Journal of Intelligent & Fuzzy Systems 30 (2016) 1957–1963
DOI:10.3233/IFS-151906
IOS Press

The theory of neutrosophic cubic sets and their applications in pattern recognition

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Abstract. In this study, we presented concept of neutrosophic cubic set by extending the concept of cubic set to neutrosophic set. We also defined internal neutrosophic cubic set (INCS) and external neutrosophic cubic set (ENCS). Then, we proposed some new type of INCS and ENCS is called $\frac{1}{3}$-INCS (or $\frac{2}{3}$-ENCS), $\frac{2}{3}$-INCS (or $\frac{1}{3}$-ENCS). Then we study some of their relevant properties. Finally, we introduce an adjustable approach to NCS based decision making by similarity measure and an illustrative example is employed to show that they can be successfully applied to problems that contain uncertainties.

Keywords: Cubic set, neutrosophic cubic set, internal (external) neutrosophic cubic set, decision making

1. Introduction

Fuzzy set is firstly introduced by Zadeh [23] to represent the degree of certainty of expert's in different statements. Zadeh also proposed the concept of a linguistic variable with application in [24]. Then Peng et al. [12] presented an application in multi-criteria decision–making problems. After Zadeh, Türkşen [18] extend fuzzy set to an interval valued fuzzy set. Interval values fuzzy sets have many applications in real life such as Sambuc [16], Kohout [9], Mukherjee and Sarkar [25] also gave its applications in Medical, Türkşen [19, 20] in interval valued logic.

Jun et al. [7] introduced cubic set which is basically the combination of fuzzy sets with interval valued fuzzy sets. They also defined internal (external) cubic sets and investigate some of their properties. The notions of cubic subalgebras/ideals in BCK/BCI algebras are also introduced in [5]. Jun et al. [6] gave cubic q-ideals in BCI-algebras and relationship between a cubic ideal and a cubic q-ideal. Also, they established conditions for a cubic ideal to be cubic q-ideal, characterizations of a cubic q-ideal and cubic extension property for a cubic q-ideal. The concept of cubic sub LA-Γ-semihypergroup is proposed by Aslam et al. [1]. They also introduced results on cubic Γ-hyperideals and cubic bi-Γ-hyperideals in left almost Γ-semihypergroups.

Smarandache [17] introduced the theory of neutrosophic logic and sets in 1995. Neutrosophy provides a foundation for a whole family of new mathematical theories with the generalization of both classical and fuzzy counterparts. In a neutrosophic set, an element has three associated defining functions such as truth membership function T, indeterminate membership function I and false membership function F defined on a universe of discourse X. These three functions are independent completely. Neutrosophic set is basically studies the origin, nature and scope of neutralities and their interactions with ideational spectra. Neutrosophic set generalizes the concept of classical fuzzy set [23], and so on. The neutrosophic

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set has vast applications in various fields [2–4, 8, 10, 13–15, 22]. After Smarandache [17], Wang et al. [21] introduced the concept of interval neutrosophic sets which are the generalization of fuzzy, interval valued fuzzy set and neutrosophic sets. Also the interval neutrosophic set studied in [26].

In this paper, we introduce neutrosophic cubic set and define some new notions such as internal (external) neutrosophic cubic sets. The notion of neutrosophic cubic set generalizes the concept of cubic set. We also investigate some of the core properties of neutrosophic cubic set. By using these new notions we then construct a decision making method called neutrosophic cubic method. We finally present an application which shows that the methods can be successfully applied to many problems containing uncertainties.

1.1. Fundamental concepts

In this section, we present basic definitions of fuzzy sets [23], interval valued fuzzy sets [18], neutrosophic sets [17], interval valued neutrosophic sets [21] and cubic sets [7]. For more detail of these sets, we refer to the earlier studies [1, 5–7, 17, 18, 21, 23].

Definition 1. [23] Let E be a universe. Then a fuzzy set μ over E is defined by

$$\mathrm{X} = \{\mu_{\mathrm{x}}\,(\mathrm{x})/\mathrm{x} : \mathrm{x} \in \mathrm{E}\}$$

where μ_x is called membership function of X and defined by μ_x: $E \to [0.1]$. For each x E, the value $\mu_x(x)$ represents the degree of x belonging to the fuzzy set X.

Definition 2. [18] Let E be a universe. Then, an interval valued fuzzy set A over E is defined by

$$A = \left\{\left[\mathrm{A}^-(\mathrm{x}), \mathrm{A}^+(\mathrm{x})\right]/\mathrm{x} : \mathrm{x} \in \mathrm{E}\right\}$$

where $\mathrm{A}^-(\mathrm{x})$ and $\mathrm{A}^+(\mathrm{x})$ are referred to as the lower and upper degrees of membership $\mathrm{x} \in \mathrm{E}$ where $0 \leq \mathrm{A}^-(\mathrm{x}) + \mathrm{A}^+(\mathrm{x}) \leq 1$, respectively.

Definition 3. [7] Let X be a non-empty set. By a cubic set, we mean a structure

$$\Xi = \{\langle x, A(x), \mu(x)\rangle \,|x \in X\}$$

in which A is an interval valued fuzzy set (IVF) and μ is a fuzzy set. It is denoted by $\langle A, \mu\rangle$.

Definition 4. [7] Let X be a non-empty set. A cubic set $\Xi = \langle A, \mu\rangle$ in X is called an internal cubic set (ICS) if $A^-(x) \leq \mu(x) \leq A^+(x)$ for all $x \in X$, where A^- and A^+ are lower fuzzy set and upper fuzzy set in X respectively.

Definition 5. [7] Let X be a non-empty set. A cubic set $\Xi = \langle A, \mu\rangle$ in X is called an external cubic set (ECS) if $\mu(x) \notin (A^-(x), A^+(x))$ for all $x \in X$.

Definition 6. [7] Let $\Xi_1 = \langle A_1, \mu_1\rangle$ and $\Xi_2 = \langle A_2, \mu_2\rangle$ be cubic sets in X. Then we define

1. (Equality) $\Xi_1 = \Xi_2$ if and only if $A_1 = A_2$ and $\mu_1 = \mu_2$.
2. (P-order) $\Xi_1 \subseteq_p \Xi_2$ if and only if $A_1 \subseteq A_2$ and $\mu_1 \leq \mu_2$.
3. (R-order) $\Xi_1 \subseteq_R \Xi_2$ if and only if $A_1 \subseteq A_2$ and $\mu_1 \geq \mu_2$.

Definition 7. [7] For any

$$\Xi_j = \left\{\left\langle x, A_j(x), \mu_j(x)\right\rangle |x \in X\right\}$$

where $j \in \Lambda$, we define

1. $\underset{j\in\Lambda}{\cup_p} \Xi_j = \left\{\left\langle x, \left(\underset{j\in\Lambda}{\cup} A_j\right)(x), \left(\underset{j\in\Lambda}{\vee} \mu_j\right)\right\rangle |x \in X\right\}$ (P-union).
2. $\underset{j\in\Lambda}{\cap_p} \Xi_j = \left\{\left\langle x, \left(\underset{j\in\Lambda}{\cap} A_j\right)(x), \left(\underset{j\in\Lambda}{\wedge} \mu_j\right)\right\rangle |x \in X\right\}$ (P-intersection).
3. $\underset{j\in\Lambda}{\cup_R} \Xi_j = \left\{\left\langle x, \left(\underset{j\in\Lambda}{\cup} A_j\right)(x), \left(\underset{j\in\Lambda}{\wedge} \mu_j\right)\right\rangle |x \in X\right\}$ (R-union).
4. $\underset{j\in\Lambda}{\cap_R} \Xi_j = \left\{\left\langle x, \left(\underset{j\in\Lambda}{\cap} A_j\right)(x), \left(\underset{j\in\Lambda}{\vee} \mu_j\right)\right\rangle |x \in X\right\}$ (R-intersection).

Definition 8. [17] Let X be an universe. Then a neutrosophic (NS) set λ is an object having the form

$$\lambda = \{< \mathrm{x} : \mathrm{T}(\mathrm{x}), \mathrm{I}(\mathrm{x}), \mathrm{F}(\mathrm{x}) >: \mathrm{x} \in \mathrm{X}\}$$

where the functions T, I, F : $\mathrm{X} \longrightarrow\ ]^-0, 1^+[$ define respectively the degree of Truth, the degree of indeterminacy, and the degree of Falsehood of the element $\mathrm{x} \in \mathrm{X}$ to the set λ with the condition.

$$^-0 \leq \mathrm{T}(\mathrm{x}) + \mathrm{I}(\mathrm{x}) + \mathrm{F}(\mathrm{x}) \leq 3^+$$

For two NS,

$$\lambda_1 = \{< \mathrm{x}, \mathrm{T}_1(\mathrm{x}), \mathrm{I}_1(\mathrm{x}), \mathrm{F}_1(\mathrm{x}) > |\mathrm{x} \in \mathrm{X}\}$$

and

$$\lambda_2 = \{< \mathrm{x}, \mathrm{T}_2(\mathrm{x}), \mathrm{I}_2(\mathrm{x}), \mathrm{F}_2(\mathrm{x}) > |\mathrm{x} \in \mathrm{X}\}$$

the operations are defined as follows:

1. $\lambda_1 \breve{\subseteq} \lambda_1$ if and only if

$$T_1(x) \leq T_2(x), I_1(x) \geq I_2(x), F_1(x) \geq F_2(x)$$

2. $\lambda_1 = \lambda_1$ if and only if,

$$T_1(x) = T_2(x), I_1(x) = I_2(x), F_1(x) = F_2(x)$$

3. $\lambda_1^{\breve{c}} = \{x,\ F_1(x),\ I_1(x),\ T_1(x) >: x \in X\}$
4. $\lambda_1 \breve{\cap} \lambda_2 = \{< x, \min\{T_1(x), T_2(x)\},\ \max\{I_1(x), I_2(x)\}, \max\{F_1(x), F_2(x)\} >: x \in X\}$
5. $\lambda_1 \breve{\cup} \lambda_2 = \{< x, \max\{T_1(x), T_2(x)\},\ \max\{I_1(x), I_2(x)\}, \min\{F_1(x), F_2(x)\} >: x \in X\}$

Definition 9. [21] Let X be a non-empty set. An interval neutrosophic set (INS) A in X is characterized by the truth-membership function A_T, the indeterminacy-membership function A_L and the falsity-membership function A_F. For each point $x \in X$, $A_T(x), A_I(x), A_F(x) \subseteq [0, 1]$.

For two INS

$$A = \left\{< x, \left[A_T^-(x), A_T^+(x)\right], \left[A_I^-(x), A_I^+(x)\right], \left[A_F^-(x), A_F^+(x)\right] >: x \in X\right\}$$

and

$$B = \left\{< x, \left[B_T^-(x), B_T^+(x)\right], \left[B_I^-(x), B_I^+(x)\right], \left[B_F^-(x), B_F^+(x)\right] >: x \in X\right\}$$

Then,

1. $A \tilde{\subseteq} B$ if and only if

$$A_T^-(x) \leq B_T^-(x), A_T^+(x) \leq B_T^+(x),$$
$$A_I^-(x) \geq B_I^-(x), A_I^+(x) \geq B_I^+(x),$$
$$A_F^-(x) \geq B_F^-(x), A_F^+(x) \geq B_F^+(x) \text{ for all } x \in X.$$

2. $A = B$ if and only if

$$A_T^-(x) = B_T^-(x), A_T^+(x) = B_T^+(x),$$
$$A_I^-(x) = B_I^-(x), A_I^+(x), B_I^+(x),$$
$$A_F^-(x) = B_F^-(x), A_F^+(x) = B_F^+(x) \text{ for all } x \in X.$$

3. $A^{\tilde{c}} = \left\{< x, \left[A_F^-(x), A_F^+(x)\right], \left[A_I^-(x), A_I^+(x)\right], \left[A_T^-(x), A_T^+(x)\right] >: x \in X\right\}$

4. $A \tilde{\cap} B = \left\{< x, \left[\min\left\{A_T^-(x), B_T^-(x)\right\}, \min\left\{A_T^+(x), B_T^+(x)\right\}\right], \left[\max\left\{A_I^-(x), B_I^-(x)\right\}, \max\left\{A_I^+(x), B_I^+(x)\right\}\right], \left[\max\left\{A_F^-(x), B_F^-(x)\right\}, \max\left\{A_F^+(x), B_F^+(x)\right\}\right] >: x \in X\right\}$

5. $A \tilde{\cup} B = \left\{< x, \left[\max\left\{A_T^-(x), B_T^-(x)\right\}, \max\left\{A_T^+(x), B_T^+(x)\right\}\right], \left[\mathrm{Min}\left\{A_I^-(x), B_I^-(x)\right\}, \min\left\{A_I^+(x), B_I^+(x)\right\}\right], \left[\min\left\{A_F^-(x), B_F^-(x)\right\}, \min\left\{A_F^+(x), B_F^+(x)\right\}\right] >: x \in X\right\}$

2. Neutrosophic cubic set

In this section, we extend the notion of cubic set to neutrosophic set and introduced neutrosophic cubic set. We also defined internal neutrosophic cubic set (INCS) and external neutrosophic cubic set (ENCS). Some of it is quoted from [1, 5–7, 25].

Definition 10. Let X be an universe. Then a neutrosophic cubic set (NCS) set $\Im$ is an object having the form

$$\Im = \{\langle x, A(x), \lambda(x)\rangle \mid x \in X\}$$

where A is an interval neutrosophic set in X and λ is a neutrosophic set in X. We simply denoted a neutrosophic cubic set as $\Im = \langle A, \lambda\rangle$.

Note that the sets of all neutrosophic cubic sets over X will be denoted by C_N^X.

Example 1. Suppose that $X = \{x_1, x_2, x_3\}$ be a universe set. Then an interval neutrosophic set A of X defined by

$$A = \left\{\begin{array}{l} \langle[0.22, 0.3], [0.3, 0.4], [0.7, 0.9]\rangle / x_1 + \\ \langle[0.1, 0.2], [0.5, 0.9], [0.1, 0.9]\rangle / x_2 + \\ \langle[0.7, 0.8], [0.1, 0.1], [0.5, 0.4]\rangle / x_3 \end{array}\right\}$$

and a neutrosophic set λ is a set of X defined by

$$\lambda = \left\{\begin{array}{l} \langle 0.01, 0.2, 0.4\rangle / x_1, \langle 0.1, 0.02, 0.2\rangle / x_2, \\ \langle 0.3, 0.1, 0.7\rangle / 3 \end{array}\right\}$$

Then $\Im = \langle A, \lambda\rangle$ is a neutrosophic cubic set in X.

Definition 11. Let $\Im = \langle A, \lambda \rangle \in C_N^X$. If $A_T^-(x) \le T(x) \le A_T^+(x)$, $A_I^-(x) \le I(x) \le A_I^+(x)$ and $A_F^-(x) \le F(x) \le A_F^+(x)$ for all $x \in X$, then $\Im$ is called an internal neutrosophic cubic set (INCS).

Example 2. Let $\Im = \langle A, \lambda \rangle \in C_N^X$. If $A(x) = \langle\langle [0.2, 0.4], [0.3, 0.5], [0.3, 0.5] \rangle\rangle$ and $\lambda(x) = \langle 0.3, 0.4, 0.4 \rangle$ for all $x \in X$, then $\Im = \langle A, \lambda \rangle$ is a INCS.

Definition 12. Let $\Im = \langle A, \lambda \rangle \in C_N^X$. If $T(x) \notin (A_T^-(x), A_T^+(x))$, $I(x) \notin (A_I^-(x), A_I^+(x))$ and $F(x) \notin (A_F^-(x), A_F^+(x))$ for all $x \in X$, then $\Im$ is called an external neutrosophic cubic set (ENCS).

Example 3. Let $\Im = \langle A, \lambda \rangle \in C_N^X$.

If $\lambda(x) = \langle 0.5, 0.2, 0.7 \rangle$ in the above Example 2, then $\Im = \langle A, \lambda \rangle$ is a ENCS.

Theorem 1. *Let $\Im = \langle A, \lambda \rangle \in C_N^X$ which is not external neutrosophic cubic set. Then, there exist $x \in X$ such that*

$$A_T^-(x) \le T(x) \le A_T^+(x), \ A_I^-(x) \le I(x) \le A_I^+(x)$$
$$\text{or } A_F^-(x) \le F(x) \le A_F^+(x).$$

Theorem 2. *Let $\Im = \langle A, \lambda \rangle \in C_N^X$. If $\Im$ is both INCS and ENCS. Then*

$$T(x) \in (U(A_T) \cup L(A_T)),$$
$$I(x) \in (U(A_I) \cup L(A_I))$$

and $F(x) \in (U(A_F) \cup L(A_F))$ for all $x \in X$, where

$$U(A_T) = \{A_T^+(x) | x \in X\},$$
$$L(A_T) = \{A_T^-(x) | x \in X\},$$
$$U(A_I) = \{A_I^+(x) | x \in X\},$$
$$L(A_I) = \{A_I^-(x) | x \in X\},$$
$$U(A_F) = \{A_F^+(x) | x \in X\}$$
$$\text{and } L(A_F) = \{A_F^-(x) | x \in X\}.$$

Proof. Suppose that $\Im = \langle A, \lambda \rangle$ is both INCS and ENCS. Then by Definitions (3.1.3) and (3.1.4), we have

$$A_T^-(x) \le T(x) \le A_T^+(x),$$
$$A_I^-(x) \le I(x) \le A_I^+(x),$$
$$A_F^-(x) \le F(x) \le A_F^+(x)$$

And

$$T(x) \notin (A_T^-(x), A_T^+(x)),$$
$$I(x) \notin (A_I^-(x), A_I^+(x)),$$
$$F(x) \notin (A_F^-(x), A_F^+(x)),$$

For all $x \in X$. Thus

$$T(x) = A_T^-(x) \text{ or } A_T^+(x) = T(x),$$
$$I(x) = A_I^-(x) \text{ or } A_I^+(x) = I(x),$$
$$F(x) = A_F^-(x) \text{ or } A_F^+(x) = F(x).$$

Hence

$$T(x) \in (U(A_T) \cup L(A_T)),$$
$$I(x) \in (U(A_I) \cup L(A_I)),$$
$$F(x) \in (U(A_F) \cup L(A_F)).$$

Definition 13. Let $\Im = \langle A, \lambda \rangle \in C_N^X$. If

$$A_T^-(x) \le T(x) \le A_T^+(x),$$
$$A_I^-(x) \le I(x) \le A_I^+(x)$$

and $F(x) \notin (A_F^-(x), A_F^+(x))$
or

$$A_T^-(x) \le T(x) \le A_T^+(x),$$
$$A_F^-(x) \le F(x) \le A_F^+(x)$$

and $I(x) \notin (A_I^-(x), A_I^+(x))$
or

$$A_F^-(x) \le F(x) \le A_F^+(x),$$
$$A_I^-(x) \le I(x) \le A_I^+(x)$$

and $T(x) \notin (A_T^-(x), A_T^+(x))$ for all $x \in X$, then $\Im$ is called $\frac{2}{3}$-INCS or $\frac{1}{3}$-ENCS.

Example 4. Let $\Im = \langle A, \lambda \rangle \in C_N^X$, where $A(x) = \langle\langle [0.1, 0.4], [0.2, 0.5], [0.3, 0.6] \rangle\rangle$ and $\lambda(x) = \langle 0.3, 0.7, 0.4 \rangle$ for all $x \in X$. Then $\Im = \langle A, \lambda \rangle$ is $\frac{2}{3}$-INCS.

Definition 14. Let $\Im = \langle A, \lambda \rangle \in C_N^X$. If

$$A_T^-(x) \le T(x) \le A_T^+(x),$$
$$I(x) \notin (A_I^-(x), A_I^+(x))$$

and $F(x) \notin (A_F^-(x), A_F^+(x))$
or

$$A_F^-(x) \le F(x) \le A_F^+(x),$$
$$T(x) \notin (A_T^-(x), A_T^+(x))$$

and $I(x) \notin (A_I^-(x), A_I^+(x))$
or

$$A_I^-(x) \le I(x) \le A_I^+(x),$$
$$T(x) \notin (A_T^-(x), A_T^+(x))$$

and $F(x) \notin (A_F^-(x), A_F^+(x))$, for all $x \in X$, then $\Im$ is called $\frac{1}{3}$-INCS or $\frac{2}{3}$-ENCS.

Example 5. Let $\Im = \langle A, \lambda\rangle \in C_N^X$, where $A(x) = \langle\langle[0.1, 0.3]_T, [0.2, 0.4]_I, [0.4, 0.6]_F\rangle\rangle$ and $\lambda(x) = \langle 0.2, 0.7, 0.7\rangle$ for all $x \in X$. Then $\Im = \langle A, \lambda\rangle$ is $\frac{1}{3}$-INCS.

Theorem 3. *Let $\Im = \langle A, \lambda\rangle \in C_N^X$. Then,*

i. *Every INCS is a generalization of the ICS.*
ii. *Every ENCS is a generalization of the ECS.*
iii. *Every NCS is the generalization of cubic set.*

Proof. The proofs are directly followed from above definitions.

Definition 15. Let $\Im_1 = \langle A_1, \lambda_1\rangle$ and $\Im_2 = \langle A_2, \lambda_2\rangle$ be neutrosophic cubic sets in X. Then

1. (*Equivality*) $\Im_1 = \Im_2$ if and only if $A_1 = A_2$ and $\lambda_1 = \lambda_2$,
2. ($P-order$) $\Im_1 \subseteq_p \Im_2$ if and only if $A_1 \tilde{\subseteq} A_2$ and $\lambda_1 \breve{\subseteq} \lambda_2$,
3. ($R-order$) $\Im_1 \subseteq_R \Im_2$ if and only if $A_1 \tilde{\subseteq} A_2$ and $\lambda_1 \breve{\supseteq} \lambda_2$.

Definition 16. Let $\Im_j = \langle A_j, \lambda_j\rangle \in C_N^X$, where $j \in \{1, 2, \ldots, n\}$, we define

1. $\underset{j\in I}{\cup_p} \Im_j = \left\{\left\langle x, \left(\underset{j\in I}{\tilde{\cup}} A_j\right)(x), \left(\underset{j\in I}{\breve{\cup}} \lambda_j\right)(x)\right\rangle | x \in X\right\}$ (P-union)
2. $\underset{j\in I}{\cap_p} \Im_j = \left\{\left\langle x, \left(\underset{j\in I}{\tilde{\cap}} A_j\right)(x), \left(\underset{j\in I}{\breve{\cap}} \lambda_j\right)(x)\right\rangle | x \in X\right\}$ (P-intersection)
3. $\underset{j\in I}{\cup_R} \Im_j = \left\{\left\langle x, \left(\underset{j\in I}{\tilde{\cup}} A_j\right)(x), \left(\underset{j\in I}{\breve{\cap}} \lambda_j\right)(x)\right\rangle | x \in X\right\}$ (R-union)
4. $\underset{j\in I}{\cap_R} \Im_j = \left\{\left\langle x, \left(\underset{j\in I}{\tilde{\cap}} A_j\right)(x), \left(\underset{j\in I}{\breve{\cup}} \lambda_j\right)(x)\right\rangle | x \in X\right\}$ (R-intersection)

Definition 17. Let $\Im = \langle A, \lambda\rangle$. The complement of $\Im = \langle A, \lambda\rangle$ is defined as

$$\Im^c = \left\{\left\langle x,\ A^{\tilde{c}}(x),\ \lambda^{\breve{c}}(x) : x \in X\right\rangle\right\}$$

Theorem 4. *Let $\Im_j = \langle A_j, \lambda_j\rangle \in C_N^X$, where $j \in \{1, 2, \ldots, n\}$, the following holds.*

1. $\left(\underset{j\in\Lambda}{\cup_p} \Im_j\right)^c = \underset{j\in\Lambda}{\cap_p} (\Im_j)^C$ and $\left(\underset{j\in\Lambda}{\cap_p} \Im_j\right)^c = \underset{j\in\Lambda}{\cup_p} (\Im_j)^C$,
2. $\left(\underset{j\in\Lambda}{\cup_R} \Im_j\right)^c = \underset{j\in\Lambda}{\cap_R} (\Im_j)^C$ and $\left(\underset{j\in\Lambda}{\cap_R} \Im_j\right)^c = \underset{j\in\Lambda}{\cup_R} (\Im_j)^C$.

Theorem 5. *Let $\Im_j = \langle A_j, \lambda_j\rangle \in C_N^X$, where $j \in \{1, 2, 3, 4\}$. Then*

1. If $\Im_1 \subseteq_p \Im_2$ and $\Im_2 \subseteq_p \Im_3$, then $\Im_1 \subseteq_p \Im_3$.
2. If $\Im_1 \subseteq_p \Im_2$, then $\Im_2^c \subseteq_p \Im_1^c$.
3. If $\Im_1 \subseteq_p \Im_2$ and $\Im_1 \subseteq_p \Im_3$, then $\Im_1 \subseteq_p (\Im_2 \cap_p \Im_3)$.
4. If $\Im_1 \subseteq_p \Im_2$ and $\Im_3 \subseteq_p \Im_2$, then $(\Im_1 \cup_p \Im_3) \subseteq_p \Im_2$.
5. If $\Im_1 \subseteq_p \Im_2$ and $\Im_3 \subseteq_p \Im_4$, then $(\Im_1 \cup_p \Im_3) \subseteq_p (\Im_2 \cup_p \Im_4)$ and $(\Im_1 \cap_p \Im_3) \subseteq_p (\Im_2 \cap_p \Im_4)$.
6. If $\Im_1 \subseteq_R \Im_2$ and $\Im_2 \subseteq_R \Im_3$, then $\Im_1 \subseteq_R \Im_3$.
7. If $\Im_1 \subseteq_R \Im_2$, then $\Im_2^c \subseteq_R \Im_1^c$.
8. If $\Im_1 \subseteq_R \Im_2$ and $\Im_1 \subseteq_R \Im_3$, then $\Im_1 \subseteq_R (\Im_2 \cap_R \Im_3)$.
9. If $\Im_1 \subseteq_R \Im_2$ and $\Im_3 \subseteq_R \Im_2$, then $(\Im_1 \cup_R \Im_3) \subseteq_R \Im_2$.
10. $\Im_1 \subseteq_R \Im_2$ and $\Im_3 \subseteq_R \Im_4$, then $(\Im_1 \cup_R \Im_3) \subseteq_R (\Im_2 \cup_R \Im_4)$ and $(\Im_1 \cap_R \Im_3) \subseteq_R (\Im_2 \cap_R \Im_4)$

Proof. Straightforward.

Theorem 6. *Let $\Im = \langle A, \lambda\rangle \in C_N^X$.*

a. $(\Im^c)^c = \Im$.
b. *If $\Im$ is an INCS, then $\Im^c$ is an INCS.*
c. *If $\Im$ is an ENCS, then $\Im^c$ is an ENCS.*

Proof. It is easy.

Definition 18. Let $\Im_1 = \langle A_1, \lambda_1\rangle$ and $\Im_2 = \langle A_2, \lambda_2\rangle$ be neutrosophic cubic sets in X. Then, distance measure between $\Im_1 = \langle A_1, \lambda_1\rangle$ and $\Im_2 = \langle A_2, \lambda_2\rangle$ is given by

$$\begin{aligned} d(\Im_1, \Im_2) &= \frac{1}{9n}\sum_{i=1}^{n}\left(\left|A_{1_T}^-(x_i) - A_{2_T}^-(x_i)\right|\right. \\ &\quad + \left|A_{1_I}^-(x_i) - A_{2_I}^-(x_i)\right| \end{aligned}$$

$$+\left|A^{-}_{1_F}(x_i)-A^{-}_{2_F}(x_i)\right|$$
$$+\left|A^{+}_{1_T}(x_i)-A^{+}_{2_T}(x_i)\right|$$
$$+\left|A^{+}_{1_I}(x_i)-A^{+}_{2_I}(x_i)\right|$$
$$+\left|A^{+}_{1_F}(x_i)-A^{+}_{2_F}(x_i)\right|+|T_1(x_i)-T_2(x_i)|$$
$$+|I_1(x_i)-I_2(x_i)|+|F_1(x_i)-F_2(x_i)|)$$

It is adapted from [25].

Theorem 7. *Let $\Im_1=\langle A_1,\lambda_1\rangle$ and $\Im_2=\langle A_2,\lambda_2\rangle$ be two neutrosophic cubic sets. The function $d(\Im_1,\ \Im_2)=C_N^X\to R^+$ given by Definition 18 is metrics, where R^+ is the set of all non-negative real numbers.*

Proof. The proof is straightforward.

Theorem 8. *Let $\Im_1=\langle A_1,\lambda_1\rangle$ and $\Im_2=\langle A_2,\lambda_2\rangle$ be neutrosophic cubic sets. Then,*

i. $d(\Im_1,\ \Im_2)\geq 0$;
ii. $d(\Im_1,\ \Im_2)=d(\Im_2,\ \Im_1)$;
iii. $d(\Im_1,\ \Im_2)=0$ *iff* $\Im_1=\Im_2$;
iv. $d(\Im_1,\ \Im_2)+d(\Im_2,\ \Im_3)\geq d(\Im_1,\ \Im_3)$.

Proof. The proof is straightforward.

3. An application in pattern recognition problem

In this section we proposed an decisin making method based on similarity measures of two NCS in pattern recognition problems which is adapted from [25]. In this method, we assume that if similarity between the ideal pattern and sample pattern is greater than or equal to 0.5, then the sample pattern belongs to the family of ideal pattern in consideration. The algorithm of this method is as follows:

Algorithm:

Step 1. Construct an ideal NCS $\Im=\langle A,\lambda\rangle$ on X.

Step 2. Construct NCSs $\Im_j=\langle A_j,\lambda_j\rangle$, $j=1,2,3,\ \ldots,\ n$, on X for the sample patterns which are to recognized.

Step 3. Calculate the distances $d(\Im,\ \Im_j)$, $j=1,\ 2,\ 3,\ldots,\ n$.

Step 4. If $d(\Im,\ \Im_j)\leq 0.5$ then the pattern $\Im_j$ is to be recognized to belong to the ideal Pattern $\Im$ and if $d(\Im,\ \Im_j)>0.5$ then the pattern $\Im_j$ is to be recognized not to belong to the ideal Pattern $\Im$.

Now we give an example which is adapted from [25].

Example 6. Here a fictitious numerical example is given to illustrate the application of similarity measures between two NCSs in pattern recognition problem. In this example we take three sample patterns which are to be recognized. Let $X=\{x_1,x_2,x_3\}$ be the universe. Also let $\Im$ be NCS set of the ideal pattern and $\Im_j=\langle A_j,\lambda_j\rangle$, $j=1,2,3$ be the NCSs of three sample patterns.

Step 1. Construct an ideal NCS $\Im=\langle A,\lambda\rangle$ on X as;

$$\Im=\left\langle\begin{Bmatrix}\langle[0.2,0.4],[0.3,0.5],[0.3,0.5]\rangle/x_1+\\ \langle[0.5,0.7],[0.0,0.5],[0.2,0.3]\rangle/x_2+\\ \langle[0.6,0.8],[0.0,0.1],[0.3,0.4]\rangle/x_3\end{Bmatrix},\begin{Bmatrix}\langle 0.1,0.2,0.4\rangle/x_1,\langle 0.1,0.2,0.2\rangle/x_2,\\ \langle 0.3,0.1,0.7\rangle/x_3\end{Bmatrix}\right\rangle$$

Step 2. Construct NCSs $\Im_j=\langle A_j,\lambda_j\rangle$, $j=1,\ 2,\ 3$ on X for the sample patterns as;

$$\Im_1=\left\langle\begin{Bmatrix}\langle[0.2,0.5],[0.4,0.5],[0.3,0.5]\rangle/x_1+\\ \langle[0.7,0.7],[0.1,0.3],[0.1,0.3]\rangle/x_2+\\ \langle[0.6,0.8],[0.5,0.6],[0.3,0.4]\rangle/x_3\end{Bmatrix},\begin{Bmatrix}\langle 0.5,0.6,0.4\rangle/x_1,\langle 0.2,0.2,0.2\rangle/x_2,\\ \langle 0.3,0.1,0.7\rangle/x_3\end{Bmatrix}\right\rangle$$

$$\Im_2=\left\langle\begin{Bmatrix}\langle[0.3,0.7],[0.3,0.5],[0.3,0.9]\rangle/x_1+\\ \langle[0.6,0.7],[0.1,0.8],[0.2,0.3]\rangle/x_2+\\ \langle[0.6,0.9],[0.9,1.0],[0.3,0.4]\rangle/x_3\end{Bmatrix},\begin{Bmatrix}\langle 0.2,0.5,0.2\rangle/x_1,\langle 0.1,0.5,0.7\rangle/x_2,\\ \langle 0.7,0.1,0.7\rangle/x_3\end{Bmatrix}\right\rangle$$

$$\Im_3=\left\langle\begin{Bmatrix}\langle[0.8,0.9],[0.1,0.2],[0.8,0.9]\rangle/x_1+\\ \langle[0.1,0.2],[0.8,0.9],[0.3,0.9]\rangle/x_2+\\ \langle[0.5,0.9],[0.1,1.0],[0.4,0.7]\rangle/x_3\end{Bmatrix},\begin{Bmatrix}\langle 0.9,0.8,0.9\rangle/x_1,\langle 0.9,0.8,0.9\rangle/x_2,\\ \langle 0.9,0.9,0.1\rangle/x_3\end{Bmatrix}\right\rangle$$

Step 3. Calculate the distances $\mathrm{d}(\Im,\ \Im_j),\ j = 1,\ 2,\ 3$ as;

$$\mathrm{d}(\Im,\ \Im_j) = 0.10$$

$$\mathrm{d}(\Im,\ \Im_j) = 0.19$$

$$\mathrm{d}(\Im,\ \Im_j) = 0.51$$

Step 4. Since $\mathrm{d}(\Im,\ \Im_1) \leq 0.5, \mathrm{d}(\Im,\ \Im_2) \leq 0.5$ and $\mathrm{d}(\Im,\ \Im_3) > 0.5$, the sample patterns whose corresponding NCS sets are represented by $\Im_1$ and $\Im_2$ are recognized as similar patterns of the family of ideal pattern whose NCS set is represented by $\Im$ and the pattern whose NCS is represented by $\Im_3$ does not belong to the family of ideal pattern $\Im$.

4. Conclusion

Neutrosophic cubic set (NCS) is a combination of a neutrosophic set with interval neutrosophic set. It is basically the generalization of cubic set. In this paper, we introduced some new type of notions with their basic properties. In the future, we will apply the sets to algebraic structures such as; sub-algebras, ideals, BCK/BCI algebras, q-ideals, and so on.

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