Hypergeometric ${}_3F_2$, Polylogarithm Li_2 , Number Pi

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Abstract

In this note we show some formulas related with: Hypergeometric function ${}_3F_2(\{2,2,2\},\{3,3\},z)$, Polylogarithm function $\mathrm{Li}_2(z)$, and Number Pi : π=3.14159...

Keyword:Hypergeometric Function,Polylogarithm Function,Number Pi

I. Introduction

Definitions

$$ {}_3F_2(\{2,2,2\},\{3,3\},z)=\sum_{n=0}^{\infty}\frac{(2)_n^3\,z^n}{(3)_n^2\,n!}\ ,\ \ |z|<1 \tag{1}$$

$$ \mathrm{Li}_2(z)=\sum_{n=1}^{\infty}\frac{z^n}{n^2}\ ,\ \ |z|\le 1 \tag{2}$$

$$ \pi=4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}=3.14159\ldots \tag{3}$$

II. Formulas

$$ \pi=3\,i\,z^2\,{}_3F_2(\{2,2,2\},\{3,3\},z)+12\,i\,\mathrm{Li}_2(z)\ ,\ \ z=1-\frac{\sqrt{2+\sqrt{3}}}{2}-i\,\frac{\sqrt{2-\sqrt{3}}}{2} \tag{4}$$

$$ \pi=\frac{3}{2}\,i\,z^2\,{}_3F_2(\{2,2,2\},\{3,3\},z)+6\,i\,\mathrm{Li}_2(z)\ ,\ \ z=1-\frac{\sqrt{3}}{2}-i\,\frac{1}{2} \tag{5}$$

$$ \pi=2\,i\,z^2\,{}_3F_2(\{2,2,2\},\{3,3\},z)+8\,i\,\mathrm{Li}_2(z)\ ,\ \ z=1-\frac{\sqrt{2+\sqrt{2}}}{2}-i\,\frac{\sqrt{2-\sqrt{2}}}{2} \tag{6}$$

Definition : $G(z)=i\,z^2\,{}_3F_2(\{2,2,2\},\{3,3\},z)+4\,i\,\mathrm{Li}_2(z)$

$$\pi = G\left(1 - \frac{2}{\sqrt{5}} - \frac{i}{\sqrt{5}}\right) + G\left(1 - \frac{3}{\sqrt{10}} - \frac{i}{\sqrt{10}}\right) \tag{7}$$

$$\pi = 4\,G\left(1 - \frac{5}{\sqrt{26}} - \frac{i}{\sqrt{26}}\right) - G\left(1 - \frac{239}{169\sqrt{2}} - \frac{i}{169\sqrt{2}}\right) \tag{8}$$

$$\pi = 2\,G\left(1 - \frac{3}{\sqrt{10}} - \frac{i}{\sqrt{10}}\right) + G\left(1 - \frac{7}{5\sqrt{2}} - \frac{i}{5\sqrt{2}}\right) \tag{9}$$

$$\pi = 5\,G\left(1 - \frac{7}{5\sqrt{2}} - \frac{i}{5\sqrt{2}}\right) + 2\,G\left(1 - \frac{79}{25\sqrt{10}} - \frac{3\,i}{25\sqrt{10}}\right) \tag{10}$$

$$\pi = G\left(1 - \frac{2}{\sqrt{5}} - \frac{i}{\sqrt{5}}\right) + G\left(1 - \frac{5}{\sqrt{26}} - \frac{i}{\sqrt{26}}\right) + G\left(1 - \frac{8}{\sqrt{65}} - \frac{i}{\sqrt{65}}\right) \tag{11}$$

$$\pi = 4\,G\left(1 - \frac{5}{\sqrt{26}} - \frac{i}{\sqrt{26}}\right) - G\left(1 - \frac{70}{13\sqrt{29}} - \frac{i}{13\sqrt{29}}\right) + G\left(1 - \frac{99}{13\sqrt{58}} - \frac{i}{13\sqrt{58}}\right) \tag{12}$$

$$\pi = 12\,G\left(1 - \frac{18}{5\sqrt{13}} - \frac{i}{5\sqrt{13}}\right) + 8\,G\left(1 - \frac{57}{5\sqrt{130}} - \frac{i}{5\sqrt{130}}\right) - 5\,G\left(1 - \frac{239}{169\sqrt{2}} - \frac{i}{169\sqrt{2}}\right) \tag{13}$$

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