New Mathematics and Natural Computation
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NEUTROSOPHIC CUBIC SETS

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Received Day Month Year
Revised Day Month Year

The aim of this paper is to extend the concept of cubic sets to the neutrosophic sets. The notions of truth-internal (indeterminacy-internal, falsity-internal) neutrosophic cubic sets and truth-external (indeterminacy-external, falsity-external) neutrosophic cubic sets are introduced, and related properties are investigated.

Keywords: Neutrosophic (cubic) set; truth-internal (indeterminacy-internal; falsity-internal) neutrosophic cubic set; truth-external (indeterminacy-external; falsity-external) neutrosophic cubic set.

1. Introduction

Fuzzy sets, which were introduced by Zadeh[9], deal with possibilistic uncertainty, connected with imprecision of states, perceptions and preferences. Based on the (interval-valued) fuzzy sets, Jun et al.[1] introduced the notion of (internal, external) cubic sets, and investigated several properties. Jun et al. applied the notion of cubic sets to BCK/BCI-algebras. They introduced the notions of cubic subalgebras/ideals, cubic ∘-subalgebras and closed cubic ideals in BCK/BCI-algebras, and then they investigated several properties (see Jun et al.[2], Jun et al.[3], Jun et al.[4] and Jun et al.[5]). The concept of neutrosophic set (NS) developed by Smarandache[6]

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and Smarandache[7] is a more general platform which extends the concepts of the classic set and fuzzy set, intuitionistic fuzzy set and interval valued intuitionistic fuzzy set. Neutrosophic set theory is applied to various part (refer to the site http://fs.gallup.unm.edu/neutrosophy.htm).

In this paper, we extend the concept of cubic sets to the neutrosophic sets. We introduce the notions of truth-internal (indeterminacy-internal, falsity-internal) neutrosophic cubic sets and truth-external (indeterminacy-external, falsity-external) neutrosophic cubic sets, and investigate related properties. We show that the P-union and the P-intersection of truth-internal (indeterminacy-internal, falsity-internal) neutrosophic cubic sets are also truth-internal (indeter-minacy-internal, falsity-internal) neutrosophic cubic sets We provide examples to show that the P-union and the P-intersection of truth-external (indeterminacy-external, falsity-external) neutrosophic cubic sets may not be truth-external (indeterminacy-external, falsity-external) neutrosophic cubic sets, and the R-union and the R-intersection of truth-internal (indeterminacy-internal, falsity-internal) neutrosophic cubic sets may not be truth-internal (indeterminacy-internal, falsity-internal) neutrosophic cubic sets. We provide conditions for the R-union of two T-internal (resp. I-internal and F-internal) neutrosophic cubic sets to be a T-internal (resp. I-internal and F-internal) neutrosophic cubic set.

2. Preliminaries

A fuzzy set in a set X is defined to be a function $\lambda : X \to [0, 1]$. Denote by $[0, 1]^X$ the collection of all fuzzy sets in a set X. Define a relation $\leq$ on $[0, 1]^X$ as follows:

$$(\forall \lambda, \mu \in [0, 1]^X)\, (\lambda \leq \mu \iff (\forall x \in X)(\lambda(x) \leq \mu(x))).$$

The join ($\vee$) and meet ($\wedge$) of λ and μ are defined by

$$(\lambda \vee \mu)(x) = \max\{\lambda(x), \mu(x)\},$$
$$(\lambda \wedge \mu)(x) = \min\{\lambda(x), \mu(x)\},$$

respectively, for all $x \in X$. The complement of λ, denoted by λ^c, is defined by

$$(\forall x \in X)\, (\lambda^c(x) = 1 - \lambda(x)).$$

For a family $\{\lambda_i \mid i \in \Lambda\}$ of fuzzy sets in X, we define the join ($\vee$) and meet ($\wedge$) operations as follows:

$$\Big(\bigvee_{i\in\Lambda} \lambda_i\Big)(x) = \sup\{\lambda_i(x) \mid i \in \Lambda\},$$
$$\Big(\bigwedge_{i\in\Lambda} \lambda_i\Big)(x) = \inf\{\lambda_i(x) \mid i \in \Lambda\},$$

respectively, for all $x \in X$.

By an interval number we mean a closed subinterval $\tilde{a} = [a^-, a^+]$ of $[0, 1]$, where $0 \leq a^- \leq a^+ \leq 1$. The interval number $\tilde{a} = [a^-, a^+]$ with $a^- = a^+$ is denoted by

a. Denote by $[[0,1]]$ the set of all interval numbers. Let us define what is known as refined minimum (briefly, rmin) of two elements in $[[0,1]]$. We also define the symbols "$\succeq$", "$\preceq$", "$=$" in case of two elements in $[[0,1]]$. Consider two interval numbers $\tilde{a}_1 := \left[a_1^-, a_1^+\right]$ and $\tilde{a}_2 := \left[a_2^-, a_2^+\right]$. Then

$$\text{rmin}\left\{\tilde{a}_1, \tilde{a}_2\right\} = \left[\min\left\{a_1^-, a_2^-\right\}, \min\left\{a_1^+, a_2^+\right\}\right],$$

$$\tilde{a}_1 \succeq \tilde{a}_2 \text{ if and only if } a_1^- \geq a_2^- \text{ and } a_1^+ \geq a_2^+,$$

and similarly we may have $\tilde{a}_1 \preceq \tilde{a}_2$ and $\tilde{a}_1 = \tilde{a}_2$. To say $\tilde{a}_1 \succ \tilde{a}_2$ (resp. $\tilde{a}_1 \prec \tilde{a}_2$) we mean $\tilde{a}_1 \succeq \tilde{a}_2$ and $\tilde{a}_1 \neq \tilde{a}_2$ (resp. $\tilde{a}_1 \preceq \tilde{a}_2$ and $\tilde{a}_1 \neq \tilde{a}_2$). Let $\tilde{a}_i \in [[0,1]]$ where $i \in \Lambda$. We define

$$\operatorname*{rinf}_{i\in\Lambda} \tilde{a}_i = \left[\inf_{i\in\Lambda} a_i^-, \inf_{i\in\Lambda} a_i^+\right] \quad \text{and} \quad \operatorname*{rsup}_{i\in\Lambda} \tilde{a}_i = \left[\sup_{i\in\Lambda} a_i^-, \sup_{i\in\Lambda} a_i^+\right].$$

For any $\tilde{a} \in [[0,1]]$, its complement, denoted by $\tilde{a}^c$, is defined be the interval number

$$\tilde{a}^c = [1 - a^+, 1 - a^-].$$

Let X be a nonempty set. A function $A : X \to [[0,1]]$ is called an interval-valued fuzzy set (briefly, an IVF set) in X. Let $IVF(X)$ stand for the set of all IVF sets in X. For every $A \in IVF(X)$ and $x \in X$, $A(x) = [A^-(x), A^+(x)]$ is called the degree of membership of an element x to A, where $A^- : X \to I$ and $A^+ : X \to I$ are fuzzy sets in X which are called a lower fuzzy set and an upper fuzzy set in X, respectively. For simplicity, we denote $A = [A^-, A^+]$. For every $A, B \in IVF(X)$, we define

$$A \subseteq B \Leftrightarrow A(x) \preceq B(x) \text{ for all } x \in X,$$

and

$$A = B \Leftrightarrow A(x) = B(x) \text{ for all } x \in X.$$

The complement A^c of $A \in IVF(X)$ is defined as follows: $A^c(x) = A(x)^c$ for all $x \in X$, that is,

$$A^c(x) = [1 - A^+(x), 1 - A^-(x)] \text{ for all } x \in X.$$

For a family $\{A_i \mid i \in \Lambda\}$ of IVF sets in X where Λ is an index set, the union $G = \bigcup_{i\in\Lambda} A_i$ and the intersection $F = \bigcap_{i\in\Lambda} A_i$ are defined as follows:

$$G(x) = \left(\bigcup_{i\in\Lambda} A_i\right)(x) = \operatorname*{rsup}_{i\in\Lambda} A_i(x)$$

and

$$F(x) = \left(\bigcap_{i\in\Lambda} A_i\right)(x) = \operatorname*{rinf}_{i\in\Lambda} A_i(x)$$

for all $x \in X$, respectively.

Let X be a non-empty set. A neutrosophic set (NS) in X (see Smarandache[6]) is a structure of the form:

$$\Lambda := \{\langle x; \lambda_T(x), \lambda_I(x), \lambda_F(x)\rangle \mid x \in X\}$$

where $\lambda_T : X \to [0,1]$ is a truth membership function, $\lambda_I : X \to [0,1]$ is an indeterminate membership function, and $\lambda_F : X \to [0,1]$ is a false membership function.

Let X be a non-empty set. An interval neutrosophic set (INS) in X (see Wang et al.[8]) is a structure of the form:

$$\mathbf{A} := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$$

where A_T, A_I and A_F are interval-valued fuzzy sets in X, which are called an interval truth membership function, an interval indeterminacy membership function and an interval falsity membership function, respectively.

3. Neutrosophic cubic sets

Jun et al.[1] have defined the cubic set as follows:

Let X be a non-empty set. A cubic set in X is a structure of the form:

$$\mathbf{C} = \{(x, A(x), \lambda(x)) \mid x \in X\}$$

where A is an interval-valued fuzzy set in X and λ is a fuzzy set in X.

We consider the notion of neutrosophic cubic sets as an extension of cubic sets.

Definition 3.1. Let X be a non-empty set. A neutrosophic cubic set (NCS) in X is a pair $\mathscr{A} = (\mathbf{A}, \Lambda)$ where $\mathbf{A} := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$ is an interval neutrosophic set in X and $\Lambda := \{\langle x; \lambda_T(x), \lambda_I(x), \lambda_F(x)\rangle \mid x \in X\}$ is a neutrosophic set in X.

Example 3.1. For $X = \{a, b, c\}$, the pair $\mathscr{A} = (\mathbf{A}, \Lambda)$ with the tabular representation in Table 1 is a neutrosophic cubic set in X.

Table 1. Tabular representation of $\mathscr{A} = (\mathbf{A}, \Lambda)$

X	$\mathbf{A}(x)$	$\Lambda(x)$
a	$([0.2, 0.3], [0.3, 0.5], [0.3, 0.5])$	$(0.1, 0.2, 0.3)$
b	$([0.4, 0.7], [0.1, 0.4], [0.2, 0.4])$	$(0.3, 0.2, 0.7)$
c	$([0.6, 0.9], [0.0, 0.2], [0.3, 0.4])$	$(0.5, 0.2, 0.3)$

Example 3.2. For a non-empty set X and any INS

$$\mathbf{A} := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$$

in X, we know that $\mathscr{C} = (\mathbf{C}, \Phi)_1 := (\mathbf{A}, \Lambda_1)$ and $\mathscr{C} = (\mathbf{C}, \Phi)_0 := (\mathbf{A}, \Lambda_0)$ are neutrosophic cubic sets in X where $\Lambda_1 := \{\langle x; 1, 1, 1\rangle \mid x \in X\}$ and $\Lambda_0 := \{\langle x; 0, 0, 0\rangle \mid x \in X\}$ in X. If we take $\lambda_T(x) = \frac{A_T^-(x)+A_T^+(x)}{2}$, $\lambda_I(x) = \frac{A_I^-(x)+A_I^+(x)}{2}$, and $\lambda_F(x) = \frac{A_F^-(x)+A_F^+(x)}{2}$, then $\mathscr{A} = (\mathbf{A}, \Lambda)$ is a neutrosophic cubic set in X

Definition 3.2. Let X be a non-empty set. A neutrosophic cubic set $\mathscr{A} = (\mathbf{A}, \Lambda)$ in X is said to be

- truth-internal (briefly, T-internal) if the following inequality is valid

$$(\forall x \in X)\left(A_T^-(x) \leq \lambda_T(x) \leq A_T^+(x)\right), \tag{3.1}$$

- indeterminacy-internal (briefly, I-internal) if the following inequality is valid

$$(\forall x \in X)\left(A_I^-(x) \leq \lambda_I(x) \leq A_I^+(x)\right), \tag{3.2}$$

- falsity-internal (briefly, F-internal) if the following inequality is valid

$$(\forall x \in X)\left(A_F^-(x) \leq \lambda_F(x) \leq A_F^+(x)\right). \tag{3.3}$$

If a neutrosophic cubic set $\mathscr{A} = (\mathbf{A}, \Lambda)$ in X satisfies (3.1), (3.2) and (3.3), we say that $\mathscr{A} = (\mathbf{A}, \Lambda)$ is an internal neutrosophic cubic in X.

Example 3.3. For $X = \{a, b, c\}$, the pair $\mathscr{A} = (\mathbf{A}, \Lambda)$ with the tabular representation in Table 2 is an internal neutrosophic cubic set in X.

Table 2. Tabular representation of $\mathscr{A} = (\mathbf{A}, \Lambda)$

X	$\mathbf{A}(x)$	$\Lambda(x)$
a	$([0.2, 0.3], [0.3, 0.5], [0.3, 0.5])$	$(0.25, 0.35, 0.40)$
b	$([0.4, 0.7], [0.1, 0.4], [0.2, 0.4])$	$(0.50, 0.30, 0.30)$
c	$([0.6, 0.9], [0.0, 0.2], [0.3, 0.4])$	$(0.70, 0.10, 0.35)$

Definition 3.3. Let X be a non-empty set. A neutrosophic cubic set $\mathscr{A} = (\mathbf{A}, \Lambda)$ in X is said to be

- truth-external (briefly, T-external) if the following inequality is valid

$$(\forall x \in X)\left(\lambda_T(x) \notin (A_T^-(x), A_T^+(x))\right), \tag{3.4}$$

- indeterminacy-external (briefly, I-external) if the following inequality is valid

$$(\forall x \in X)\left(\lambda_I(x) \notin (A_I^-(x), A_I^+(x))\right), \tag{3.5}$$

- falsity-external (briefly, F-external) if the following inequality is valid

$$(\forall x \in X) \left(\lambda_F(x) \notin (A_F^-(x), A_F^+(x)) \right). \tag{3.6}$$

If a neutrosophic cubic set $\mathscr{A} = (\mathbf{A}, \Lambda)$ in X satisfies (3.4), (3.5) and (3.6), we say that $\mathscr{A} = (\mathbf{A}, \Lambda)$ is an external neutrosophic cubic in X.

Proposition 3.1. Let $\mathscr{A} = (\mathbf{A}, \Lambda)$ be a neutrosophic cubic set in a non-empty set X which is not external. Then there exists $x \in X$ such that $\lambda_T(x) \in (A_T^-(x), A_T^+(x))$, $\lambda_I(x) \in (A_I^-(x), A_I^+(x))$, or $\lambda_F(x) \in (A_F^-(x), A_F^+(x))$.

Proof. Straightforward. □

Proposition 3.2. Let $\mathscr{A} = (\mathbf{A}, \Lambda)$ be a neutrosophic cubic set in a non-empty set X. If $\mathscr{A} = (\mathbf{A}, \Lambda)$ is both T-internal and T-external, then

$$(\forall x \in X) \left(\lambda_T(x) \in \{A_T^-(x) \mid x \in X\} \cup \{A_T^+(x) \mid x \in X\} \right). \tag{3.7}$$

Proof. Two conditions (3.1) and (3.4) imply that $A_T^-(x) \leq \lambda_T(x) \leq A_T^+(x)$ and $\lambda_T(x) \notin (A_T^-(x), A_T^+(x))$ for all $x \in X$. It follows that $\lambda_T(x) = A_T^-(x)$ or $\lambda_T(x) = A_T^+(x)$, and so that $\lambda_T(x) \in \{A_T^-(x) \mid x \in X\} \cup \{A_T^+(x) \mid x \in X\}$. □

Similarly, we have the following propositions.

Proposition 3.3. Let $\mathscr{A} = (\mathbf{A}, \Lambda)$ be a neutrosophic cubic set in a non-empty set X. If $\mathscr{A} = (\mathbf{A}, \Lambda)$ is both I-internal and I-external, then

$$(\forall x \in X) \left(\lambda_I(x) \in \{A_I^-(x) \mid x \in X\} \cup \{A_I^+(x) \mid x \in X\} \right). \tag{3.8}$$

Proposition 3.4. Let $\mathscr{A} = (\mathbf{A}, \Lambda)$ be a neutrosophic cubic set in a non-empty set X. If $\mathscr{A} = (\mathbf{A}, \Lambda)$ is both F-internal and F-external, then

$$(\forall x \in X) \left(\lambda_F(x) \in \{A_F^-(x) \mid x \in X\} \cup \{A_F^+(x) \mid x \in X\} \right). \tag{3.9}$$

Definition 3.4. Let $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ be neutrosophic cubic sets in a non-empty set X where

$\mathbf{A} := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$,
$\Lambda := \{\langle x; \lambda_T(x), \lambda_I(x), \lambda_F(x)\rangle \mid x \in X\}$,
$\mathbf{B} := \{\langle x; B_T(x), B_I(x), B_F(x)\rangle \mid x \in X\}$,
$\Psi := \{\langle x; \psi_T(x), \psi_I(x), \psi_F(x)\rangle \mid x \in X\}$.

Then we define the equality, P-order and R-order as follows:

(a) (Equality) $\mathscr{A} = \mathscr{B} \;\Leftrightarrow\; \mathbf{A} = \mathbf{B}$ and $\Lambda = \Psi$.
(b) (P-order) $\mathscr{A} \subseteq_P \mathscr{B} \;\Leftrightarrow\; \mathbf{A} \subseteq \mathbf{B}$ and $\Lambda \leq \Psi$.
(b) (R-order) $\mathscr{A} \subseteq_R \mathscr{B} \;\Leftrightarrow\; \mathbf{A} \subseteq \mathbf{B}$ and $\Lambda \geq \Psi$.

We now define the P-union, P-intersection, R-union and R-intersection of neutrosophic cubic sets as follows:

Definition 3.5. For any neutrosophic cubic sets $\mathscr{A}_i = (\mathbf{A}_i, \Lambda_i)$ in a non-empty set X where
$\mathbf{A}_i := \{\langle x; A_{iT}(x), A_{iI}(x), A_{iF}(x)\rangle \mid x \in X\}$,
$\Lambda_i := \{\langle x; \lambda_{iT}(x), \lambda_{iI}(x), \lambda_{iF}(x)\rangle \mid x \in X\}$
for $i \in J$ and J is any index set, we define

(a) $\bigcup_{P\, i\in J} \mathscr{A}_i = \left(\bigcup_{i\in J} \mathbf{A}_i, \bigvee_{i\in J} \Lambda_i\right)$ (P-union)

(b) $\bigcap_{P\, i\in J} \mathscr{A}_i = \left(\bigcap_{i\in J} \mathbf{A}_i, \bigwedge_{i\in J} \Lambda_i\right)$ (P-intersection)

(c) $\bigcup_{R\, i\in J} \mathscr{A}_i = \left(\bigcup_{i\in J} \mathbf{A}_i, \bigwedge_{i\in J} \Lambda_i\right)$ (R-union)

(d) $\bigcap_{R\, i\in J} \mathscr{A}_i = \left(\bigcap_{i\in J} \mathbf{A}_i, \bigvee_{i\in J} \Lambda_i\right)$ (R-intersection)

where

$$\bigcup_{i\in J} \mathbf{A}_i = \left\{\left\langle x; \left(\bigcup_{i\in J} A_{iT}\right)(x), \left(\bigcup_{i\in J} A_{iI}\right)(x), \left(\bigcup_{i\in J} A_{iF}\right)(x)\right\rangle \mid x \in X\right\},$$
$$\bigvee_{i\in J} \Lambda_i = \left\{\left\langle x; \left(\bigvee_{i\in J} \lambda_{iT}\right)(x), \left(\bigvee_{i\in J} \lambda_{iI}\right)(x), \left(\bigvee_{i\in J} \lambda_{iF}\right)(x)\right\rangle \mid x \in X\right\},$$
$$\bigcap_{i\in J} \mathbf{A}_i = \left\{\left\langle x; \left(\bigcap_{i\in J} A_{iT}\right)(x), \left(\bigcap_{i\in J} A_{iI}\right)(x), \left(\bigcap_{i\in J} A_{iF}\right)(x)\right\rangle \mid x \in X\right\},$$
$$\bigwedge_{i\in J} \Lambda_i = \left\{\left\langle x; \left(\bigwedge_{i\in J} \lambda_{iT}\right)(x), \left(\bigwedge_{i\in J} \lambda_{iI}\right)(x), \left(\bigwedge_{i\in J} \lambda_{iF}\right)(x)\right\rangle \mid x \in X\right\}.$$

The complement of $\mathscr{A} = (\mathbf{A}, \Lambda)$ is defined to be the neutrosophic cubic set $\mathscr{A}^c = (\mathbf{A}^c, \Lambda^c)$ where $\mathbf{A}^c := \{\langle x; A^c_T(x), A^c_I(x), A^c_F(x)\rangle \mid x \in X\}$ is an interval neutrosophic set in X and $\Lambda^c := \{\langle x; \lambda^c_T(x), \lambda^c_I(x), \lambda^c_F(x)\rangle \mid x \in X\}$ is a neutrosophic set in X.

Obviously, $(\mathscr{A}^c)^c = \mathscr{A}$, $\left(\bigcup_{P\, i\in J} \mathscr{A}_i\right)^c = \bigcap_{P\, i\in J} \mathscr{A}_i^c$, $\left(\bigcap_{P\, i\in J} \mathscr{A}_i\right)^c = \bigcup_{P\, i\in J} \mathscr{A}_i^c$, $\left(\bigcup_{R\, i\in J} \mathscr{A}_i\right)^c = \bigcap_{R\, i\in J} \mathscr{A}_i^c$, and $\left(\bigcap_{R\, i\in J} \mathscr{A}_i\right)^c = \bigcup_{R\, i\in J} \mathscr{A}_i^c$.

The following proposition is clear.

Proposition 3.5. For any neutrosophic cubic sets $\mathscr{A} = (\mathbf{A}, \Lambda)$, $\mathscr{B} = (\mathbf{B}, \Psi)$, $\mathscr{C} = (\mathbf{C}, \Phi)$, and $\mathscr{D} = (\mathbf{D}, \Omega)$ in a non-empty set X, we have

(1) if $\mathscr{A} \subseteq_P \mathscr{B}$ and $\mathscr{B} \subseteq_P \mathscr{C}$ then $\mathscr{A} \subseteq_P \mathscr{C}$.
(2) if $\mathscr{A} \subseteq_P \mathscr{B}$ then $\mathscr{B}^c \subseteq_P \mathscr{A}^c$.
(3) if $\mathscr{A} \subseteq_P \mathscr{B}$ and $\mathscr{A} \subseteq_P \mathscr{C}$ then $\mathscr{A} \subseteq_P \mathscr{B} \cap_P \mathscr{C}$.
(4) if $\mathscr{A} \subseteq_P \mathscr{B}$ and $\mathscr{C} \subseteq_P \mathscr{B}$ then $\mathscr{A} \cup_P \mathscr{C} \subseteq_P \mathscr{B}$.

(5) if $\mathscr{A} \subseteq_P \mathscr{B}$ and $\mathscr{C} \subseteq_P \mathscr{D}$ then $\mathscr{A} \cup_P \mathscr{C} \subseteq_P \mathscr{B} \cup_P \mathscr{D}$ and $\mathscr{A} \cap_P \mathscr{C} \subseteq_P \mathscr{B} \cap_P \mathscr{D}$
(6) if $\mathscr{A} \subseteq_R \mathscr{B}$ and $\mathscr{B} \subseteq_R \mathscr{C}$ then $\mathscr{A} \subseteq_R \mathscr{C}$.
(7) if $\mathscr{A} \subseteq_R \mathscr{B}$ then $\mathscr{B}^c \subseteq_R \mathscr{A}^c$.
(8) if $\mathscr{A} \subseteq_R \mathscr{B}$ and $\mathscr{A} \subseteq_R \mathscr{C}$ then $\mathscr{A} \subseteq_R \mathscr{B} \cap_R \mathscr{C}$.
(9) if $\mathscr{A} \subseteq_R \mathscr{B}$ and $\mathscr{C} \subseteq_R \mathscr{B}$ then $\mathscr{A} \cup_R \mathscr{C} \subseteq_R \mathscr{B}$.
(10) if $\mathscr{A} \subseteq_R \mathscr{B}$ and $\mathscr{C} \subseteq_R \mathscr{D}$ then $\mathscr{A} \cup_R \mathscr{C} \subseteq_R \mathscr{B} \cup_R \mathscr{D}$ and $\mathscr{A} \cap_R \mathscr{C} \subseteq_R \mathscr{B} \cap_R \mathscr{D}$.

Theorem 3.1. Let $\mathscr{A} = (\mathbf{A}, \Lambda)$ be a neutrosophic cubic set in a non-empty set X. If $\mathscr{A} = (\mathbf{A}, \Lambda)$ is I-internal (resp. I-external), then the complement $\mathscr{A}^c = (\mathbf{A}^c, \Lambda^c)$ of $\mathscr{A} = (\mathbf{A}, \Lambda)$ is an I-internal (resp. I-external) neutrosophic cubic set in X.

Proof. If $\mathscr{A} = (\mathbf{A}, \Lambda)$ is an I-internal (resp. I-external) neutrosophic cubic set in a non-empty set X, then $A_I^-(x) \leq \lambda_I(x) \leq A_I^+(x)$ (resp., $\lambda_I(x) \notin (A_I^-(x), A_I^+(x))$) for all $x \in X$. It follows that $1 - A_I^+(x) \leq 1 - \lambda_I(x) \leq 1 - A_I^-(x)$ (resp., $1 - \lambda_I(x) \notin (1 - A_I^+(x), 1 - A_I^-(x))$). Therefore $\mathscr{A}^c = (\mathbf{A}^c, \Lambda^c)$ is an I-internal (resp. I-external) neutrosophic cubic set in X. □

Similarly, we have the following theorems.

Theorem 3.2. Let $\mathscr{A} = (\mathbf{A}, \Lambda)$ be a neutrosophic cubic set in a non-empty set X. If $\mathscr{A} = (\mathbf{A}, \Lambda)$ is T-internal (resp. T-external), then the complement $\mathscr{A}^c = (\mathbf{A}^c, \Lambda^c)$ of $\mathscr{A} = (\mathbf{A}, \Lambda)$ is a T-internal (resp. T-external) neutrosophic cubic set in X.

Theorem 3.3. Let $\mathscr{A} = (\mathbf{A}, \Lambda)$ be a neutrosophic cubic set in a non-empty set X. If $\mathscr{A} = (\mathbf{A}, \Lambda)$ is F-internal (resp. F-external), then the complement $\mathscr{A}^c = (\mathbf{A}^c, \Lambda^c)$ of $\mathscr{A} = (\mathbf{A}, \Lambda)$ is an F-internal (resp. F-external) neutrosophic cubic set in X.

Corollary 3.1. Let $\mathscr{A} = (\mathbf{A}, \Lambda)$ be a neutrosophic cubic set in a non-empty set X. If $\mathscr{A} = (\mathbf{A}, \Lambda)$ is internal (resp. external), then the complement $\mathscr{A}^c = (\mathbf{A}^c, \Lambda^c)$ of $\mathscr{A} = (\mathbf{A}, \Lambda)$ is an internal (resp. external) neutrosophic cubic set in X.

Theorem 3.4. If $\{\mathscr{A}_i = (\mathbf{A}_i, \Lambda_i) \mid i \in J\}$ is a family of F-internal neutrosophic cubic sets in a non-empty set X, then the P-union and the P-intersection of $\{\mathscr{A}_i = (\mathbf{A}_i, \Lambda_i) \mid i \in J\}$ are F-internal neutrosophic cubic sets in X.

Proof. Since $\mathscr{A}_i = (\mathbf{A}_i, \Lambda_i)$ is an F-internal neutrosophic cubic set in a non-empty set X, we have $A_{iF}^-(x) \leq \lambda_{iF}(x) \leq A_{iF}^+(x)$ for $i \in J$. It follows that

$$\left(\bigcup_{i \in J} \mathbf{A}_{iF}\right)^-(x) \leq \left(\bigvee_{i \in J} \Lambda_{iF}\right)(x) \leq \left(\bigcup_{i \in J} \mathbf{A}_{iF}\right)^+(x)$$

and

$$\left(\bigcap_{i \in J} \mathbf{A}_{iF}\right)^-(x) \leq \left(\bigwedge_{i \in J} \Lambda_{iF}\right)(x) \leq \left(\bigcap_{i \in J} \mathbf{A}_{iF}\right)^+(x).$$

Therefore $\bigcup_{P_{i\in J}} \mathscr{A}_i = \left(\bigcup_{i\in J} \mathbf{A}_i, \bigvee_{i\in J} \Lambda_i\right)$ and $\bigcap_{P_{i\in J}} \mathscr{A}_i = \left(\bigcap_{i\in J} \mathbf{A}_i, \bigwedge_{i\in J} \Lambda_i\right)$ are F-internal neutrosophic cubic sets in X. □

Similarly, we have the following theorems.

Theorem 3.5. If $\{\mathscr{A}_i = (\mathbf{A}_i, \Lambda_i) \mid i \in J\}$ is a family of T-internal neutrosophic cubic sets in a non-empty set X, then the P-union and the P-intersection of $\{\mathscr{A}_i = (\mathbf{A}_i, \Lambda_i) \mid i \in J\}$ are T-internal neutrosophic cubic sets in X.

Theorem 3.6. If $\{\mathscr{A}_i = (\mathbf{A}_i, \Lambda_i) \mid i \in J\}$ is a family of I-internal neutrosophic cubic sets in a non-empty set X, then the P-union and the P-intersection of $\{\mathscr{A}_i = (\mathbf{A}_i, \Lambda_i) \mid i \in J\}$ are I-internal neutrosophic cubic sets in X.

Corollary 3.2. If $\{\mathscr{A}_i = (\mathbf{A}_i, \Lambda_i) \mid i \in J\}$ is a family of internal neutrosophic cubic sets in a non-empty set X, then the P-union and the P-intersection of $\{\mathscr{A}_i = (\mathbf{A}_i, \Lambda_i) \mid i \in J\}$ are internal neutrosophic cubic sets in X.

The following example shows that P-union and P-intersection of F-external (resp. I-external and T-external) neutrosophic cubic sets may not be F-external (resp. I-external and T-external) neutrosophic cubic sets.

Example 3.4. Let $\mathscr{A} = (\mathbf{A}, \Lambda)$, and $\mathscr{B} = (\mathbf{B}, \Psi)$ be neutrosophic cubic sets in $[0,1]$ where

$\mathbf{A} = \{\langle x; [0.2, 0.5], [0.5, 0.7], [0.3, 0.5]\rangle \mid x \in [0,1]\}$,
$\Lambda = \{\langle x; 0.3, 0.4, 0.8\rangle \mid x \in [0,1]\}$,
$\mathbf{B} = \{\langle x; [0.6, 0.8], [0.4, 0.7], [0.7, 0.9]\rangle \mid x \in [0,1]\}$,
$\Psi = \{\langle x; 0.7, 0.3, 0.4\rangle \mid x \in [0,1]\}$.

Then $\mathscr{A} = (\mathbf{A}, \Lambda)$, and $\mathscr{B} = (\mathbf{B}, \Psi)$ are F-external neutrosophic cubic sets in $[0,1]$, and $\mathscr{A} \cup_P \mathscr{B} = (\mathbf{A} \cup \mathbf{B}, \Lambda \vee \Psi)$ with

$\mathbf{A} \cup \mathbf{B} = \{\langle x; [0.6, 0.8], [0.5, 0.7], [0.7, 0.9]\rangle \mid x \in [0,1]\}$,
$\Lambda \vee \Psi = \{\langle x; 0.7, 0.4, 0.8\rangle \mid x \in [0,1]\}$

is not an F-external neutrosophic cubic set in $[0,1]$ since

$$(\lambda_F \vee \psi_F)(x) = 0.8 \in (0.7, 0.9) = ((A_F \cup B_F)^-(x), (A_F \cup B_F)^+(x)).$$

Also $\mathscr{A} \cap_P \mathscr{B} = (\mathbf{A} \cap \mathbf{B}, \Lambda \wedge \Psi)$ with

$\mathbf{A} \cap \mathbf{B} = \{\langle x; [0.2, 0.5], [0.4, 0.7], [0.3, 0.5]\rangle \mid x \in [0,1]\}$,
$\Lambda \wedge \Psi = \{\langle x; 0.3, 0.3, 0.4\rangle \mid x \in [0,1]\}$

is not an F-external neutrosophic cubic set in $[0,1]$ since

$$(\lambda_F \wedge \psi_F)(x) = 0.4 \in (0.3, 0.5) = ((A_F \cap B_F)^-(x), (A_F \cap B_F)^+(x)).$$

Example 3.5. For $X = \{a, b, c\}$, let $\mathscr{A} = (\mathbf{A}, \Lambda)$, and $\mathscr{B} = (\mathbf{B}, \Psi)$ be neutrosophic cubic sets in X with the tabular representations in Tables 3 and 4, respectively.

Table 3. Tabular representation of $\mathscr{A} = (\mathbf{A}, \Lambda)$

X	$\mathbf{A}(x)$	$\Lambda(x)$
a	$([0.2, 0.3], [0.3, 0.5], [0.3, 0.5])$	$(0.35, 0.25, 0.40)$
b	$([0.4, 0.7], [0.1, 0.4], [0.2, 0.4])$	$(0.35, 0.50, 0.30)$
c	$([0.6, 0.9], [0.0, 0.2], [0.3, 0.4])$	$(0.50, 0.60, 0.55)$

Table 4. Tabular representation of $\mathscr{B} = (\mathbf{B}, \Psi)$

X	$\mathbf{B}(x)$	$\Psi(x)$
a	$([0.3, 0.7], [0.3, 0.5], [0.1, 0.5])$	$(0.25, 0.25, 0.60)$
b	$([0.5, 0.8], [0.5, 0.6], [0.2, 0.5])$	$(0.45, 0.30, 0.30)$
c	$([0.4, 0.9], [0.4, 0.7], [0.3, 0.5])$	$(0.35, 0.10, 0.45)$

Table 5. Tabular representation of $\mathscr{A} \cup_P \mathscr{B} = (\mathbf{A} \cup \mathbf{B}, \Lambda \vee \Psi)$

X	$(\mathbf{A} \cup \mathbf{B})(x)$	$(\Lambda \vee \Psi)(x)$
a	$([0.3, 0.7], [0.3, 0.5], [0.3, 0.5])$	$(0.35, 0.25, 0.60)$
b	$([0.5, 0.8], [0.5, 0.6], [0.2, 0.5])$	$(0.45, 0.50, 0.30)$
c	$([0.6, 0.9], [0.4, 0.7], [0.3, 0.5])$	$(0.50, 0.60, 0.55)$

Table 6. Tabular representation of $\mathscr{A} \cap_P \mathscr{B} = (\mathbf{A} \cap \mathbf{B}, \Lambda \wedge \Psi)$

X	$(\mathbf{A} \cap \mathbf{B})(x)$	$(\Lambda \wedge \Psi)(x)$
a	$([0.2, 0.3], [0.3, 0.5], [0.1, 0.5])$	$(0.25, 0.25, 0.40)$
b	$([0.4, 0.7], [0.1, 0.4], [0.2, 0.4])$	$(0.35, 0.30, 0.30)$
c	$([0.4, 0.9], [0.0, 0.2], [0.3, 0.4])$	$(0.35, 0.10, 0.45)$

Then $\mathscr{A} = (\mathbf{A}, \Lambda)$, and $\mathscr{B} = (\mathbf{B}, \Psi)$ are both T-external and I-external neutrosophic cubic sets in X. Note that the tabular representation of $\mathscr{A} \cup_P \mathscr{B} = (\mathbf{A} \cup \mathbf{B}, \Lambda \vee \Psi)$ and $\mathscr{A} \cap_P \mathscr{B} = (\mathbf{A} \cap \mathbf{B}, \Lambda \wedge \Psi)$ are given by Tables 5 and 6, respectively.
Then $\mathscr{A} \cup_P \mathscr{B} = (\mathbf{A} \cup \mathbf{B}, \Lambda \vee \Psi)$ is neither an I-external neutrosophic cubic set nor

a T-external neutrosophic cubic set in X since

$$(\lambda_I \vee \psi_I)(c) = 0.60 \in (0.4, 0.7) = ((A_I \cup B_I)^-(c), (A_I \cup B_I)^+(c))$$

and

$$(\lambda_T \vee \psi_T)(a) = 0.35 \in (0.3, 0.7) = ((A_T \cup B_T)^-(a), (A_T \cup B_T)^+(a)).$$

Also $\mathscr{A} \cap_P \mathscr{B} = (\mathbf{A} \cap \mathbf{B}, \Lambda \wedge \Psi)$ is neither an I-external neutrosophic cubic set nor a T-external neutrosophic cubic set in X since

$$(\lambda_I \wedge \psi_I)(b) = 0.30 \in (0.1, 0.4) = ((A_I \cap B_I)^-(b), (A_I \cap B_I)^+(b))$$

and

$$(\lambda_T \wedge \psi_T)(a) = 0.25 \in (0.2, 0.3) = ((A_T \cap B_T)^-(a), (A_T \cap B_T)^+(a)).$$

We know that R-union and R-intersection of T-internal (resp. I-internal and F-internal) neutrosophic cubic sets may not be T-internal (resp. I-internal and F-internal) neutrosophic cubic sets as seen in the following examples.

Example 3.6. Let $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ be neutrosophic cubic sets in $[0,1]$ where

$\mathbf{A} = \{\langle x; [0.3, 0.5], [0.5, 0.7], [0.3, 0.5]\rangle \mid x \in [0,1]\}$,
$\Lambda = \{\langle x; 0.4, 0.4, 0.8\rangle \mid x \in [0,1]\}$,
$\mathbf{B} = \{\langle x; [0.7, 0.9], [0.4, 0.7], [0.7, 0.9]\rangle \mid x \in [0,1]\}$,
$\Psi = \{\langle x; 0.8, 0.3, 0.8\rangle \mid x \in [0,1]\}$.

Then $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ are T-internal neutrosophic cubic sets in $[0,1]$. The R-union $\mathscr{A} \cup_R \mathscr{B} = (\mathbf{A} \cup \mathbf{B}, \Lambda \wedge \Psi)$ of $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ is given as follows:

$\mathbf{A} \cup \mathbf{B} = \{\langle x; [0.7, 0.9], [0.5, 0.7], [0.7, 0.9]\rangle \mid x \in [0,1]\}$,
$\Lambda \wedge \Psi = \{\langle x; 0.4, 0.3, 0.8\rangle \mid x \in [0,1]\}$.

Note that $(\lambda_T \wedge \psi_T)(x) = 0.4 < 0.7 = (A_T \cup B_T)^-(x)$ and $(\lambda_I \wedge \psi_I)(x) = 0.3 < 0.5 = (A_I \cup B_I)^-(x)$. Hence $\mathscr{A} \cup_R \mathscr{B} = (\mathbf{A} \cup \mathbf{B}, \Lambda \wedge \Psi)$ is neither a T-internal neutrosophic cubic set nor an I-internal neutrosophic cubic set in $[0,1]$. But, we know that $\mathscr{A} \cup_R \mathscr{B} = (\mathbf{A} \cup \mathbf{B}, \Lambda \wedge \Psi)$ is an F-internal neutrosophic cubic set in $[0,1]$. Also, the R-intersection $\mathscr{A} \cap_R \mathscr{B} = (\mathbf{A} \cap \mathbf{B}, \Lambda \vee \Psi)$ of $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ is given as follows:

$\mathbf{A} \cap \mathbf{B} = \{\langle x; [0.3, 0.5], [0.4, 0.7], [0.3, 0.5]\rangle \mid x \in [0,1]\}$,
$\Lambda \vee \Psi = \{\langle x; 0.8, 0.4, 0.8\rangle \mid x \in [0,1]\}$.

Since $(A_I \cap B_I)^-(x) \leq (\lambda_I \vee \psi_I)(x) \leq (A_I \cap B_I)^+(x)$ for all $x \in [0,1]$, $\mathscr{A} \cap_R \mathscr{B} = (\mathbf{A} \cap \mathbf{B}, \Lambda \vee \Psi)$ is an I-internal neutrosophic cubic set in $[0,1]$. But it is neither a T-internal neutrosophic cubic set nor an F-internal neutrosophic cubic set in $[0,1]$.

Example 3.7. Let $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ be neutrosophic cubic sets in $[0,1]$ where

$\mathbf{A} = \{\langle x; [0.1, 0.3], [0.5, 0.7], [0.3, 0.5]\rangle \mid x \in [0,1]\}$,

$\Lambda = \{\langle x; 0.4, 0.6, 0.8\rangle \mid x \in [0,1]\}$,
$\mathbf{B} = \{\langle x; [0.7, 0.9], [0.4, 0.5], [0.7, 0.9]\rangle \mid x \in [0,1]\}$,
$\Psi = \{\langle x; 0.5, 0.45, 0.2\rangle \mid x \in [0,1]\}$,
Then $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ are I-internal neutrosophic cubic sets in $[0,1]$. The R-union $\mathscr{A} \cup_R \mathscr{B} = (\mathbf{A} \cup \mathbf{B}, \Lambda \wedge \Psi)$ of $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ is given as follows:
$\mathbf{A} \cup \mathbf{B} = \{\langle x; [0.7, 0.9], [0.5, 0.7], [0.7, 0.9]\rangle \mid x \in [0,1]\}$,
$\Lambda \wedge \Psi = \{\langle x; 0.4, 0.45, 0.2\rangle \mid x \in [0,1]\}$.
Since $(\lambda_I \wedge \psi_I)(x) = 0.45 < 0.5 = (A_I \cup B_I)^-(x)$, we know that $\mathscr{A} \cup_R \mathscr{B}$ is not an I-internal neutrosophic cubic set in $[0,1]$. Also, the R-intersection $\mathscr{A} \cap_R \mathscr{B} = (\mathbf{A} \cap \mathbf{B}, \Lambda \vee \Psi)$ of $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ is given as follows:
$\mathbf{A} \cap \mathbf{B} = \{\langle x; [0.1, 0.3], [0.4, 0.5], [0.3, 0.5]\rangle \mid x \in [0,1]\}$,
$\Lambda \vee \Psi = \{\langle x; 0.5, 0.6, 0.8\rangle \mid x \in [0,1]\}$,
and it is not an I-internal neutrosophic cubic set in $[0,1]$.

Example 3.8. Let $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ be neutrosophic cubic sets in $[0,1]$ where
$\mathbf{A} = \{\langle x; [0.1, 0.3], [0.5, 0.7], [0.3, 0.8]\rangle \mid x \in [0,1]\}$,
$\Lambda = \{\langle x; 0.4, 0.6, 0.4\rangle \mid x \in [0,1]\}$,
$\mathbf{B} = \{\langle x; [0.4, 0.7], [0.4, 0.7], [0.5, 0.8]\rangle \mid x \in [0,1]\}$,
$\Psi = \{\langle x; 0.5, 0.3, 0.6\rangle \mid x \in [0,1]\}$,
Then $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ are F-internal neutrosophic cubic sets in $[0,1]$. The R-union $\mathscr{A} \cup_R \mathscr{B} = (\mathbf{A} \cup \mathbf{B}, \Lambda \wedge \Psi)$ of $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ is given as follows:
$\mathbf{A} \cup \mathbf{B} = \{\langle x; [0.4, 0.7], [0.5, 0.7], [0.5, 0.8]\rangle \mid x \in [0,1]\}$,
$\Lambda \wedge \Psi = \{\langle x; 0.4, 0.3, 0.4\rangle \mid x \in [0,1]\}$,
which is not an F-internal neutrosophic cubic set in $[0,1]$. If $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ are neutrosophic cubic sets in $\mathscr{R}$ with
$\mathbf{A} = \{\langle x; [0.2, 0.6], [0.3, 0.7], [0.7, 0.8]\rangle \mid x \in \mathscr{R}\}$,
$\Lambda = \{\langle x; 0.7, 0.6, 0.75\rangle \mid x \in \mathscr{R}\}$,
$\mathbf{B} = \{\langle x; [0.3, 0.7], [0.6, 0.7], [0.2, 0.6]\rangle \mid x \in \mathscr{R}\}$,
$\Psi = \{\langle x; 0.5, 0.3, 0.5\rangle \mid x \in \mathscr{R}\}$,
then $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ are F-internal neutrosophic cubic sets in $\mathscr{R}$ and the R-intersection $\mathscr{A} \cap_R \mathscr{B} = (\mathbf{A} \cap \mathbf{B}, \Lambda \vee \Psi)$ of $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ which is given as follows:
$\mathbf{A} \cap \mathbf{B} = \{\langle x; [0.2, 0.6], [0.3, 0.7], [0.2, 0.6]\rangle \mid x \in \mathscr{R}\}$,
$\Lambda \vee \Psi = \{\langle x; 0.7, 0.6, 0.75\rangle \mid x \in \mathscr{R}\}$,
is not an F-internal neutrosophic cubic set in $[0,1]$.

We provide conditions for the R-union of two T-internal (resp. I-internal and F-internal) neutrosophic cubic sets to be a T-internal (resp. I-internal and F-internal) neutrosophic cubic set.

Theorem 3.7. Let $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ be T-internal neutrosophic cubic sets in a non-empty set X such that

$$(\forall x \in X)\left(\max\{A_T^-(x), B_T^-(x)\} \leq (\lambda_T \wedge \psi_T)(x)\right). \tag{3.10}$$

Then the R-union of $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ is a T-internal neutrosophic cubic set in X.

Proof. Let $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ be T-internal neutrosophic cubic sets in a non-empty set X which satisfy the condition (3.10). Then

$$A_T^-(x) \leq \lambda_T(x) \leq A_T^+(x) \text{ and } B_T^-(x) \leq \psi_T(x) \leq B_T^+(x),$$

and so $(\lambda_T \wedge \psi_T)(x) \leq (A_T \cup B_T)^+(x)$. It follows from (3.10) that

$$(A_T \cup B_T)^-(x) = \max\{A_T^-(x), B_T^-(x)\} \leq (\lambda_T \wedge \psi_T)(x) \leq (A_T \cup B_T)^+(x).$$

Hence $\mathscr{A} \cup_R \mathscr{B} = (\mathbf{A} \cup \mathbf{B}, \Lambda \wedge \Psi)$ is a T-internal neutrosophic cubic set in X. □

Similarly, we have the following theorems.

Theorem 3.8. Let $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ be I-internal neutrosophic cubic sets in a non-empty set X such that

$$(\forall x \in X)\left(\max\{A_I^-(x), B_I^-(x)\} \leq (\lambda_I \wedge \psi_I)(x)\right). \tag{3.11}$$

Then the R-union of $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ is an I-internal neutrosophic cubic set in X.

Theorem 3.9. Let $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ be F-internal neutrosophic cubic sets in a non-empty set X such that

$$(\forall x \in X)\left(\max\{A_F^-(x), B_F^-(x)\} \leq (\lambda_F \wedge \psi_F)(x)\right). \tag{3.12}$$

Then the R-union of $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ is an F-internal neutrosophic cubic set in X.

Corollary 3.3. If two internal neutrosophic cubic sets $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ satisfy conditions (3.10), (3.11) and (3.12), then the R-union of $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ is an internal neutrosophic cubic set in X.

We provide conditions for the R-intersection of two T-internal (resp. I-internal and F-internal) neutrosophic cubic sets to be a T-internal (resp. I-internal and F-internal) neutrosophic cubic set.

Theorem 3.10. Let $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ be I-internal neutrosophic cubic sets in a non-empty set X such that

$$(\forall x \in X)\left((\lambda_I \vee \psi_I)(x) \leq \min\{A_I^+(x), B_I^+(x)\}\right). \tag{3.13}$$

Then the R-intersection of $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ is an I-internal neutrosophic cubic set in X.

Proof. Assume that the condition (3.13) is valid. Then

$$A_I^-(x) \leq \lambda_I(x) \leq A_I^+(x) \text{ and } B_I^-(x) \leq \psi_I(x) \leq B_I^+(x)$$

for all $x \in X$. It follows from (3.13) that

$$(A_I \cap B_I)^-(x) \leq (\lambda_I \vee \psi_I)(x) \leq \min\{A_I^+(x), B_I^+(x)\} = (A_I \cap B_I)^+(x)$$

for all $x \in X$. Therefore $\mathscr{A} \cap_R \mathscr{B} = (\mathbf{A} \cap \mathbf{B}, \Lambda \vee \Psi)$ is an I-internal neutrosophic cubic set in X. □

Similarly, we have the following theorems.

Theorem 3.11. Let $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ be T-internal neutrosophic cubic sets in a non-empty set X such that

$$(\forall x \in X)\left((\lambda_T \vee \psi_T)(x) \leq \min\{A_T^+(x), B_T^+(x)\}\right). \tag{3.14}$$

Then the R-intersection of $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ is a T-internal neutrosophic cubic set in X.

Theorem 3.12. Let $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ be F-internal neutrosophic cubic sets in a non-empty set X such that

$$(\forall x \in X)\left((\lambda_F \vee \psi_F)(x) \leq \min\{A_F^+(x), B_F^+(x)\}\right). \tag{3.15}$$

Then the R-intersection of $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ is an F-internal neutrosophic cubic set in X.

Corollary 3.4. If two internal neutrosophic cubic sets $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ satisfy conditions (3.13), (3.14) and (3.15), then the R-intersection of $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ is an internal neutrosophic cubic set in X.

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