. when the frame Θ and/or the model $\mathcal{M}$ is changing with time.

In any fusion problems, we consider as primordial at the very beginning and before combining information expressed as belief functions to define clearly the proper frame Θ of the given problem and to choose explicitly its corresponding model one wants to work with. Once this is done, the second important point is to select the proper set 2^Θ, D^Θ or S^Θ on which the belief functions will be defined. The third point concerns the choice of an efficient rule of combination of belief functions and finally the criteria adopted for decision-making.

2.3. *Generalized belief functions*

From a general frame Θ, we define a map $m(.) : G^\Theta \to [0, 1]$ associated to a given body of evidence $\mathcal{B}$ as

$$m(\emptyset) = 0 \quad \text{and} \quad \sum_{A \in G^\Theta} m(A) = 1. \tag{1}$$

The quantity $m(A)$ is called the *generalized basic belief assignment/mass* (gbba) of A. The *generalized belief and plausibility functions* are defined in almost the same manner as within DST, i.e.

$$\mathrm{Bel}(A) = \sum_{\substack{B \subseteq A \\ B \in G^\Theta}} m(B) \quad \mathrm{Pl}(A) = \sum_{\substack{B \cap A \neq \emptyset \\ B \in G^\Theta}} m(B). \tag{2}$$

We recall that G^Θ is the generic notation for the set on which the gbba is defined (G^Θ can be 2^Θ, D^Θ or even S^Θ depending on the model chosen for Θ). G^Θ is called the fusion space. These definitions are compatible with the definitions of the classical belief functions in DST framework when $G^\Theta = 2^\Theta$ for fusion problems where Shafer's model $\mathcal{M}^0(\Theta)$ holds. We still have $\forall A \in G^\Theta$, $\mathrm{Bel}(A) \leq \mathrm{Pl}(A)$. Note that when working with the free DSm model $\mathcal{M}^f(\Theta)$, one has always $\mathrm{Pl}(A) = 1 \, \forall A \neq \emptyset \in (G^\Theta = D^\Theta)$ which is normal.

3. Combination of Belief Functions with the Proportional Conflict Redistribution Rule

In the development of DSmT, several rules have been proposed to combine distinct sources of evidence providing their bba's defined on the same fusion space G^Θ. The most simple one is the DSmC (Classical/conjunctive rule) when DSm free model holds, or the DSmH (Hybrid rule) for working with DSm hybrid models of the frame.[20] DSmH is an extension of Dubois and Prade (DP) rule of combination,[8] and consists to apply a direct transfer of partial conflicts onto partial uncertainties. The most recent and effective combination rule proposed in DSmT for managing the conflicting mass of belief is the Proportional Conflict Redistribution rule (PCR) that is presented in this section. The idea behind PCR[22,23] is to transfer (total or partial) conflicting masses to non-empty sets involved in the conflicts proportionally with respect to the masses assigned to them by sources as follows:

(1) Calculation the conjunctive rule of the belief masses of sources;
(2) Calculation the total or partial conflicting masses (the product of bba's committed to the empty set);
(3) Redistribution of the (total or partial) conflicting masses to the non-empty sets involved in the conflicts proportionally with respect to their masses assigned by the sources.

The way the conflicting mass is redistributed yields actually several versions of PCR rules. These PCR fusion rules work for any degree of conflict, for any DSm models (Shafer's model, free DSm model or any hybrid DSm model) and both in DST and DSmT frameworks for static or dynamical fusion situations. We present below only the most sophisticated proportional conflict redistribution rule denoting PCR5 in Refs. 22 and 23. PCR5 rule is what we feel to be the most efficient PCR fusion rule developed so far. This rule redistributes the partial conflicting mass to the elements involved in the partial conflict, considering the conjunctive normal form of the partial conflict. PCR5 is what we think the most mathematically exact redistribution of conflicting mass to non-empty sets following the logic of the conjunctive rule. It does a better redistribution of the conflicting mass than Dempster's rule since PCR5 goes backwards on the tracks of the conjunctive rule and redistributes the conflicting mass only to the sets involved in the conflict and proportionally to their masses put in the conflict. PCR5 rule is quasi-associative and preserves the neutral impact of the vacuous belief assignment because in any partial conflict, as well in the total conflict (which is a sum of all partial conflicts), the conjunctive normal form of each partial conflict does not include Θ since Θ is a neutral element for intersection (conflict). Therefore, Θ gets no mass after the redistribution of the conflicting mass. We have proved in Ref. 23 the continuity property of the fusion result with continuous variations of bba's to combine.

3.1. *PCR formula*

The PCR5 formula for the combination of two sources ($s = 2$) is given by: $m_{PCR5}(\emptyset) = 0$ and $\forall X \in G^{\Theta} \setminus \{\emptyset\}$

$$m_{PCR5}(X) = m_{12}(X) + \sum_{\substack{Y \in G^{\Theta} \setminus \{X\} \\ X \cap Y = \emptyset}} \left[\frac{m_1(X)^2 m_2(Y)}{m_1(X) + m_2(Y)} + \frac{m_2(X)^2 m_1(Y)}{m_2(X) + m_1(Y)} \right] \tag{3}$$

where all sets involved in formula are in canonical form and where G^{Θ} corresponds to classical power set 2^{Θ} if Shafer's model is used, or to a constrained hyper-power set D^{Θ} if any other hybrid DSm model is used instead, or to the super-power set S^{Θ} if the minimal refinement Θ^{ref} of Θ is used; $m_{12}(X) \equiv m_{\cap}(X)$ corresponds to the conjunctive consensus on X between the $s = 2$ sources and where all denominators are different from zero. If a denominator is zero, that fraction is discarded. A general formula of PCR5 for the fusion of $s > 2$ sources has been proposed in Ref. 23, but a more intuitive PCR formula (denoted PCR6) which provides good results in practice has been proposed by Martin and Osswald.[7,23, p. 69–88] $m_{12\ldots s}(X) \equiv m_{\cap}(X)$ corresponds to the conjunctive consensus on X between the $s > 2$ sources. If a denominator is zero, that fraction is discarded because all masses $m_i(X_i) = 0$ so the numerator is also zero, i.e. no conflicting mass (nothing to redistribute). For two sources ($s = 2$), PCR5 and PCR6 formulas coincide. The implementation of PCR6 is easier than PCR5 and can be found in Ref. 32.

We compare here the solutions for well-known Zadeh's example[35,36] provided by several fusion rules. A detailed presentation with more comparisons can be found in Refs. 20 and 23. Let's consider $\Theta = \{M, C, T\}$ as the frame of three potential origins about possible diseases of a patient (M standing for *meningitis*, C for *concussion* and T for *tumor*), the Shafer's model and the two following belief assignments provided by two independent doctors after examination of the same patient: $m_1(M) = 0.9$, $m_1(C) = 0$, $m_1(T) = 0.1$ and $m_2(M) = 0$, $m_2(C) = 0.9$ and $m_2(T) = 0.1$. The total conflicting mass in this example is high since it is $m_1(M)m_2(C) + m_1(M)m_2(T) + m_2(C)m_1(T) = 0.99$.

- with Dempster's rule (i.e. the normalized conjunctive rule) and Shafer's model (DS), one gets the counter-intuitive result (see justifications in Refs. 8, 20, 30, 34, 35): $m_{DS}(T) = 1$.
- with Yager's rule[34] and Shafer's model, all the conflicting mass is transferred to the total ignorance Θ, so that: $m_Y(M \cup C \cup T) = 0.99$ and $m_Y(T) = 0.01$.
- with DSmH and Shafer's model, the partial conflicting masses are transferred to the corresponding partial ignorances: $m_{DSmH}(M \cup C) = 0.81$, $m_{DSmH}(T) = 0.01$, and $m_{DSmH}(M \cup T) = m_{DSmH}(C \cup T) = 0.09$.
- The Dubois & Prade's rule (DP)[8] based on Shafer's model provides in Zadeh's example the same result as DSmH, because DP and DSmH coincide in all static fusion problems.[e]
- with PCR5 and Shafer's model: $m_{PCR5}(M) = m_{PCR5}(C) = 0.486$ and $m_{PCR5}(T) = 0.028$.

One sees that when the total conflict between sources becomes high, DSmT is able (upon authors opinion) to manage more adequately through DSmH or PCR5 rules the combination of information than Dempster's rule, even when working with Shafer's model — which is only a specific hybrid model. DSmH rule is in agreement with DP rule for the static fusion, but DSmH and DP rules differ in general (for non degenerate cases) for dynamic fusion while PCR5 rule is the most exact proportional conflict redistribution rule. Besides this particular example, we showed in Ref. 20 that there exist several infinite classes of counter-examples to Dempster's rule which can be solved by DSmT.

[e] Indeed, DP rule has been developed for static fusion only while DSmH has been developed to take into account the possible dynamicity of the frame itself and also its associated model.

In summary, DST based on Dempster's rule provides counter-intuitive results in Zadeh's example, or in non-Bayesian examples similar to Zadeh's and no result when the conflict is 1. Only *ad hoc* discounting techniques allow to circumvent troubles of Dempster's rule or we need to switch to another model of representation/frame; in the later case, the solution obtained doesn't fit with the Shafer's model one originally wanted to work with. We want also to emphasize that in dynamic fusion when the conflict becomes high, both DST[18] and Smets' Transferable Belief Model (TBM)[28] approaches fail to respond to new information provided by new sources as shown in Ref. 7.

4. The DSmP Transformation

In the theories of belief functions, the mapping from the belief to the probability domain is a controversial issue. The original purpose of such mappings was to take (hard) decision, but contrariwise to erroneous widespread idea/claim, this is not the only interest for using such mappings nowadays. Actually the probabilistic transformations of belief mass assignments are for example very useful in modern multi-target multisensor tracking systems (or in any other systems) where one deals with soft decisions (i.e. where all possible solutions are kept for state estimation with their likelihoods). For example, in a Multiple Hypotheses Tracker using both kinematical and attribute data, one needs to compute all probabilities values for deriving the likelihoods of data association hypotheses and then mixing them altogether to estimate states of targets. Therefore, it is very relevant to use a mapping which provides a highly probabilistic information content (PIC), in order to reduce the uncertainty and facilitate decision-making for expecting better performances in the systems. The PIC is the dual of the normalized Shannon's entropy.[7] In this section, we briefly recall a new probabilistic transformation, denoted *DSmP* introduced in DSmT.[5] All details on DSmP transformation can be found with examples in Ref. 25. *DSmP* is straight and different from other transformations. Its basic consists in a new way of proportionalizations of the mass of each partial ignorance such as $A_1 \cup A_2$ or $A_1 \cup (A_2 \cap A_3)$ or $(A_1 \cap A_2) \cup (A_3 \cap A_4)$, etc. and the mass of the total ignorance $A_1 \cup A_2 \cup \ldots \cup A_n$, to the elements involved in the ignorances. DSmP takes into account both the values of the masses and the cardinality of elements in the proportional redistribution process.

Let's consider a discrete frame Θ with a given model (free DSm model, hybrid DSm model or Shafer's model), the *DSmP* mapping is defined by $DSmP_\epsilon(\emptyset) = 0$ and $\forall\, X \in G^\Theta \setminus \{\emptyset\}$ by

$$DSmP_\epsilon(X) = \sum_{Y \in G^\Theta} \frac{\sum_{\substack{Z \subseteq X \cap Y \\ \mathcal{C}(Z)=1}} m(Z) + \epsilon \cdot \mathcal{C}(X \cap Y)}{\sum_{\substack{Z \subseteq Y \\ \mathcal{C}(Z)=1}} m(Z) + \epsilon \cdot \mathcal{C}(Y)} m(Y) \tag{4}$$

where $\epsilon \geq 0$ is a tuning parameter and G^Θ corresponds to the generic set (2^Θ, S^Θ or D^Θ including eventually all the integrity constraints (if any) of the model $\mathcal{M}$);

$\mathcal{C}(X \cap Y)$ and $\mathcal{C}(Y)$ denote the DSm cardinals[f] of the sets $X \cap Y$ and Y respectively. The *DSm cardinality* of any element A of hyper-power set G^Θ, denoted $\mathcal{C}_\mathcal{M}(A)$, corresponds to the number of parts of A in the corresponding fuzzy/vague Venn diagram of the problem (model $\mathcal{M}$) taking into account the set of integrity constraints (if any), i.e. all the possible intersections due to the nature of the elements θ_i. This *intrinsic cardinality* depends on the model $\mathcal{M}$ (free, hybrid or Shafer's model). $\mathcal{M}$ is the model that contains A, which depends both on the dimension $n = |\Theta|$ and on the number of non-empty intersections present in its associated Venn diagram.[20] The DSm cardinality measures the complexity of any element of G^Θ and we may say that for the element θ_i not even $|\Theta|$ counts, but only its structure (= how many other singletons intersect θ_i). Simple illustrative examples are given in Chapters 3 and 7.[20] ϵ allows to reach the maximum PIC value of the approximation of $m(.)$ into a subjective probability measure. The smaller ϵ, the better/bigger PIC value. In some particular degenerate cases however, the $DSmP_{\epsilon=0}$ values cannot be derived, but the $DSmP_{\epsilon>0}$ values can however always be derived by choosing ϵ as a very small positive number, say $\epsilon = 1/1000$ for example in order to be as close as we want to the maximum of the PIC. When $\epsilon = 1$ and when the masses of all elements Z having $\mathcal{C}(Z) = 1$ are zero, (4) reduces to classical Smet's betting probability $BetP$.[27, p. 284] The passage from a free DSm model to a Shafer's model involves the passage from a structure to another one, and the cardinals change as well in formula (4). $DSmP$ works for all models (free, hybrid and Shafer's). In order to apply classical transformation (Pignistic, Cuzzolin's one, Sudano's ones, etc.[25]), we need at first to refine the frame (on the cases when it is possible!) in order to work with Shafer's model, and then apply their formulas. In the case where refinement makes sense, then one can apply the other subjective probabilities on the refined frame. $DSmP$ works on the refined frame as well and gives the same result as it does on the non-refined frame. Thus $DSmP$ with $\epsilon > 0$ works on any models and so is very general and appealing. $DSmP$ does a redistribution of the ignorance mass with respect to both the singleton masses and the singletons' cardinals in the same time. Now, if all masses of singletons involved in all ignorances are different from zero, then we can take $\epsilon = 0$, and $DSmP$ gives the best result, i.e. the best PIC value. In summary, $DSmP$ does an "improvement" over previous known probabilistic transformations in the sense that $DSmP$ mathematically makes a more accurate redistribution of the ignorance masses to the singletons involved in ignorances. $DSmP$ and $BetP$ work in both theories: DST (= Shafer's model) and DSmT (= free or hybrid models) as well.

5. Dealing with Imprecise Beliefs Assignments

In many fusion problems, it seems very difficult (if not impossible) to have precise sources of evidence generating precise basic belief assignments (especially when belief functions are provided by human experts), and a more flexible plausible and

[f] We have omitted the index of the model $\mathcal{M}$ for the notation convenience.

paradoxical theory supporting imprecise information becomes necessary. DSmT offers also the possibility to deal with *admissible imprecise generalized basic belief assignments* $m^I(.)$ defined as real subunitary intervals of $[0, 1]$, or even more general as real subunitary sets [i.e. sets, not necessarily intervals]. An imprecise belief assignment $m^I(.)$ over G^Θ is said *admissible* if and only if there exists for every $X \in G^\Theta$ at least one real number $m(X) \in m^I(X)$ such that $\sum_{X \in G^\Theta} m(X) = 1$. To manipulate imprecise bba's, we have introduced the following set operators

- **Addition of sets:** $S_1 \boxplus S_2 = S_2 \boxplus S_1 \triangleq \{x | x = s_1 + s_2, s_1 \in S_1, s_2 \in S_2\}$
- **Subtraction of sets:** $S_1 \boxminus S_2 \triangleq \{x | x = s_1 - s_2, s_1 \in S_1, s_2 \in S_2\}$
- **Multiplication of sets:** $S_1 \boxdot S_2 \triangleq \{x | x = s_1 \cdot s_2, s_1 \in S_1, s_2 \in S_2\}$
- **Division of sets:** If 0 doesn't belong to S_2, $S_1 \boxslash S_2 \triangleq \{x | x = s_1 / s_2, s_1 \in S_1, s_2 \in S_2\}$

Based on these set operators, all the rules of combination developed in DST, or DSmT for fusioning precise quantitative bba's can be directly extended to manipulate and combine imprecise quantitative bba's. Details and examples can be found in Ref. 20 (Chap. 6).

6. Dealing with Qualitative Beliefs Assignments

DSmT offers also the possibility to deal with qualitative belief assignments to model beliefs of human experts expressed in natural language (with linguistic labels). A detailed presentation can be found in Refs. 23 and 25. The derivations are based on a new arithmetic on linguistic labels which allows a direct extension of all quantitative rules of combination (and conditioning). Computing with words (CW) and qualitative information is more vague, less precise than computing with numbers, but it offers the advantage of robustness if done correctly. Here is a general arithmetic we propose for computing with words (i.e. with linguistic labels). Let's consider a finite frame $\Theta = \{\theta_1, \dots, \theta_n\}$ of n (exhaustive) elements θ_i, $i = 1, 2, \dots, n$, with an associated model $\mathcal{M}(\Theta)$ on Θ (either Shafer's model $\mathcal{M}^0(\Theta)$, free-DSm model $\mathcal{M}^f(\Theta)$, or more general any Hybrid-DSm model[20]). A model $\mathcal{M}(\Theta)$ is defined by the set of integrity constraints on elements of Θ (if any); Shafer's model $\mathcal{M}^0(\Theta)$ assumes all elements of Θ truly exclusive, while free-DSm model $\mathcal{M}^f(\Theta)$ assumes no exclusivity constraints between elements of the frame Θ. Let's define a finite set of linguistic labels $\tilde{L} = \{L_1, L_2, \dots, L_m\}$ where $m \geq 2$ is an integer. $\tilde{L}$ is endowed with a total order relationship $\prec$, so that $L_1 \prec L_2 \prec \cdots \prec L_m$. To work on a close linguistic set under linguistic addition and multiplication operators, we extends $\tilde{L}$ with two extreme values L_0 and L_{m+1} where L_0 corresponds to the minimal qualitative value and L_{m+1} corresponds to the maximal qualitative value, in such a way that $L_0 \prec L_1 \prec L_2 \prec \cdots \prec L_m \prec L_{m+1}$, where $\prec$ means inferior to, or less (in quality) than, or smaller (in quality) than, etc. hence a relation of order from a qualitative point of view. But if we make a correspondence between qualitative labels and quantitative values on the scale $[0,1]$, then $L_{\min} = L_0$ would correspond to the numerical value 0, while $L_{\max} = L_{m+1}$ would correspond to the numerical value 1, and each L_i would belong to $[0,1]$, i.e. $L_{\min} = L_0 < L_1 < L_2 < \cdots < L_m < L_{m+1} = L_{\max}$. We work on extended ordered set L of qualitative values $L = \{L_0, \tilde{L}, L_{m+1}\} = \{L_0, L_1, L_2, \ldots, L_m, L_{m+1}\}$ and we define the following accurate operators for qualitative labels (see the chapter of Ref. 25 devoted on the DSm Field and Linear Algebra of Refined Labels (FLARL)). In FLARL, we can replace the "qualitative quasi-normalization" of qualitative operators we used in our previous papers by "qualitative normalization" since in FLARL we have exact qualitative calculations and exact normalization. So, we use mainly the

- Label addition: $L_a + L_b = L_{a+b}$ since $\frac{a}{m+1} + \frac{b}{m+1} = \frac{a+b}{m+1}$.
- Label multiplication: $L_a \times L_b = L_{(ab)/(m+1)}$ since $\frac{a}{m+1} \cdot \frac{b}{m+1} = \frac{(ab)/(m+1)}{m+1}$.
- Label division (when $L_b \neq L_0$): $L_a \div L_b = L_{(a/b)(m+1)}$ since $\frac{a}{m+1} \div \frac{b}{m+1} = \frac{a}{b} = \frac{(a/b)(m+1)}{m+1}$.

If one really needs to stay within the original set of labels, an approximation will be necessary at the very end of the calculations. A qualitative belief assignment[g] (qba) is a mapping function $qm(.) : G^{\Theta} \mapsto L$ where G^{Θ} corresponds either to 2^{Θ}, to D^{Θ} or even to S^{Θ} depending on the model of the frame Θ we choose to work with. In the case when the labels are equidistant, i.e. the qualitative distance between any two consecutive labels is the same, we get an exact qualitative result, and a qualitative basic belief assignment (bba) is considered normalized if the sum of all its qualitative masses is equal to $L_{\max} = L_{m+1}$. If the labels are not equidistant, we still can use all qualitative operators defined in the FLARL, but the qualitative result is approximate, and a qualitative bba is considered quasi-normalized if the sum of all its masses is equal to $L_{\max}$. Using the qualitative operator of FLARL, we can easily extend all the combination and conditioning rules from quantitative to qualitative. When dealing with qualitative beliefs within the DSm Field and Linear Algebra of Refined Labels[25] we get an exact qualitative result no matter what fusion rule is used (DSm fusion rules, Dempster's rule, Smets's rule, Dubois-Prade's rule, etc.). The exact qualitative result will be a refined label (but the user can round it up or down to the closest integer index label). Examples of qualitative PCR5 rule and qualitative DSmP transformation are found in Ref. 7 for convenience.

7. Belief Conditioning Rules

Until very recently, the most commonly used conditioning rule for belief revision was the one proposed by Shafer[18] and referred as Shafer's Conditioning Rule (SCR). It consists in combining the prior bba $m(.)$ with a specific bba focused on A with

[g] We call it also *qualitative belief mass* or *q-mass* for short.

Dempster's rule of combination for transferring the conflicting mass to non-empty sets in order to provide the revised bba. In other words, the conditioning by a proposition A, is obtained by $m_{SCR}(.|A) = [m \oplus m_S](.)$, where $m(.)$ is the prior bba to update, A is the conditioning event, $m_S(.)$ is the bba focused on A defined by $m_S(A) = 1$ and $m_S(X) = 0$ for all $X \neq A$ and $\oplus$ denotes Dempster's rule of combination.[18] The SCR approach remains *subjective* since actually in such belief revision process both sources are subjective and in our opinions SCR doesn't manage satisfactorily the objective nature/absolute truth carried by the conditioning term. Indeed, when conditioning *a prior* mass $m(.)$, *knowing* (or assuming) that the truth is in A, means that we have in hands an absolute (not subjective) knowledge, i.e. the truth in A has occurred (or is assumed to have occurred), thus A is realized (or is assumed to be realized) and this is (or at least must be interpreted as) an absolute truth. The conditioning term "Given A" must therefore be considered as an absolute truth, while $m_S(A) = 1$ introduced in SCR cannot refer to an absolute truth actually, but only to a *subjective certainty* on the possible occurrence of A from a *virtual* second source of evidence. In DSmT, we have also proposed several approaches for belief conditioning. The simplest approach consists to follow Shafer's idea but in changing Dempster's rule by the more efficient PCR5 rule.[6] However, more sophisticated belief conditioning rules (BCR) based on Hyper-Power Set Decomposition (HPSD) have also been proposed and can be found in details with examples in Ref. 23. Of course these BCR have also been extended to deal with imprecise or qualitative bba's as well.[7,24]

8. Conclusion

A short presentation of the foundations of DSmT has been proposed in this introduction. DSmT proposes new quantitative and qualitative rules of combination for uncertain, imprecise and highly conflicting sources of information. Several applications of DSmT have been proposed recently in the literature and show the potential and the efficiency of this new theory. DSmT offers the possibility to work in different fusion spaces depending on the nature of problem under consideration. Thus, one can work either in $2^\Theta = (\Theta, \cup)$ (i.e. in the classical power set as in DST framework), in $D^\Theta = (\Theta, \cup, \cap)$ (the hyper-power set — also known as Dedekind's lattice) or in the super-power set $S^\Theta = (\Theta, \cup, \cap, c(.))$, which includes 2^Θ and D^Θ and which represents the power set of the minimal refinement of the frame Θ when the refinement is possible (because for vague elements whose frontiers are not well known, the refinement is not possible). We have enriched the DSmT with a subjective probability ($DSmP_\epsilon$) that gets the best Probabilistic Information Content (PIC) in comparison with other existing subjective probabilities. Also, we have defined and developed the DSm Field and Linear Algebra of Refined Labels that permit the transformation of any fusion rule to a corresponding qualitative fusion rule which gives an exact qualitative result (i.e. a refined label), so far the best in literature.

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