Geodetic Games

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Abstract. We show that the geodetic game, introduced by Fraenkel and Harary, is decidable in polynomial time on AT-free graphs.

1 Introduction

Time is a game played beautifully by children. [5]

2 Geodetic games on graphs

De Bruijn gives the following example of a game for which it is easy to see that the first player has a winning strategy, however, nobody knows what it is. Write down the numbers 1 up to n. A move is the selection of a number. The effect of the move is that the selected number plus all its divisors are removed. Assume that the first player chooses the number 1. If the new position is winning for the second player, then there is a winning move. But this winning move could have been made by the first player instead.

According to Úlehla, the game of Hackendot was invented by Von Neumann. (Úlehla gave the game its name.) Von Neumann's intention was to show that, in a 2-player game, it can be easy to decide whether there is a winning strategy for the first player, whilst it can be hard to find an optimal strategy.

The Hackendot game is played on a rooted tree which is oriented outward, that is, away from the root. During the game points are removed from the tree,

[5] This is attributed to the weeping philosopher, Heraclitus.

leaving a forest as the playground for the next move. Each tree in the forest naturally inherits a root, namely the point without incoming arcs. A move consists of selecting a point in a tree of the forest. The effect of the move is that all points (including the endpoints) on the path from the selected point to the root of that tree are removed. Players move alternately. The first player who can't move (because there are no points left) loses the game.

Von Neumann shows, as follows, that the first player has a winning strategy. First of all, according to the Sprague-Grundy theory, one of the two players has a winning strategy (which is computable in exponential time). Assume, to the contrary, that the second player has a winning strategy. Then let the first player choose the root. Assume the second player plays a point x and assume that he wins the game by playing that move. Instead of playing the root in the first move, the first player could have played x instead, making him the winner of the game.

Back in 1980, Úlehla gave a polynomial-time algorithm to decide the Hackendot game. Deuber and Thomassé provided an easier algorithm in 1996. In 2013, Grier showed that the problem to decide the winner is PSPACE-complete for abitrary, finite posets.

Fraenkel and Harary define the 'geodetic game' on a tree as follows. Two players play the game on an undirected tree T. During the game the points of T get labeled. Initially, the set of labeled points is empty. A move is the selection of an unlabeled point. This labels the point plus all the points in T that are on geodesics between the selected point and previously labeled points.

Fraenkel and Harary reduce the geodetic game on trees to Hackendot as follows. Try all points in the tree as a first move. This first point simulates a root, and orients the tree, making it an input for Hackendot. Use the algorithm of Úlehla, to decide whether there is a winning strategy for the first player, when playing the root first. If the first player loses, in all those games of Hackendot, when he chooses the root, the second player wins the geodetic game.

Fraenkel and Harary analyze the geodetic game for cycles as follows.

Lemma 1. *If the cycle is even, the first player loses the game.*

Proof. The second player chooses the opposite vertex of the cycle and ends the game. □

Theorem 1. *The first player wins the game on a cycle C_n if and only if*

$$n = 2^k - 1 \quad \textit{for some } k \in \mathbb{N}.$$

To avoid ambiguity, we carefully define the geodetic game on an arbitrary graph as follows. The initial playground is a graph G. Player one moves first and he chooses some vertex r. This labels the chosen vertex r as a 'root.' In the remaining game, the playground is a (changing) graph with at most one root in each component. Players move alternately. A 'move' is the selection of an unlabeled vertex, say x. If x is in a component that already has a root r', then the move labels all vertices that are on geodesics from x to r', including x and r'. If x is in a component that has no root yet, then x becomes the unique root for that component. Any connected subgraph on labeled vertices is now contracted to one new vertex and this new vertex becomes the new root of the component that contains it. This ends the description of the new playground, which is now ready for the next move.

The player who can't move (because there are no more unlabeled vertices) loses the game and his opponent wins, that is, we consider what is called 'normal play.' (If the outcome were reversed, the game would be called 'misère play.')

3 The geodetic game on blockgraphs

Definition 1. *A graph is a blockgraph if every biconnected component is a clique.*

Alternatively, a blockgraph is characterized as a chordal graph without induced diamond, that is, blockgraphs are chordal and every two maximal cliques have at most one point in common. Blockgraphs form a subclass of the Ptolemaic graphs (the chordal, distance-hereditary graphs). The connected, claw-free blockgraphs are the linegraphs of trees.

To decide the geodetic game on blockgraphs it suffices to give an algorithm that computes the nim function for each component of the graph. Thus, in the following we assume that the graph is connected. The nim function for the geodetic game assigns a value to each position, ie, each feasible playground, of the game. The nim value of a position is defined as the smallest integer in $\mathbb{N} \cup \{0\}$ which is not attained by any position that can be reached via one move. Thus, if a position has no followers, then its nim value is zero; that is, the game is won for the player that made the last move.

The first move in a geodetic game on a blockgraph G is the selection of a point, which becomes the root, r, of G.

Lemma 2. *Let G be a rooted, connected blockgraph. Let x be a vertex which is not equal to the root r. There is a unique geodetic from x to r.*

Proof. A blockgraph is Ptolemaic. Thus all chordless paths between two nonadjacent vertices have the same length. Since all minimal separators are articulation points, and all biconnected components are cliques, each geodetic is unique. $\square$

Lemma 3. *Let* G *be a rooted, connected blockgraph. The subgraph* H *containing all edges that are in geodesics from vertices in* G *to the root, is a tree.*

Proof. By Lemma 2, H is the BFS-tree rooted at r. $\square$

Lemma 4. *Blockgraphs are closed under edge contractions.*

Proof. Let G be a blockgraph and let $e = \{x, y\} \in E(G)$. Let G' be the graph obtained from G by contracting the edge e to one new point xy. First of all, G' is chordal, since the class of chordal graphs is closed under edge contractions (see, eg, the textbook of Kloks and Wang). Any two maximal cliques in G share at most one vertex. Contracting an edge maintains this property. $\square$

Lemma 5. *Let* G *be a connected blockgraph with a root* r*. Let* G' *be the graph obtained from* G *by contracting the geodetic* P *from a vertex* x *to* r *to a new root* r'*. Let* $\{a, b\} \in E(G)$ *such that* b *lies on the geodetic from* a *to* r*.*

(1) *If* $\{a, b\} \in E(P)$ *then* a *and* b *are identified with* r' *in* G'*.*
(2) *If* $a \notin V(P)$ *and* $b \in V(P)$ *then* a *is connected to* r' *in* G'*.*
(3) *If* $\{a, b\} \cap V(P) = \varnothing$*, then* $\{a, b\} \in E(G')$ *and* b *lies on the geodetic from* a *to* r' *in* G'*.*

Proof. The only claim which is not obviously true is, perhaps, the last one. Consider the BFS-tree H rooted at r. Contracting the geodesic P in H to r' yields a BFS-tree for G'. $\square$

Theorem 2. *There exists a polynomial-time algorithm to decide the geodetic game for blockgraphs.*

Proof. Consider choosing a root r as a first move in a connected blockgraph. This defines an oriented BFS-tree. The rooted tree can be considered as the input of the Hackendot game, and, by the previous observations, this Hackendot game is equivalent to the geodetic game on the blockgraph. Úlehla's algorithm computes the nim value and decides if choosing the root is a winning move. $\square$

4 The geodetic game on cographs

Definition 2. *A graph is a cograph if it has no induced* P_4*.*

There are many characterizations of cographs. For example, a graph is a cograph if and only if every induced subgraph with at least two vertices has a twin. We refer to the textbook of Kloks and Wang for some other characterizations and for a description of the tree decomposition of cographs.

In this section we analyze the geodetic game on cographs.

Lemma 6. *Let* G *be a connected cograph. Let* a *and* b *be two vertices of* G*. Let* G' *be the graph obtained from* G *by contracting the subgraph induced by the vertices that are on* a,b*-geodesics to one vertex. Then* G' *is a cograph. Let* r *denote the vertex in* G' *that replaces* $V(G) \setminus V(G')$*. Then* r *is a universal vertex of* G'*.*

Proof. We may assume that G is the join of two smaller cographs G_1 and G_2, that is,

$$G = G_1 \otimes G_2.$$

Assume that a and b are adjacent and assume $a \in V(G_1)$ and $b \in V(G_2)$. Then a is adjacent to all vertices of G_2 and b is adjacent to all vertices in G_1. Thus, contracting the edge $\{a, b\}$ yields a graph in which the vertex r, that replaces $\{a, b\}$, is universal. The graph G' is a join between the universal vertex $\{r\}$ and the cograph $G - \{a, b\}$ and so G' is a cograph.

Assume that a and b are adjacent and that $\{a, b\} \subseteq V(G_1)$. Then, by induction, contracting the edge $\{a, b\}$ in G_1 yields a graph G_1' in which the vertex r is universal. By induction also, G_1' is a cograph. Since all vertices of G_2 are adjacent to all vertices of G_1, r is universal in

$$G' = G_1' \otimes G_2.$$

Assume that a and b are not adjacent and that a and b are both in G_1. Since $V(G_2) \neq \varnothing$, all a, b-geodesics have length two, and $V(G_2)$ is in the common neighborhood of a and b. The new vertex r replaces

$$\{\ a,\ b\ \} \cup V(G_2) \cup (\ N_{G_1}(a) \cap N_{G_1}(b)\ ).$$

Since every vertex in G_2 is adjacent to every vertex in G_1, the new vertex r is universal in G'.

The proves the lemma. □

Lemma 7. *Let* G *be a connected cograph with at least two vertices. Assume player* A *starts the geodetic game and plays vertex* a*. Next, player* B *plays vertex* b*. Let* G' *be the graph with a labeled vertex* r*, which is the contraction of the set of vertices on* a, b*-geodesics. Then, in the remaining game, each move reduces the number of vertices of* G' *by one, that is, player* A *wins the game if* G' *has an odd number of vertices (excluding* r*) and, otherwise, player* B *wins the game.*

Proof. By Lemma 6, after the first move of player B, the a, b-geodesics are contracted to a universal vertex r. Any subsequent move, selects an unlabeled vertex which is adjacent to r, so it effectively removes that vertex from the graph. □

Theorem 3. *There exists a polynomial-time algorithm to decide the geodetic game on cographs.*

Proof. By Lemma 7, there is a polynomial-time algorithm that computes the nim-function for a connected cograph. According to the Sprague-Grundy theorem, for cographs in general, the nim-function is the nim-sum of its components. □

5 The geodetic game on interval graphs

A graph is an interval graph if it is the intersection graph of a collection of intervals on the real line. Interval graphs can be recognized in linear time. They form a subclass of the chordal graphs.

Theorem 4. *A graph is an interval graph if and only if it has a consecutive clique arrangement, that is, a linear arrangement of the maximal cliques such that, for each vertex, the maximal cliques that contain it occur consecutively in the sequence.*

Lemma 8. *The class of interval graphs is closed under contracting edges.*

Proof. Let G be an interval graph and let $\{x, y\} \in E(G)$. Let G' be the graph obtained from G by contracting the edge $\{x, y\}$. Consider an interval model for G and let I_x and I_y be the intervals that represent x and y. Consider the interval model obtained by replacing the intervals I_x and I_y by one new interval which is the union $I_x \cup I_y$. This new collection of intervals represents G'. □

Lemma 9. *Let* G *be an interval graph and let*

$$[\ C_1,\ C_2,\ \dots,\ C_t\ ] \tag{1}$$

be a consecutive clique arrangement for G*. Let* $x \in V(G)$ *and assume that* x *is in cliques with indices in the closed interval* $[i, j]$*, where* $1 \leqslant i \leqslant j \leqslant t$*. Let* $z \in N(x)$*. Then* z *is on an* x, y*-geodesic for some* $y \notin N[x]$ *if and only if*

$$z \in C_i \cap C_{i-1} \quad \textit{or} \quad z \in C_j \cap C_{j+1},$$

or, equivalently, $N(z) \setminus N(x) \neq \varnothing$*.*

Proof. Consider the components of $G - N[x]$. Let W be one of those components and let $y \in W$. Then $N(W)$ is the unique minimal x, y-separator contained in $N(x)$. Furthermore, since G is chordal, W contains a vertex p that is adjacent to all vertices in $N(W)$. This implies that $N(W)$ is the set of common neighbors of p and x, that is, $N(W)$ is exactly the set of vertices that are on p, x-geodesics.

Suppose a vertex $q \in N(x) \setminus N(W)$ were in a geodesic P from x to some vertex in W. Then P passes through $N(W)$, which implies that P has a shortcut. This is a contradiction. So, the only vertices that are possibly on geodesics from x to some vertex in W are in $N(W)$.

When G is an interval graph, with a consecutive clique arrangement as in (1), then the minimal separators of G are the intersections of consecutive maximal cliques;

$$C_a \cap C_{a+1} \quad \text{for } a \in \{1, \dots, t-1\}.$$

This proves the lemma. □

Let G be an interval graph and let $r \in V(G)$. We denote the graph G rooted at the vertex r by G_r. A geodetic game played on G_r takes as a playground the interval graph G with one labeled vertex r.

Lemma 10. *Let G_r be a rooted interval graph and assume that the root r is a universal vertex. Then the geodetic game played on G_r wins for the first player to move, if and only if $|V(G) \setminus \{r\}|$ is odd.*

Proof. Each move consists of the selection of an unlabeled vertex, say x. Since r is universal, the effect of the move is that the edge $\{x, r\}$ is contracted to one point, that is, the new playground is $G_r - x$. □

Theorem 5. *There exists a polynomial-time algorithm to decide the geodetic game on interval graphs.*

Proof. First assume that the graph is connected. Consider an interval model for G. Let a vertex x be represented by the interval I_x. An instance of the game is represented by a pair (I, k) where I is the interval of the root r, and where k is the number of unlabeled vertices whose interval is contained in that of the root.

The Grundy value of an instance (I, k) is recursively defined as follows. If

$$I = \bigcup_{x \in V(G)} I_x$$

then the root is universal, and the grundy value of (I, k) is

$$f(I, k) = k \bmod 2.$$

Consider an arbitrary instance (I, k) and let y be a vertex for which I_y is not contained in I. Let $\gamma(r, y)$ be the set of vertices that are on geodesics between r and y. Then the new root, J, is defined as

$$J = \bigcup_{z \in \gamma(r,y)} I_z. \tag{2}$$

Let $(J, \ell)_{r,y,k}$ denote the instance of the game obtained by contracting the graph induced by vertices on geodesics between r and y, that is, J is defined as in Equation (2) and

$$\ell = k + |\{ z \mid N(z) \subseteq N[J] \setminus (N(r) \cup \gamma(r, y)) \}|, \tag{3}$$

where we abuse notation by writing $N[J]$ for the closed neighborhood of the new root, represented by the interval J.

The nim-function value for the position (I, k) in the game is the smallest integer not attained by the successors of (I, k), that is,

$$\begin{aligned} f(I,k) = \min \{ \; q \in \mathbb{N} \cup \{0\} \;\mid\; (\; k > 0 \;\; \Rightarrow \;\; f(I, k-1) \neq q \;) \quad \text{and} \\ \forall_y \;\; (\; N(y) \setminus N[r] \neq \varnothing \;\; \Rightarrow \;\; f((J,\ell)_{(r,y,k)}) \neq q \;) \; \}, \end{aligned} \tag{4}$$

where the pair (J, ℓ) is obtained, as described above, from (I, k) and y. (See, eg, [15] for an excellent, brief guide into the Sprague-Grundy theorem.)

Player one has a winning strategy if he can start with a vertex x such that

$$f(I_x, k) \neq 0, \quad \text{where } k = |\, \{z \mid N[z] \subseteq N[x] \,\} \,|.$$

When the graph has more than one component, the Grundy value is obtained by taking the nim sum over the components. □

6 The geodetic game on cocomparability graphs

A graph is a comparability graph if it has a transitive orientation. The complement of a comparability graph is called cocomparability. Cocomparability graphs were characterized by Golumbic et al., as follows.

Theorem 6. *A graph is a cocomparability graphs if there is an intersection model in which each vertex x is represented by a continuous function $f_x : [0, 1] \to \mathbb{R}$.*

Lemma 11. *The class of cocomparability graphs is closed under contractions.*

Proof. Let G be a cocomparability graph and let $\{x, y\} \in E(G)$. Consider an intersection model for G in which a vertex a is represented by a continuous function $f_a : [0, 1] \to \mathbb{R}$.

Replace the two functions f_x and f_y by a new continuous function which rapidly zigzags between f_x and f_y. This yields an intersection model for the graph G' obtained from G by the contraction of $\{x, y\}$. □

Theorem 7. *There exists a polynomial-time algorithm to decide the geodetic game on cocomparability graphs.*

Proof. The proof is similar to the proof of Theorem 5.

Assume that the cocomparability graph G is connected. For a vertex x let $\delta(x)$ denote the set of vertices whose neighborhood is contained in $N[x]$, that is,

$$\delta(x) = \{ \; y \mid y \neq x \quad \text{and} \quad N[y] \subseteq N[x] \; \}.$$

Consider a game instance (r, k), where r is the current root and k is the number of vertices in $\delta(r)$ that are not yet labeled.

The nim-function value for the game instance (r, k) is the smallest integer in $\mathbb{N} \cup \{0\}$ which does not appear as a nim-value at one of the successors of (r, k).

If $k > 0$, then one of the successors is $(r, k-1)$, that is, the player picks one unlabeled vertex of $\delta(r)$. By definition of $\delta(x)$, the sole effect of this move is that the chosen vertex gets labeled. So, the encoding of the successor is indeed $(r, k-1)$.

Consider a move that selects a vertex $y \notin \delta(r)$. Let $\gamma(r, y)$ be the set of vertices that are on r, y-geodesics. The move changes the playground; it contracts the vertices in $\gamma(r, y)$ to a single vertex, which is the new root r'. The function $\rho : [0, 1] \to \mathbb{R}$, to represent r', is a function that rapidly zigzags between the functions r and y. The closed neighborhood of r' is

$$N[r'] = \{r'\} \cup \left(\bigcup_{z \in \gamma(r,y)} N(z) \ \setminus \gamma(r, y) \right).$$

The new instance is represented as $(r', |\delta(r')|)$. Notice that

$$|\delta(r')| = k + \ |\{ z \mid N(z) \setminus N[r] \neq \varnothing \quad \text{and} \quad z \neq r' \\ \text{and} \quad z \notin \gamma(r, y) \quad \text{and} \quad N[z] \subseteq N[r']\}| . \tag{5}$$

The first player has a winning move if there exists a vertex $x \in V(G)$ such that

$$f((x, |\delta(x)|) \neq 0.$$

A winning move is then a successor of $(x, |\delta(x)|)$ with nim-value zero. In case the graph has more than one component, the nim-value is the nim-sum of the nim-values of the components. □

7 The geodetic game on AT-free graphs

Recall that an asteroidal triple is an independent set of 3 vertices having a path between every pair that avoids the closed neighborhood of the third. AT's were introduced by Lekkerkerker and Boland to characterize interval graphs. A graph is AT-free if it has no asteroidal triple. The class generalizes the class of cocomparability graphs. Although the graphs still expose a linear structure (via a dominating path), they are, in general, no longer perfect; for example C_5 is AT-free. Also, as far as we know, unlike famous subfamiliae, such as interval graphs, permutation graphs, trapezoid graphs and cocomparability graphs, the class lacks a nice intersection model.

Lemma 12. *The class of AT-free graphs is closed under edge contractions.*

Proof. Let G be AT-free and let H be obtained from G by contraction an edge $\{a, b\} \in E(G)$. Assume that H has an asteroidal triple $\{p, q, r\}$.

First assume that none of p, q or r represents the contracted edge $\{a, b\}$ in H. Fix paths P, Q and R, between pairs that avoid the closed neighborhood of p, q and r, respectively. If none of the paths contains the contracted edge, then the triple is an AT in G. Suppose P contains the contracted edge. Then there is a path P' in G, obtained by replacing the contracted edge by, either one endpoint, or the edge $\{a, b\}$, connecting q and r, and avoiding $N_G[p]$. So, $\{p, q, r\}$ is an asteroidal triple in G.

Finally, assume that p is the contracted edge $\{a, b\}$ in H. Then $\{a, q, r\}$ is an asteroidal triple in G. To see that, first observe that the q, r-path P avoiding p in H is a q, r-path avoiding $N[a]$ in G. The p, q-path avoiding $N[r]$ may be replaced by an a, q-path (avoiding $N[r]$) in G.

This proves the lemma. □

Remark 1. Lemma 12 remains true for the larger class of 'hereditary dominating pair graphs' introduced by Pržulj et al. The class is defined by the property that every connected induced subgraph has a pair of vertices such that every path that connects the pair is a dominating set in the graph. This class properly contains AT-free graphs; note that C_6 is a hereditary dominating pair graph which is not AT-free. It remains an open problem whether the geodetic game on hereditary dominating pair graphs is decidable in polynomial time. The class of graphs has not been widely studied; one major obstruction is that, as far as we know, the recognition of hereditary dominating pair graphs is still open.

Definition 3. *Let G be AT-free and let x and y be two nonadjacent vertices in G. A vertex z is between x and y if there exists an x, z-path that avoids $N[y]$ and an y, z-path that avoids $N[x]$.*

For two vertices, we denote by $B(x, y)$ the 'between-set' of vertices, ie, the set of vertices that are between x and y. For adjacent vertices x and y, we define $B(x, y) = \varnothing$. When x and y are not adjacent we write

$$\bar{B}(x, y) = B(x, y) \cup N[x] \cup N[y],$$

and we call this the closure of the between-set $B(x, y)$.

Lemma 13. *Let x and y be nonadjacent and let*

$$u \in N(x) \cap N(y).$$

Then u is adjacent to all vertices in $B(x, y)$.

Proof. Consider a vertex $z \in B(x, y)$ and $z \notin N(u)$. By definition of the betweenness, there exist paths connecting z with x and y that avoid $N[y]$ and $N[x]$, repectively. The path $[x, u, y]$ runs between x and y and avoids $N[z]$. This implies that $\{x, y, z\}$ is an asteroidal triple, which is a contradiction. □

Lemma 14. *Consider an instance in a geodetic game, played on an* AT*-free graph* G*. Let* x *and* y *be two labeled vertices in* G *such that* $N(x) \cap N(y) = \varnothing$*. An unlabeled vertex* $z \in B(x, y)$ *has a labeled neighbor.*

Proof. Consider an x, y-path P consisting of labeled vertices. We claim that z has a neighbor in P. Assume not. Since $z \in B(x, y)$ there are paths from z to x and y that avoid $N[y]$ and $N[x]$, respectively. If $N[z] \cap P = \varnothing$, then P is a path avoiding $N[z]$, which implies that $\{x, y, z\}$ is an asteroidal triple. This contradicts the assumption that G is AT-free. □

The next result follows from the previous two lemmas.

Corollary 1. *Assume* x *and* y *are nonadjacent and assume they are labeled in some instance of the geodetic game. Then every vertex of* $\bar{B}(x, y)$ *has a labeled neighbor.*

For two nonadjacent vertices x and y we write $C^x(y)$ for the component of $G - N[x]$ that contains the vertex y.

Definition 4. *A pair of nonadjacent vertices* x *and* y *is extreme if* $|C^x(y)|$ *and* $|C^y(x)|$ *are maximal. To refine this; a nonadjacent pair* $\{x, y\}$ *is extreme if*

$$\forall_{x' \notin N[y]} \; |C^{x'}(y)| \leqslant |C^x(y)| \quad \textit{and} \quad \forall_{y' \notin N[x]} \; |C^{y'}(x)| \leqslant |C^y(x)|.$$

Lemma 15. *Let* G *be a connected* AT*-free graph. Consider the situation where two labeled vertices* x *and* y *form an extreme pair. Let* k *be the number of unlabeled vertices. Then the game is won for the first player to move if and only if* k *is odd.*

Proof. Let

$$\Delta = N(C^x(y)) \quad \text{and} \quad A = V(G) \setminus (\Delta \cup C^x(y)).$$

Then $x \in A$. We claim that all vertices of A are adjacent to all vertices of Δ. Assume not, and let $a \in A$ and $\delta \in \Delta$ be nonadjacent. Then

$$C^x(y) \cup \{\delta\} \subseteq C^a(y)$$

which contradicts the assumption that the pair $\{x, y\}$ is extreme.

We claim that every unlabeled vertex has a labeled neighbor. This follows from Corollary 1 and the observation above. This proves the lemma. □

Theorem 8. *There exists a polynomial-time algorithm to decide the geodetic game on* AT*-free graphs.*

Proof. By Corollary 1 we can encode the status of an interval with labeled endpoints by the number of unlabeled vertices contained in it. The remainder of the proof is analogous to the proof of Theorem 5 on page 7. □

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