3×3-Kronecker-Pauli Matrices

Christian Rakotonirina*
Civil Engineering Department,
Institut Supérieur de Technologie d'Antananarivo, IST-T, Madagascar
Simon Rasamizafy†
Department of Physics
University of Toliara, Madagascar

March 29, 2016

Abstract

The properties of what we call inverse-symmetric-matrices have helped us for constructing a basis of $\mathbb{C}^{3\times 3}$ which satisfy four properties of the Kronecker generalized Pauli matrices. In using some properties of the Kronecker commutation matrices, bases of $\mathbb{C}^{5\times 5}$ and $\mathbb{C}^{6\times 6}$ which share the same properties has also constructed. The Pauli groups of these bases have been defined.

Keywords: Kronecker product, Pauli matrices, Kronecker commutation matrices, Kronecker generalized Pauli matrices.

1 Introduction

In a few words this paper tries to solve a problem evoked in [1], of searching nine 3×3-matrices satisfied a relation with the $3 \otimes 3$ Kronecker commutation matrix.
The usefulness of the Kronecker permutation matrices, particularly the Kronecker commutation matrices (KCMs) in mathematical physics can be seen in [2], [3], [4], [5]. In these papers the $2 \otimes 2$-Kronecker commutation matrix is written in terms of the Pauli matrices, which are 2×2 matrices, by the following way

$$K_{2\otimes 2} = \frac{1}{2}\sum_{i=0}^{3} \sigma_i \otimes \sigma_i \qquad (1)$$

The generalization of this formula in terms of generalized Gell-Mann matrices, which are a generalization of the Pauli matrices, is the topic of [6]. But there

*rakotonirinachristianpierre@gmail.com
†rasamizafy_simon@yahoo.fr

are other generalization of the Pauli matrices in other sense than the generalized Gell-Mann matrices, among others the Kibler matrices [7], the Kronecker generalization of the Pauli matrices, see for example [8]. The more generalized relation giving the $2^k \otimes 2^k$-Kronecker commutation matrix in terms of this last can be seen in [1]. That makes us still to search for the generalization of the Pauli matrices in this sense, like the generalization of the Gell-Mann matrices to the rectangle Gell-Mann matrices in [9] where the $n \otimes n$-KCM $K_{n\otimes n}$ is expressed in terms of the $n \times n$-Gell-Mann matrices, for generalizing it to the expression of $n \otimes p$-KCM we have introduced the $n \times p$-Gell-Mann matrices. That is we search for 3×3 matrices which have got some properties of the Kronecker generalization of the Pauli matrices, which are $2^k \times 2^k$-matrices. We will call these matrices 3×3-Kronecker Pauli matrices (KPMs).
These properties of $n \times n$-KPM $(\Sigma_i)_{0\leq i\leq (2^k)^2-1}$, with $n = 2^k$ are

- i) $(\Sigma_i)_{0\leq i\leq n^2-1}$ is a basis of $\mathbb{C}^{n\times n}$;
- ii)
$$K_{n\otimes n} = \frac{1}{n}\sum_{i=1}^{n^2} \Sigma_i \otimes \Sigma_i \tag{2}$$
- iii)
$$\Sigma_j^{+} = \Sigma_j \text{ (hermiticity)} \tag{3}$$
- iv)
$$\Sigma_j^2 = I_n \text{ (Square root of unity)} \tag{4}$$
- v)
$$Tr(\Sigma_j^{+}\Sigma_k) = n\delta_{jk} \text{ (Orthogonality)} \tag{5}$$
- vi)
$$Tr(\Sigma_j) = 0 \text{ (Tracelessness)} \tag{6}$$

However, there is no 3×3 matrix, formed by zeros in the diagonal which satisfy both the relations (3) and (4) [1]. Thus, at a first time for the 3×3-KPMs we do not demand tracelessness. We would like to take this opportunity to point out that the last sentence in [1] is wrong and beg the reader to not regard it.
For the $2^k \otimes 2^k$-Kronecker matrices or Kronecker generalized Pauli matrices, we give this calling by the fact that they are obtained by Kronecker product of the Pauli Matrices. That is why we will call 3×3-Kronecker Pauli matrices (KPMs) the set of 3×3-matrices which satisfy the five properties above, tracelessness moved apart.
Thus, in this paper we will talk at first about Kronecker commutation matrices, in the next section we will talk about what we call *inverse-symmetric matrices*. These matrices have got interesting properties for constructing the 3×3-KPMs. After, we will give the set of 3×3-KPMs, which are inverse-symmetric matrices, and some of their properties. Finally, some way to the generalization will be discussed.

We know that the set of Kronecker generalized Pauli matrices is a group for the usual matrix product. So, we will try to define the Pauli group of the 3×3-KPMs.
Some calculations such as the expression of 3×3-KCMs request calculations with software. We have used SCILAB for those calculations.

2 Kronecker Commutation matrices

The Kronecker product of matrices is not commutative, but there is a permutation matrix which, in multiplying to the product, commutes the product. We call such matrix *Kronecker commutation matrix.*

Definition 1 *The permutation matrix $K_{n\otimes p}$ such that for any matrices $a \in \mathbb{C}^{n\times 1}$, $b \in \mathbb{C}^{p\times 1}$*

$$K_{n\otimes p}(a\otimes b) = b\otimes a$$

is called $n\otimes p$ Kronecker commutation matrix.

$$K_{2\otimes 2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad K_{3\otimes 3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Proposition 2 *Suppose $K_{n\otimes m} = \sum_{i,j=1}^{s} A_i \otimes B_j$ and $K_{p\otimes q} = \sum_{k,l=1}^{r} C_k \otimes D_l$. Then, $K_{np\otimes mq} = \sum_{i,j=1}^{s}\sum_{k,l=1}^{r} A_i \otimes C_k \otimes B_j \otimes D_l$.*

Proof. Let (a_α), (c_β), (b_γ) and (d_δ) be, respectively, bases of $\mathbb{C}^{n\times 1}$, $\mathbb{C}^{p\times 1}$, $\mathbb{C}^{m\times 1}$ and $\mathbb{C}^{q\times 1}$. Then, $(a_\alpha \otimes c_\beta \otimes b_\gamma \otimes d_\delta)$ is a basis of $\mathbb{C}^{npmq\times 1}$ It is enough to prove that $\sum_{i,j=1}^{s}\sum_{k,l=1}^{r} A_i \otimes C_k \otimes B_j \otimes D_l(a_\alpha \otimes c_\beta \otimes b_\gamma \otimes d_\delta) = b_\gamma \otimes d_\delta \otimes a_\alpha \otimes c_\beta$. We use the proposition 10. $\sum_{i,j=1}^{s} A_i \otimes B_j(a_\alpha \otimes b_\gamma) = b_\gamma \otimes a_\alpha$. Thus, $\sum_{k,l=1}^{r}\sum_{i,j=1}^{s} A_i a_\alpha \otimes C_k c_\beta \otimes B_j b_\gamma \otimes D_l d_\delta = \sum_{k,l=1}^{r} b_\gamma \otimes C_k c_\beta \otimes a_\alpha \otimes D_l d_\delta = b_\gamma \otimes \sum_{k,l=1}^{r} C_k c_\beta \otimes a_\alpha \otimes D_l d_\delta$. However, $\sum_{k,l=1}^{r} C_k c_\beta \otimes a_\alpha \otimes D_l d_\delta = d_\delta \otimes a_\alpha \otimes c_\beta$. Hence $\sum_{i,j=1}^{s}\sum_{k,l=1}^{r} A_i \otimes C_k \otimes B_j \otimes D_l(a_\alpha \otimes c_\beta \otimes b_\gamma \otimes d_\delta) = b_\gamma \otimes d_\delta \otimes a_\alpha \otimes c_\beta$

■

3 Inverse-symmetric matrices

In this section we introduce what we call inverse-symmetric matrices. We think that this term will be useful for the continuation.

Definition 3 *Let us call inverse-symmetric matrix an invertible complex matrix* $A = (A^i_j)$ *such that* $A^j_i = \frac{1}{A^i_j}$ *if* $A^i_j \neq 0$.

That is, in difference with an antisymmetric matrix, for the non-zero element of the matrix its symmetric with respect to the diagonal is its inverse.

Example 4 *The* 2×2 *unit matrix* $I_2 = \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *and the Pauli matrices* $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ *are inverse-symmetric matrices.*

Proposition 5 *Let* $A = (A^i_j)$ *and* $B = (B^i_j)$ *be inverse-symmetric matrices. Then* $A \otimes B$ *is an inverse-symmetric matrix.*

Proof. $A \otimes B$ is an invertible matrix. If $A^i_j \neq 0$ and $B^k_l \neq 0$, $(A \otimes B)^{ik}_{jl} = A^i_j B^k_l \neq 0$ is a non zero element of $A \otimes B$, its symmetric with respect to the diagonal $(A \otimes B)^{jl}_{ik} = A^j_i B^l_k = \frac{1}{A^i_j} \frac{1}{B^k_l} = \frac{1}{(A \otimes B)^{ik}_{jl}}$. ■

Proposition 6 *For any* $n \otimes n$*-inverse-symmetric matrix* A *with only* n *non zero elements,* $A^2 = I_n$.

Proof. Let $A = (A^i_j)_{1 \leq i,j \leq n}$ and then $A^2 = (\sum_{k=1}^{n} A^i_k A^k_j)_{1 \leq i,j \leq n}$. In order that A is invertible there must be only a non zero element in each row and in each column. Let A^i_m be the non zero element in the row i and A^p_j the non zero element in the column j, $(A^2)^i_j = \sum_{k=1}^{n} A^i_k A^k_j = A^i_m A^m_j + A^p_j A^i_p$.

If $i \neq j$, $A^m_i = \frac{1}{A^i_m} \neq 0$ thus $A^m_j = 0$. $A^p_j \neq 0$ thus $A^i_p = 0$. Hence, $(A^2)^i_j = 0$ for $i \neq j$

If $i = j$, $(A^2)^i_j = \sum_{k=1}^{n} A^i_k A^k_i = A^i_m A^m_i = 1$. ■

Therefore, let us take some inverse-symmetric matrices formed by only three non zero elements for the nine 3×3 matrices we would like to search for, in order that (4) is satisfied.

4 Kronecker generalization of the Pauli matrices

The Kronecker generalization of the Pauli matrices are the matrices $(\sigma_i \otimes \sigma_j)_{0 \leq i,j \leq 3}$ [10], [11] $(\sigma_i \otimes \sigma_j \otimes \sigma_k)_{0 \leq i,j,k \leq 3}$, $(\sigma_{i_1} \otimes \sigma_{i_2} \otimes \ldots \otimes \sigma_{i_n})_{0 \leq i_1, i_2, \ldots, i_n \leq 3}$ [8] obtained by Kronecker product of the Pauli matrices and the 2×2 unit matrix. According to the propositions above, They are inverse-symmetric matrices and share many of the properties of the Pauli matrices: basis of $\mathbb{C}^{n \times n}$, (2), (3), (4), (5) and (6), for $n = 2^k$ [8]. Denote the set of $(\sigma_{i_1} \otimes \sigma_{i_2} \otimes \ldots \otimes \sigma_{i_n})_{0 \leq i_1, i_2, \ldots, i_n \leq 3}$ by $\mathcal{K}_n$.

$$\mathcal{K}_n = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}^{\otimes n} = \{\sigma_{i_1} \otimes \sigma_{i_2} \otimes \ldots \otimes \sigma_{i_n} \setminus 0 \leq i_1, i_2, \ldots, i_n \leq 3\} \quad (7)$$

The set

$$\mathcal{G}_n = \mathcal{K}_n \otimes \{-1, +1, -i, +i\} \quad (8)$$

is a group called the *Pauli group* of $(\sigma_{i_1} \otimes \sigma_{i_2} \otimes \ldots \otimes \sigma_{i_n})_{0 \leq i_1, i_2, \ldots, i_n \leq 3}$ [8].

We can check easily, in using the relation $\sigma_j \sigma_k = \delta_{jk} \sigma_0 + i \sum_{l=1}^{3} \epsilon_{jkl} \sigma_l$, for $j,\ k \in \{1,\ 2,\ 3\}$ that $\mathcal{G}_n$ is equal to the set of the products of two elements of $\mathcal{K}_n$, up to multiplicative phases, which are elements of $\{-1, +1, -i, +i\}$

$$\mathcal{G}_n = \mathcal{K}_n^2 \otimes \{-1, +1, -i, +i\} = \mathcal{K}_n \mathcal{K}_n \otimes \{-1, +1, -i, +i\} \quad (9)$$

where δ_{jk} is the Kronecker symbol, ϵ_{jkl} is totally antisymmetric, $\epsilon_{123} = +1$ $\epsilon_{213} = -1$.
It is normal to think that there should be nine 3×3 matrices which share many of the properties of the Kronecker generalization of the Pauli matrices, which are $2^k \times 2^k$ matrices.

5 3×3-Kronecker-Pauli Matrices

Now, we are going to construct the nine 3×3-matrices which satisfy the six properties cited in the introduction, tracelessness moved apart. As we have said above these matrices should be among the inverse-symmetric matrices formed by only three non zero elements. In order that the hermiticity (3) to be verified, let us take the 3×3-inverse-symmetric matrices formed by the cubic roots of unit, 1, $j = e^{\frac{2i\pi}{3}}$ and $j^2 = e^{\frac{4i\pi}{3}}$. Our choice of the cubic roots of unit have been inspired by [7], [12].

$$\tau_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},\ \tau_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & j \\ 0 & j^2 & 0 \end{pmatrix},\ \tau_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & j^2 \\ 0 & j & 0 \end{pmatrix}$$

$$\tau_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ \tau_5 = \begin{pmatrix} 0 & 0 & j \\ 0 & 1 & 0 \\ j^2 & 0 & 0 \end{pmatrix},\ \tau_6 = \begin{pmatrix} 0 & 0 & j^2 \\ 0 & 1 & 0 \\ j & 0 & 0 \end{pmatrix}$$

$$\tau_7 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \tau_8 = \begin{pmatrix} 0 & j & 0 \\ j^2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \tau_9 = \begin{pmatrix} 0 & j^2 & 0 \\ j & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The set $\mathcal{K}_3$ of them is a basis of $\mathbb{C}^{3\times 3}$. We can check easily that these matrices satisfy the two other properties, orthogonality (5) and (2), for $n = 3$. In contrast with the Kronecker generalized Pauli matrices the 3×3-KPMs are not traceless, but according to the orthogonality (5) and hermiticity (3) any product of two different 3×3-KPMs is traceless. Thus $\mathcal{K}_3$ up to multiplicative phases can not be a group. However, according to the relation (9), for defining the Pauli group $\mathcal{G}_3$ of the 3×3-KPMs we suggest to take the set of the products of two elements of the 3×3-KPMs up to multiplicative phases. We have got the following relations between these products

$$\tau_7\tau_3 = j^2\tau_9\tau_2 = j\tau_8\tau_1 = \tau_6\tau_7 = j\tau_5\tau_8 = j^2\tau_4\tau_9 = j\tau_3\tau_6 = j^2\tau_2\tau_4 = \tau_1\tau_5 \quad (10)$$

$$\tau_7\tau_6 = j^2\tau_8\tau_5 = j\tau_9\tau_4 = \tau_6\tau_3 = j\tau_4\tau_2 = j^2\tau_5\tau_1 = j\tau_2\tau_9 = j^2\tau_1\tau_8 = \tau_3\tau_7 \quad (11)$$

$$\tau_8\tau_3 = j^2\tau_4\tau_8 = j\tau_7\tau_2 = \tau_9\tau_1 = j\tau_5\tau_7 = j^2\tau_3\tau_4 = j\tau_1\tau_6 = j^2\tau_2\tau_5 = \tau_6\tau_9 \quad (12)$$

$$\tau_9\tau_3 = j^2\tau_8\tau_2 = j\tau_7\tau_1 = \tau_5\tau_9 = j\tau_4\tau_7 = j^2\tau_6\tau_8 = j\tau_1\tau_4 = j^2\tau_2\tau_6 = \tau_3\tau_5 \quad (13)$$

$$\tau_8\tau_6 = j^2\tau_9\tau_5 = j\tau_7\tau_4 = \tau_2\tau_8 = j\tau_4\tau_1 = j^2\tau_5\tau_3 = j\tau_1\tau_7 = j^2\tau_3\tau_9 = \tau_6\tau_2 \quad (14)$$

$$\tau_9\tau_6 = j^2\tau_7\tau_5 = j\tau_8\tau_4 = \tau_6\tau_1 = j\tau_4\tau_3 = j^2\tau_5\tau_2 = j\tau_3\tau_8 = j^2\tau_2\tau_7 = \tau_1\tau_9 \quad (15)$$

$$\tau_9\tau_7 = j^2\tau_3\tau_1 = j\tau_6\tau_5 = \tau_8\tau_9 = j\tau_5\tau_4 = j^2\tau_2\tau_3 = j\tau_4\tau_6 = j^2\tau_1\tau_2 = \tau_7\tau_8 \quad (16)$$

$$\tau_9\tau_8 = j^2\tau_6\tau_4 = j\tau_3\tau_2 = \tau_8\tau_7 = j\tau_2\tau_1 = j^2\tau_5\tau_6 = j\tau_1\tau_3 = j^2\tau_4\tau_5 = \tau_7\tau_9 \quad (17)$$

Therefore we can check easily that

$$\mathcal{G}_3 = \mathcal{K}_3^2 \otimes \{1,\ j,\ j^2\} = \mathcal{K}_3\mathcal{K}_3 \otimes \{1,\ j,\ j^2\} \quad (18)$$

is a group, the Pauli group of the 3×3-KPMs.

6 Roads to generalization

We talk here two roads to generalization, the first one is by Kronecker product, which do not include the case of prime number, thus the second one is the case of prime number.

6.1 Kronecker generalization

In this subsection, we give two examples of 6×6-KPMs, obtained by Kronecker product. The first one is $(\tau_j \otimes \sigma_k)_{1\leq j\leq 9, 0\leq k\leq 3}$ and the second one is $(\sigma_j \otimes \tau_k)_{0\leq j\leq 3, 1\leq k\leq 9}$

6.2 prime number $\times$ prime number-KPMs

The case of 3×3-KPMs suggests us how to construct a 5×5-KPMs.
For starting, let us take 5×5 ones matrices, all elements are equals to $+1$. Decompose this matrices as a sum of five inverse-symmetric matrices the only five non zero elements are equals to $+1$,

$$\begin{pmatrix}1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\end{pmatrix}=\begin{pmatrix}1&0&0&0&0\\0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\end{pmatrix}+\begin{pmatrix}0&0&1&0&0\\0&1&0&0&0\\1&0&0&0&0\\0&0&0&0&1\\0&0&0&1&0\end{pmatrix}+\begin{pmatrix}0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\\1&0&0&0&0\end{pmatrix}+$$
$$\begin{pmatrix}0&1&0&0&0\\1&0&0&0&0\\0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\end{pmatrix}+\begin{pmatrix}0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\\1&0&0&0&0\\0&0&0&0&1\end{pmatrix}.$$

For each of these five inverse-symmetric matrices replace the $+1$s by the five fifth roots of unity $1,\ u,\ u^2,\ u^3,\ u^4$. But we arrange them in order that the orthogonality (5) is satisfied and they are inverse-symmetric matrices. Then, we have got the following twenty five inverse-symmetric matrices, which are 5×5-KPMs. That is they share also the five properties of the Kronecker generalized Pauli matrices.

$$Q_1=\begin{pmatrix}1&0&0&0&0\\0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\end{pmatrix},\ Q_2=\begin{pmatrix}1&0&0&0&0\\0&0&0&0&u^4\\0&0&0&u^3&0\\0&0&u^2&0&0\\0&u&0&0&0\end{pmatrix},\ Q_3=\begin{pmatrix}1&0&0&0&0\\0&0&0&0&u^2\\0&0&0&u^4&0\\0&0&u&0&0\\0&u^3&0&0&0\end{pmatrix},$$
$$Q_4=\begin{pmatrix}1&0&0&0&0\\0&0&0&0&u^3\\0&0&0&u&0\\0&0&u^4&0&0\\0&u^2&0&0&0\end{pmatrix},\ Q_5=\begin{pmatrix}1&0&0&0&0\\0&0&0&0&u\\0&0&0&u^2&0\\0&0&u^3&0&0\\0&u^4&0&0&0\end{pmatrix}$$
$$Q_6=\begin{pmatrix}0&0&1&0&0\\0&1&0&0&0\\1&0&0&0&0\\0&0&0&0&1\\0&0&0&1&0\end{pmatrix},\ Q_7=\begin{pmatrix}0&0&u&0&0\\0&1&0&0&0\\u^4&0&0&0&0\\0&0&0&0&u^2\\0&0&0&u^3&0\end{pmatrix},\ Q_8=\begin{pmatrix}0&0&u^3&0&0\\0&1&0&0&0\\u^2&0&0&0&0\\0&0&0&0&u^4\\0&0&0&u&0\end{pmatrix},$$
$$Q_9=\begin{pmatrix}0&0&u^2&0&0\\0&1&0&0&0\\u^3&0&0&0&0\\0&0&0&0&u\\0&0&O&u^4&0\end{pmatrix},\ Q_{10}=\begin{pmatrix}0&0&u^4&0&0\\0&1&0&0&0\\u&0&0&0&0\\0&0&0&0&u^3\\0&0&0&u^2&0\end{pmatrix}$$

$$Q_{11}=\begin{pmatrix}0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\\1&0&0&0&0\end{pmatrix}, Q_{12}=\begin{pmatrix}0&0&0&0&u\\0&0&0&u^2&0\\0&0&1&0&0\\0&u^3&0&0&0\\u^4&0&0&0&0\end{pmatrix}, Q_{13}=\begin{pmatrix}0&0&0&0&u\\0&0&0&u^2&0\\0&0&1&0&0\\0&u^3&0&0&0\\u^4&0&0&0&0\end{pmatrix},$$

$$Q_{14}=\begin{pmatrix}0&0&0&0&u^2\\0&0&0&u^4&0\\0&0&1&0&0\\0&u&0&0&0\\u^3&0&0&0&0\end{pmatrix}, Q_{15}=\begin{pmatrix}0&0&0&0&u^4\\0&0&0&u^3&0\\0&0&1&0&0\\0&u^2&0&0&0\\u&0&0&0&0\end{pmatrix}$$

$$Q_{16}=\begin{pmatrix}0&1&0&0&0\\1&0&0&0&0\\0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\end{pmatrix}, Q_{17}=\begin{pmatrix}0&u&0&0&0\\u^4&0&0&0&0\\0&0&0&0&u^2\\0&0&0&1&0\\0&0&u^3&0&0\end{pmatrix}, Q_{18}=\begin{pmatrix}0&u^3&0&0&0\\u^2&0&0&0&0\\0&0&0&0&u\\0&0&0&1&0\\0&0&u^4&0&0\end{pmatrix},$$

$$Q_{19}=\begin{pmatrix}0&u^2&0&0&0\\u^3&0&0&0&0\\0&0&0&0&u^4\\0&0&0&1&0\\0&0&u&0&0\end{pmatrix}, Q_{20}=\begin{pmatrix}0&u^4&0&0&0\\u&0&0&0&0\\0&0&0&0&u^3\\0&0&0&1&0\\0&0&u^2&0&0\end{pmatrix}$$

$$Q_{21}=\begin{pmatrix}0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\\1&0&0&0&0\\0&0&0&0&1\end{pmatrix}, Q_{22}=\begin{pmatrix}0&0&0&u&0\\0&0&u^2&0&0\\0&u^3&0&0&0\\u^4&0&0&0&0\\0&0&0&0&1\end{pmatrix}, Q_{23}=\begin{pmatrix}0&0&0&u^2&0\\0&0&u^4&0&0\\0&u&0&0&0\\u^3&0&0&0&0\\0&0&0&0&1\end{pmatrix},$$

$$Q_{24}=\begin{pmatrix}0&0&0&u^3&0\\0&0&u&0&0\\0&u^4&0&0&0\\u^2&0&0&0&0\\0&0&0&0&1\end{pmatrix}, Q_{25}=\begin{pmatrix}0&0&0&u^4&0\\0&0&u^3&0&0\\0&u^2&0&0&0\\u&0&0&0&0\\0&0&0&0&1\end{pmatrix}$$

The decomposition of 5×5 ones matrices as a sum of five inverse-symmetric matrices the only five non zero elements are equals to $+1$ is not unique, thus we can construct other 5×5-KPMs than above.

Conclusion

For concluding, we think having given solution to the problem evoked of searching 3×3 matrices sharing five properties of the Kronecker generalized Pauli matrices, tracelessness moved apart. We call these matrices 3×3-Kronecker-Pauli matrices. For the definition of the Pauli group we would prefer to call Pauli group of the generalized Pauli matrices the group of the set of the products of two elements up to multiplicative phases in order that it can be extended to the 3×3-KPMs.
The 3×3-KPMs we have obtained suggest us how to construct 5×5-KPMs.
We have introduced what we call inverse-symmetric matrices. Their properties

and those of Kronecker matrices have made more obvious the construction of the 3×3-KPMs and some ways to generalization.

References

[1] Rakotonirina C. and Rakotondramavonirina J., *Tensor Commutation Matrices and Some Generalizations of the Pauli Matrices*, Gen. Math. Notes, **23**(1), (2014), 1-8.

[2] Faddeev L.D.*Algebraic Aspects of Bethe Ansatz*, Int. J. Mod. Phys. A, 10, 1845 (1995). DOI: 10.1142/S0217751X95000905

[3] Zvonarev, M. *Notes on Bethe Ansatz* (2010).

[4] Fujii, K. *Introduction to coherent states and quantum information theory, Prepared for 10th Numazu Meeting on Integral System, Noncommutative Geometry and Quantum theory*, Numazu, Shizuoka, Japan, 7-9 Mai (2002).

[5] Verstraete, F., *A study of entanglement in quantum information theory*, Thèse de Doctorat, Katholieke Universiteit Leuven, (2002), 90-93.

[6] Rakotonirina, C. *Expression of a tensor commutation matrix in terms of the generalized Gell-Mann matrices*, International Journal of Mathematics and Mathematical Sciences, Article ID 20672 (2007).

[7] Kibler, M.R., *An angular momentum approach to quadratic Fourier transform, Hadamard matrices, Gauss sums, mutually unbiased bases, the unitary group and the Pauli group*, J. Phys. A: Math. Theor., 42(353001) (2009), 28.

[8] C. Rigetti, C., Mosseri, R. and Devoret, M., *Geometric approach to digital quantum information*, Quantum Information Processing, 3(6)(December) (2004), 351-380.

[9] Rakotonirina, C. *Rectangle Gell-Mann matrices*, Int. Math. Forum, Vol. 6, 2011, no. 1-4, 57-62.

[10] Saniga, M., Planat, M. and Pracna, P. *Projective ring line encompassing two-qubits*, Hal-00111733, version 5 - 28 December (2006).

[11] Gil, J.J. and José, I.S., *Invariant indices of polarimetric purity generalized indices of purity for $n \otimes n$ covariance matrices*, Opt. Commun., 284(2011), 38.

[12] Volkov, G., *Ternary quaternions and ternary $TU(3)$ algebra*, arXiv: 1006.5627 (2010).

A Kronecker Product

Definition 7 *For any matrices* $A = \left(A^i_j\right)_{1\leq i\leq n, 1\leq j\leq p} \in \mathbb{C}^{n\times p}$, $B = \left(B^i_j\right)_{1\leq i\leq m, 1\leq j\leq q} \in \mathbb{C}^{m\times q}$ *the Kronecker product of the matrix* A *by the matrix* B *is the matrix* $A \otimes B \in \mathbb{C}^{nm\times pq}$

$$A \otimes B = \begin{pmatrix} A^1_1 B & A^1_2 B & \cdots & A^1_p B \\ A^2_1 B & A^2_2 B & \cdots & A^2_p B \\ \vdots & \vdots & \cdots & \vdots \\ A^m_1 B & A^m_2 B & \cdots & A^m_p B \end{pmatrix}$$

Properties 8 .

- *i)* $\otimes$ *is associative.*
- *ii)* $\otimes$ *is distributive with respect to the addition.*
- *iii) For any matrices* A, B, C *and* D
$$(A \otimes B)(C \otimes D) = AC \otimes BD$$
- *iv) For any invertible matrices* A *and* B
$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$$
- *v)*
$$(A \otimes B)^{+} = A^{+} \otimes B^{+}$$
- *vi)*
$$Tr(A \otimes B) = Tr(A)Tr(B)$$

Proposition 9 *Let* $(A_i)_{1\leq i\leq np}$ *and* $(B_j)_{1\leq j\leq mq}$ *respectively be some bases of* $\mathbb{C}^{n\times p}$ *and* $\mathbb{C}^{m\times q}$. *Then,* $(A_i \otimes B_j)_{1\leq i\leq np, 1\leq j\leq mq}$ *is a basis of* $\mathbb{C}^{nm\times pq}$.

Proposition 10 .
If $\sum_{j=1}^{m} M_j \otimes \ N_j = \sum_{i=1}^{n} A_i \otimes \ B_i$ *then,* $\sum_{j=1}^{m} M_j \otimes \ K \otimes \ N_j = \sum_{i=1}^{n} A_i \otimes \ K \otimes \ B_i$, *for any matrix* K.

[1]